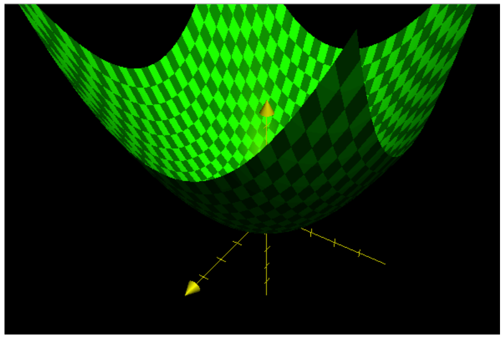
Partial Derivatives and the Del Operators

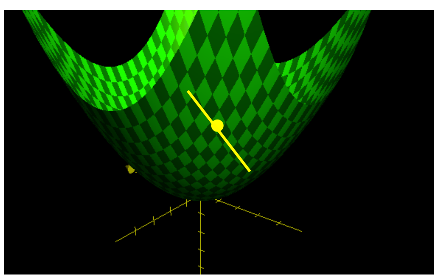
Let's say you have a multivariable function like: $z = f(x,y) = (y^2 + x^3)/5$

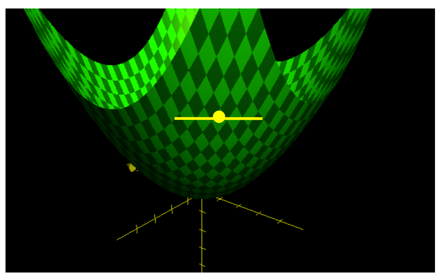
(look at 3-d depiction)

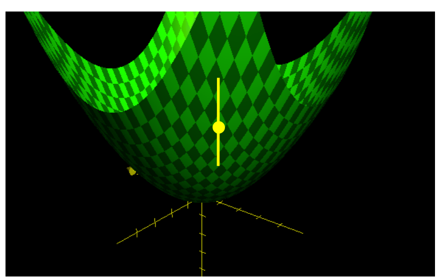
We'd like to know something about how the function changes spatially. That is, how does the function's value vary as we traverse in a particular direction in the 3-d space. Put altogether differently, how do we determine the derivative of the function with respect to a particular direction?



The problem is that at any given point (see yellow dot on sketch), there are an infinite number of slopes (three are shown).







So how do we denote the slope in, say, the x-direction?

What we need is an operation that takes the derivative of the function in a particular direction while keeping all other variables constant. The derivative that does that is given a special name and symbol. It is called a *partial derivative*, and it's notion is shown below:

$$\frac{\partial f}{\partial x}.$$

This is termed "the partial derivative of *f* with respect to *x*." It essentially directs the user to treat the function as though *x* is the only active variable with all other variables held constant.

For our problem, the rate at which the function changes with changes in *x* is:

$$\frac{\partial f}{\partial x} = \frac{\partial\left[(y^2 + x^3)/5\right]}{\partial x}$$
$$= 3x^2/5$$

Why is this useful? If have an electric field E and you travel some distance dx in the direction of the field, you will observe a change of voltage dV that is related to E and dx by the relationship:

$$\vec{E}_x \bullet d\vec{x} = -dV$$
$$\Rightarrow \quad |E_x||dx|\cos 0^\circ = -dV$$
$$\Rightarrow \quad E_x \; dx = -dV$$

If this is true, it must also be true that:

$$E_x = -\frac{dV}{dx}$$

In other words, the *x-component* of the electric field is related the rate at which the voltage field changes as you traverse in the *x-direction* (give or take a minus sign). And if we wanted to write that out using unit vectors, we would come up with:

$$(E_x)\hat{i} = -\left(\frac{dV}{dx}\right)\hat{i}$$

This is all fine for a one-dimensional world, but what if we want E in a three dimensional world? In that case, we have to do the derivative operation for *each direction*, then multiply by the appropriate unit vector. Additionally, when we take those derivatives we also have to keep the other variables constant so that the only variability is in the direction of interest. In other words, we need to use *partial derivatives*. Doing that yields:

$$\bar{E} = \left[(E_x)\hat{i} + (E_y)\hat{j} + (E_z)\hat{k}\right]$$

$$\Rightarrow \quad \vec{E} = -\left[\left(\frac{\partial V}{\partial x}\right)\hat{i} + \left(\frac{\partial V}{\partial y}\right)\hat{j} + \left(\frac{\partial V}{\partial z}\right)\hat{k}\right]$$

