Chapter 24: Gauss’s Law

Summary: Gauss’s law is used to calculate the electric field surrounding a given charge distribution. The law, which is based on the concept of electric flux, is most convenient for calculating the electric fields of symmetrical charge distributions.

Key Equations:

\[ \Phi = EA \cos \theta \]
\[ \oint E \cdot dA = \frac{q_{\text{enc}}}{\varepsilon_0} \]

Major Topics:

- Electric flux
  - Defined as the component of the electric field passing through a surface
  - Proportional to the number of electric field lines penetrating the surface.
  - The net electric flux through a closed surface that surrounds no charge is zero.
- Gauss’s Law
  - Right side: the net flux through any closed surface surrounding a point charge \( q \) is given by \( q/\varepsilon_0 \) and is independent of the shape of that surface.
  - Left side: \( E \) represents the total electric field, which is effected by charges both inside and outside the surface.
  - Gauss’s Law is most commonly applied using spherical or cylindrical charge distributions. It can also be used for planes of charge with a given charge density.
- Notes about conductors:
  - The electric field inside a conductor is zero
  - Any net charge on a conductor resides on the surface
  - The electric field just outside the conductor has a magnitude \( \sigma/\varepsilon_0 \)
Practice Problems:

1. A pyramid with a horizontal square base, 6.00 m on each side, and a height of 4.00 m is placed in a vertical electric field of 52.0 N/C. Calculate the total electric flux through the pyramid’s four slanted surfaces.

Solution 1:

This problem deals with the concept of electric flux. One could use geometry to determine all the angles and areas of the 4 sides of the pyramid, but the far simpler way to do this problem is to recognize that because flux is a dot product, the electric flux through the base will equal the flux through the four triangular side surfaces.

\[ \Phi_{E-base} = \Phi_{E-sides} \]

\[ \Phi_E = E \cdot A = EA \cos \theta \]

\[ \Phi_E = (52.0 \text{ N/C})(6.00\text{m})^2(\cos0) \]

\[ \Phi_E = 1.87 \text{ kN} \cdot \text{m}^2/\text{C} \]

2. A solid sphere of radius 40.0 cm has a total positive charge of 26.0 \( \mu \)C uniformly distributed throughout its volume. Calculate the magnitude of the electric field at a) 0 cm, b) 10.0 cm, c) 40.0 cm, d) 60.0 cm from the center of the sphere.

Solution 2:

This is a Gauss’s law problem, using spherical symmetry. Note that the charge is uniformly distributed throughout the sphere’s volume.

Begin with the equation form of Gauss’s law.

\[ \int E \cdot dA = \frac{Q_{in}}{\varepsilon_0} \]

Because \( E \) is constant for a specific radius, it can be pulled out of the integral. \( E \) and \( dA \) will also have parallel vectors, meaning that the angle is zero degrees, and the cos of the angle is 1, meaning we can eliminate the dot product function.

\[ E \int dA = \frac{Q_{in}}{\varepsilon_0} \]

To complete the left side, we know that for spherical symmetry, the integral of \( dA \) is always \( 4\pi r^2 \), the surface area of the Gaussian sphere.
For this problem, \( Q_{in} \) will vary as the radius changes.

a) When \( r = 0 \) cm, there is no charge enclosed, so \( E = 0 \).

b) When \( r = 10.0 \) cm, because the charge is evenly distributed, the charge enclosed will be the total charge multiplied by the ratio of the volume enclosed by the Gaussian surface to the total volume of the sphere.

\[
V = \frac{4}{3} \pi R^3
\]

\[
Q_{in} = Q \cdot \frac{V}{\frac{4}{3} \pi R^3} = Q \frac{r^3}{R^3}
\]

Plug this equation in for the enclosed charge, and solve for \( E \)

\[
E(4\pi r^2) = \frac{Q r^3}{\epsilon_0}
\]

\[
E = \frac{Q r}{4\pi \epsilon_0 R^3}
\]

This equation will be true for all values of \( r < R \). Plug in values for \( r = 10.0 \) cm.

\[
E = \frac{(26 \times 10^{-6} \text{ C})(.10 \text{ m})/[4\pi(8.85 \times 10^{-12} \text{ C}^2/\text{N}\cdot\text{m}^2)(.40\text{m})^3]} = 365 \text{ kN/C}
\]

c) For \( r = 40.0 \) cm, we use spherical symmetry again, except now \( r = R \). Therefore, the total charge enclosed will just equal \( Q \). This gives the final equation:

\[
E = \frac{Q}{4\pi \epsilon_0 r^2}
\]

Note that this is also the electric field equation for a point charge that was shown in the previous chapter.

Plugging in the values, we get:

\[
E = \frac{(26 \times 10^{-6} \text{ C})/[4\pi(8.85 \times 10^{-12} \text{ C}^2/\text{N}\cdot\text{m}^2)(.40\text{m})^3]} = 1.46 \text{ MN/C}
\]

d) For \( r = 60.0 \) cm, \( r > R \), so we can use the point charge function in part c.
\[ E = \frac{Q}{4\pi \epsilon_0 r^2} \]

\[ E = \frac{(26 \times 10^{-6} \text{ C})}{4\pi (8.85 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2)(0.60\text{ m})^2} = 649 \text{ kN/C} \]

Note: As the enclosed charge is positive, all electric fields are directed radially outward.

3. A long, straight metal rod has a radius of 5.00 cm and a charge per unit length of 30.0 nC/m. Find the electric field a) 3.00 cm, b) 10.0 cm, and c) 100 cm from the axis of the rod, where distances are measured perpendicular to the rod.

**Solution 3:**

In this Gauss’s law problem, the rod is a conductor. Also, we will use cylindrical, not spherical symmetry. By the same process explained above…

\[ \int E \cdot dA = \frac{Q_{in}}{\epsilon_0} \]

\[ E(2\pi r L) = \frac{Q_{in}}{\epsilon_0} \]

Where \( 2\pi r L \) is the surface area of the cylinder through which the flux passes. (Realize that no flux passes through the end caps because they are perpendicular to the electric field)

a) For \( r = 3.00 \text{ cm} \), \( r < R \) so the electric field is zero because this is inside a conductor. \[ E = 0 \]

b) For \( r = 10.0 \text{ cm} \), since we are not given a length but a charge per unit length, we use the charge density function \( \lambda = \frac{Q}{L} \)

\[ E = \frac{Q_{in}}{\epsilon_0 (2\pi r L)} = \frac{\lambda}{\epsilon_0 2\pi r} \]

Plug in values:

\[ E = \frac{(30.0 \times 10^{-9} \text{ C/m})}{[(8.85 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2)2\pi(0.10\text{ m})]} = 5.40 \text{ kN/C outward} \]

c) For \( r = 100 \text{ cm} \), we just plug in to the same equation as in b

\[ E = \frac{\lambda}{\epsilon_0 2\pi r} = \frac{(30.0 \times 10^{-9} \text{ C/m})}{[(8.85 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2)2\pi(1.00\text{ m})]} = 540 \text{ N/C outward} \]