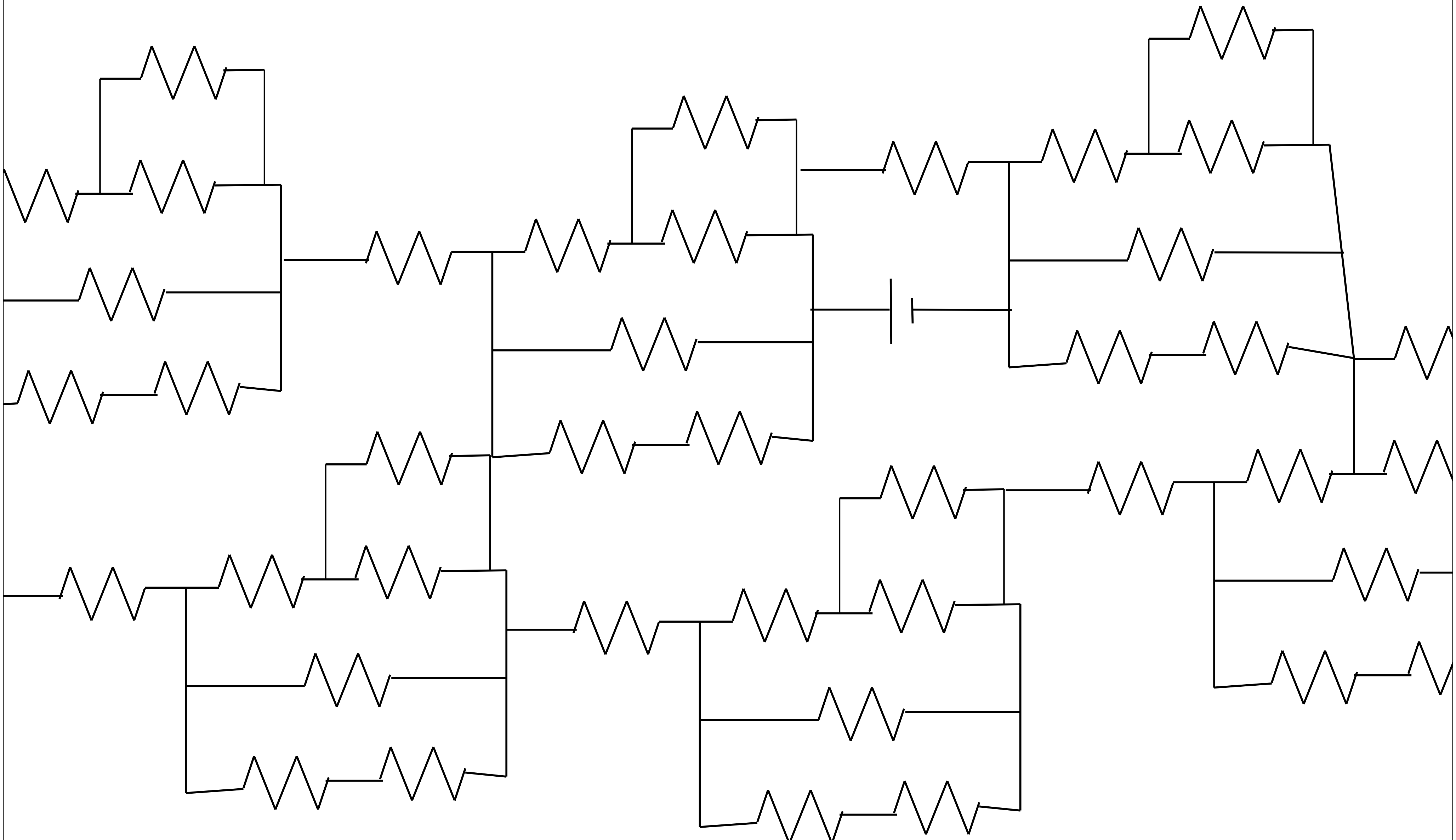
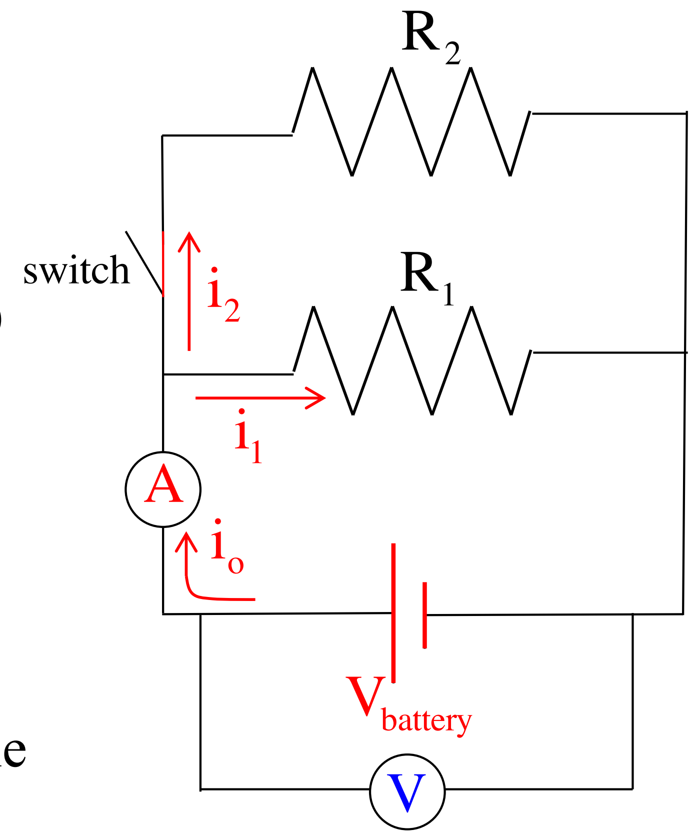
CHAPTER 28:
Complex DC Electrical Circuits