As is often the case when a mathematical operation is used over and over again within the world of physics, a special notation has been defined to allow us to shorthand it (think dot and cross products). It makes use of what is called in the world of multi-dimensional Calculus the *del operator* (symbol $\vec{\nabla}$), and it looks like:

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Minor note: The del operator acts on vectors and scalars in a specific, mathematical way. It has no meaning intrinsic, unto itself. It is an *operation*. In any case, as applied to the electric field/electric potential situation, we can write:

$$\bar{E} = \left[\ (E_x)\hat{i} + \ (E_x)\hat{j} + (E_x)\hat{k}\right]$$

$$\Rightarrow \quad \vec{E} = -\left[\left(\frac{\partial V}{\partial x}\right)\hat{i} + \left(\frac{\partial V}{\partial x}\right)\hat{j} + \left(\frac{\partial V}{\partial x}\right)\hat{k}\right]$$

$$\Rightarrow \quad \vec{E} = -\left[\left(\frac{\partial}{\partial x}\right)\hat{i} + \left(\frac{\partial}{\partial x}\right)\hat{j} + \left(\frac{\partial}{\partial x}\right)\hat{k}\right]V$$

$$\Rightarrow \quad \vec{E} = -\vec{\nabla}V$$

So let's say our f(x,y) function had been a voltage function such that:

$$V = (k_1y^2 + k_2x^3)/5$$

where the *k* terms are constants whose units make each entity's units *volts*.

What electric field belongs to this potential function?

$$\bar{E} = \left[\ (E_x)\hat{i} + \ (E_x)\hat{j} + (E_x)\hat{k}\right]$$

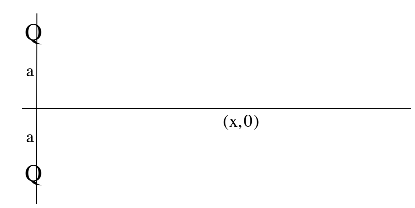
$$\Rightarrow \quad \vec{E} = -\left[\left(\frac{\partial V}{\partial x}\right)\hat{i} + \left(\frac{\partial V}{\partial x}\right)\hat{j} + \left(\frac{\partial V}{\partial x}\right)\hat{k}\right]$$

$$\Rightarrow \quad \vec{E} = -\left[\left(\frac{\partial\left[\frac{(k_1y^2 + k_2x^3)}{5}\right]}{\partial x}\right)\hat{i} + \left(\frac{\partial\left[\frac{(k_1y^2 + k_2x^3)}{5}\right]}{\partial x}\right)\hat{j} + \left(\frac{\partial\left[\frac{(k_1y^2 + k_2x^3)}{5}\right]}{\partial x}\right)\hat{k}\right]$$

$$\Rightarrow \quad \vec{E} = -\left[\quad \frac{3k_2x^2}{5}\hat{i} \quad + \quad \left(\frac{2k_1y}{5}\right)\hat{j} \quad + \quad 0\hat{k} \quad \right]$$

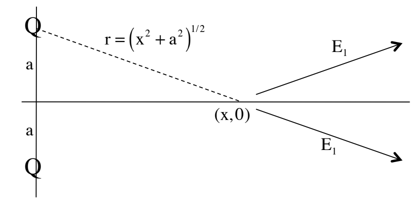
Nifty, eh?

Consider now the following: Two equal charges are symmetrically placed on an axis as shown. Using the del operator approach, derive an expression for the electric field at (x,0), where *x* is an arbitrary point down the x-axis?



$$V = k\frac{Q}{r} + k\frac{Q}{r}$$

$$= k\frac{2Q}{(x^2 + a^2)^{1/2}}$$



Note that if nothing else, the electric field components in the *y-direction* should add to zero leaving only a net *x-component* for the field.

Using the del operator to determine the electric field vector, we get:

$$\vec{E} = -\vec{\nabla} V$$

$$= -\left[\left(\frac{\partial}{\partial x}\right)\hat{i} + \left(\frac{\partial}{\partial x}\right)\hat{j} + \left(\frac{\partial}{\partial x}\right)\hat{k}\right]V$$

$$= -\left[\left(\frac{\partial V}{\partial x}\right)\hat{i} + \left(\frac{\partial V}{\partial x}\right)\hat{j} + \left(\frac{\partial V}{\partial x}\right)\hat{k}\right]$$

$$= -\left(\frac{\partial\left(k\frac{2Q}{\left(x^2+a^2\right)^{1/2}}\right)}{\partial x}\hat{i} + \frac{\partial\left(k\frac{2Q}{\left(x^2+a^2\right)^{1/2}}\right)}{\partial y}\hat{j} + \frac{\partial\left(k\frac{2Q}{\left(x^2+a^2\right)^{1/2}}\right)}{\partial z}\hat{k}\right)$$

$$= -2kQ\left(\frac{\partial\left(\left(x^2+a^2\right)^{-1/2}\right)}{\partial r}\right)\hat{i}$$

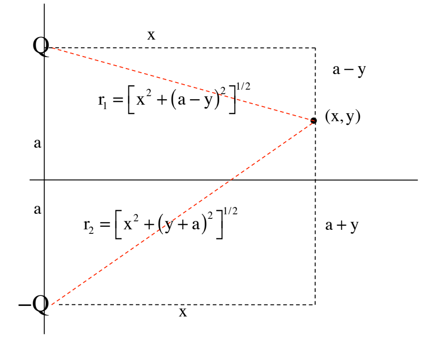
$$= -\left[2kQ\left(-\frac{1}{2}(2x)\right)\left(x^2+a^2\right)^{-3/2}\right]\hat{i}$$

$$= \left[2kQ\frac{x}{\left(x^2+a^2\right)^{3/2}}\right]\hat{i}$$

What it all comes down to is the fact that we didn't determine a general expression for the electrical potential function (that is, one that is good for *any* arbitrary point in space). Technically, *that* should have had its derivative taken. That voltage function would looks like:f

$$V(x,y) = k\frac{Q}{r_1} + k\frac{-Q}{r_2}$$

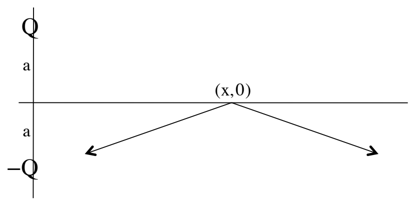
$$= k\frac{Q}{\left(x^2+(a-y)^2\right)^{1/2}} - k\frac{Q}{\left(x^2+(a+y)^2\right)^{1/2}}$$



The relationship $\vec{E} = \left[2kQ\frac{x}{\left(x^2+a^2\right)^{3/2}}\right]\hat{i}$ seems reasonable (at least the units work out).

There is a rub, though, and it is easily seen if we alter the problem just a bit. Let's now assume the charges are equal but opposite.

The electric field is clearly non-zero, but the electrical potential at (x,0) IS zero. So what to do?