1.)

EMF and Terminal Voltage

Example 1: Consider the circuit to the right. If the resistors represent light bulbs:



a.) What does the *ammeter* read when the switch is open? (step 1—redraw without the meters)

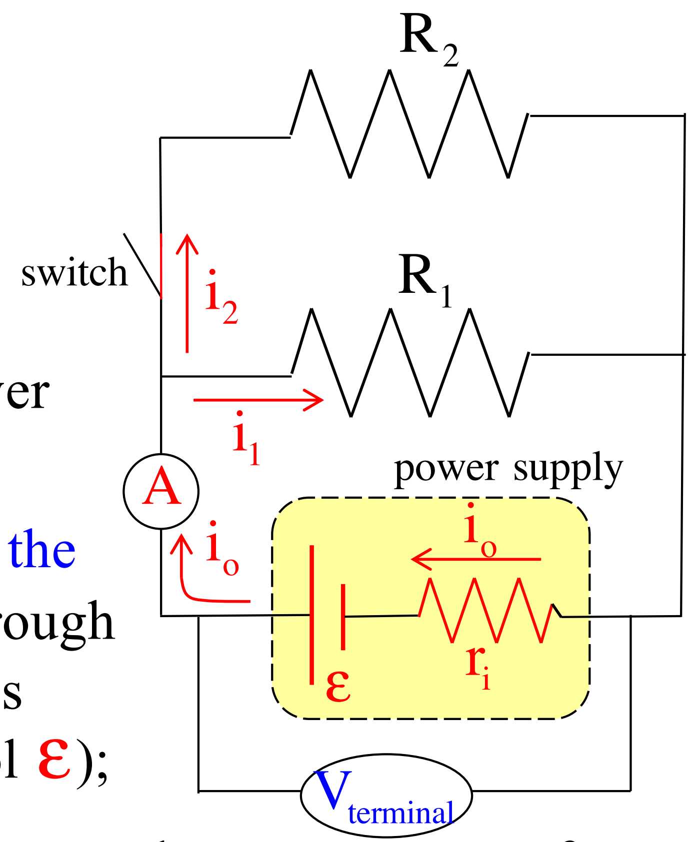
All the battery's voltage drop happens across the resistor R_1, and the current through the ammeter is just i_1, so:

$$V_{bat} = i_1 R_1 = i_o R_1 \quad \Rightarrow \quad i_o = \frac{V_{bat}}{R_1}$$

b.) In an ideal world, what should happen to the current through R_1 when the switch is thrown?

Nothing, and this is tricky. If you think of the current from the battery as fixed, you'll conclude R_1's current will drop as some current will now be needed for R_2. But au, contraire. Current through R_1 is governed by the voltage across it and its resistance, neither of which changes when you throw the switch. So what happens? The battery outputs MORE CURRENT so $i_o = i_i + i_2$ and R_1, in theory, should remain unaffected.

c.) Here's the rub: if you carry this experiment out in the real world, the light bulb associated with R_1 will *actually dim* suggesting that the current through R_1 has diminished. So what's going on?



The problem lies in the internal workings of power supplies. There are actually two parts to a p.s.:

1.) There is the part of the battery that creates the electric field that motivates charge to move through the wires. This part has the units of *volts* and is called *the electromotive force,* or EMF (symbol ε);

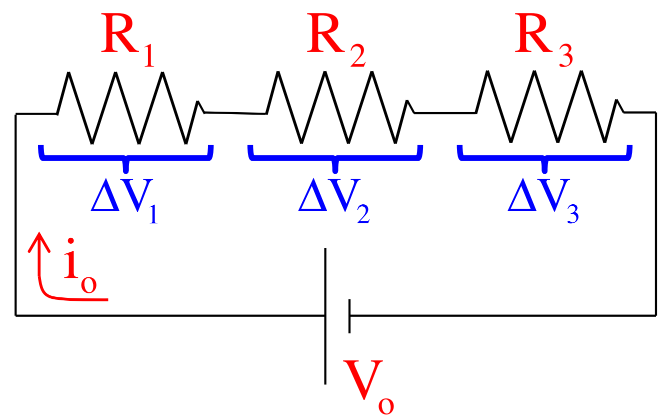
2.) Although it is possible to "rectify" a power supply to compensate for this, a power supply in its natural state also has internal resistance r_i.