Going completely wild, we can use the del operator on this getting?

$$\vec{E} = -\vec{\nabla} V$$

$$= -kQ\left(\frac{\partial\left(\frac{1}{\left(x^2+(a-y)^2\right)^{1/2}} - \frac{1}{\left(x^2+(a+y)^2\right)^{1/2}}\right)}{\partial x}\hat{i} + \frac{\partial\left(\frac{1}{\left(x^2+(a-y)^2\right)^{1/2}} - \frac{1}{\left(x^2+(a+y)^2\right)^{1/2}}\right)}{\partial y}\hat{j} + 0\hat{k}\right)$$

$$= -kQ\left[\frac{\partial\left(\left(x^2+(a-y)^2\right)^{-1/2} - \left(x^2+(a+y)^2\right)^{-1/2}\right)}{\partial x}\hat{i} + \frac{\partial\left(\left(x^2+(a-y)^2\right)^{-1/2} - \left(x^2+(a+y)^2\right)^{-1/2}\right)}{\partial y}\hat{j}\right]$$

$$= -kQ\left[\left(-\frac{1}{2}\right)\left(x^2+(a-y)^2\right)^{-3/2}(2x) - \left(-\frac{1}{2}\right)\left(x^2+(a+y)^2\right)^{-3/2}(2x)\right]\hat{i}$$

$$-\frac{Q}{k}\left[\left(-\frac{1}{2}\right)\left(x^2+(a-y)^2\right)^{-3/2}\left(2(a-y)(-1)\right) - \left(-\frac{1}{2}\right)\left(x^2+(a+y)^2\right)^{-3/2}\left(2(a+y)(1)\right)\right]\hat{j}$$

$$= kQ\left[\frac{x}{\left(x^2+(a-y)^2\right)^{3/2}} - \frac{x}{\left(x^2+(a+y)^2\right)^{3/2}}\right]\hat{i} + \frac{Q}{k}\left[\frac{-(a-y)}{\left(x^2+(a-y)^2\right)^{3/2}} - \frac{(a+y)}{\left(x^2+(a+y)^2\right)^{3/2}}\right]\hat{j}$$

If we take our final relationship for E and evaluate it at (x, 0), we get:

$$\vec{E} = kQ\left[\frac{x}{\left(x^2+(a-y)^2\right)^{3/2}} - \frac{x}{\left(x^2+(a+y)^2\right)^{3/2}}\right]\hat{i} + \frac{Q}{k}\left[\frac{-(a-y)}{\left(x^2+(a-y)^2\right)^{3/2}} - \frac{(a+y)}{\left(x^2+(a+y)^2\right)^{3/2}}\right]\hat{j}$$

$$= kQ\left[\frac{x}{\left(x^2+(a-0)^2\right)^{3/2}} - \frac{x}{\left(x^2+(a+0)^2\right)^{3/2}}\right]\hat{i} + \frac{Q}{k}\left[\frac{-(a-0)}{\left(x^2+(a-0)^2\right)^{3/2}} - \frac{(a+0)}{\left(x^2+(a+0)^2\right)^{3/2}}\right]\hat{j}$$

$$= kQ\left[\frac{x}{\left(x^2+a^2\right)^{3/2}} - \frac{x}{\left(x^2+a^2\right)^{3/2}}\right]\hat{i} + \frac{Q}{k}\left[\frac{-a}{\left(x^2+a^2\right)^{3/2}} - \frac{a}{\left(x^2+a^2\right)^{3/2}}\right]\hat{j}$$

$$= kQ[0]\hat{i} + \frac{Q}{k}\left[\frac{-2a}{\left(x^2+a^2\right)^{3/2}}\right]\hat{j}$$

$$= -kQ\left[\frac{2a}{\left(x^2+a^2\right)^{3/2}}\right]\hat{j}$$

This is exactly what we'd expect: an electric field whose units turn out to be inverse square in meters and that is in the –j direction.

Dang! WE ARE GOOD!!!

And what was the deal with the electrical potential being zero at (x,0)? Not such a shock. The electric field isn't related to the electrical potential at a point, it's related to the rate of change of electrical potential at a point. The point in question can have V=0, but if there is a non-zero value just below or above that point, then dV/dy is not zero and there IS an electric field in that direction.