Including the voltage drop due to r_i, this means a VOLTMETER will read what is called *the terminal voltage* (i.e., the voltage as measured at the terminals of the p.s.) equal to: $V_{terminal} = \varepsilon - i_o r_i$

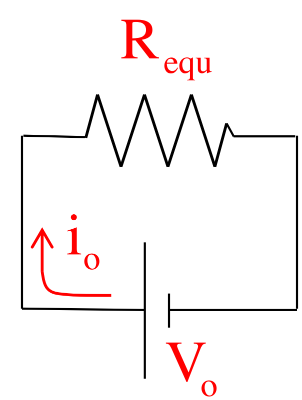
Soooo, if you increase i_o by throwing the switch, $V_{terminal}$ does DOWN across the resistors and the bulbs will dim!

Series Resistor Combinations

Example 2: Derive an expression for the *equivalent resistance* (the single resistor that can take the place of the resistor combination) for the series combination shown to the right. Assume an ideal power supply with no internal resistance.



The idea behind R_{equ} is to find the single resistor that can take the place of all the resistors in the system. In other words, the single resistor that, when put across V_o will draw i_o
.



Using the idea that the sum of the voltage drops across all the resistors will equal the voltage drop across the power supply, and including Ohm's Law in the mix, we can write:

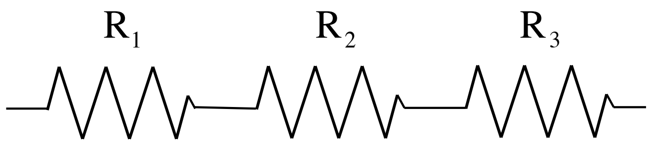
$$V_o \;=\; \Delta V_i + \Delta V_2 + \Delta V_3$$

$$\cancel{i_o} R_{eq} = \cancel{i_o} R_1 + \cancel{i_o} R_2 + \cancel{i_o} R_3$$

$$\Rightarrow \quad R_{eq} = R_1 + R_2 + R_3$$

Characteristics of a Series Combinations

--Each element in a series combination is attached to its neighbor in *one place only*.



--Current is common to each element in a series combination.

--There are no nodes (junctions—places where current can slit up) internal to series combinations.

--The equivalent resistance for a series combination is: $R_{eq} = R_1 + R_2 + R_3 + ...$

--This means the equivalent resistance is larger than the largest resistor in the combination;

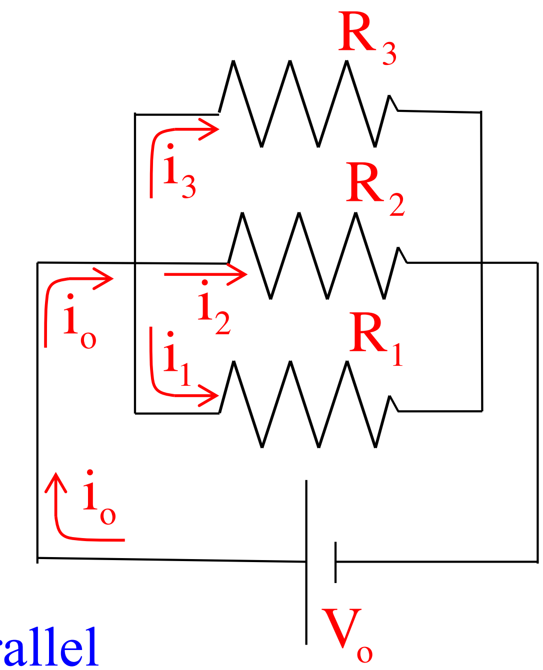
--This means that if you add a resistor to the combination, R_{eq} will *increase* and the current through the combination (for a given voltage) will *decrease*.

Example 3: What's the equivalent resistance of a 5 Ω, 6 Ω and 7 Ω resistor in series?

$$R_{eq} = (5\Omega) + (6\Omega) + (7\Omega)$$
$$= 18\ \Omega$$

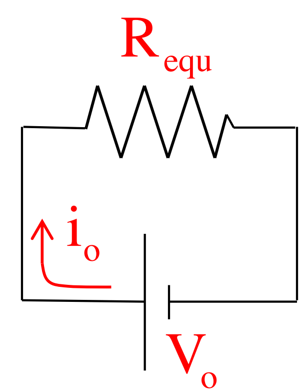
Parallel Resistor Combinations

Example 3: Derive an expression for the *equivalent resistance* (the single resistor that can take the place of the resistor combination) for the parallel combination shown to the right. Assume an ideal power supply with no internal resistance.



What's common in a parallel combination is the voltage drop across each element.

Also, in this case, the sum of the currents through the parallel combination must equal the current drawn from the power supply. Using that and Ohm's Law, we can write:

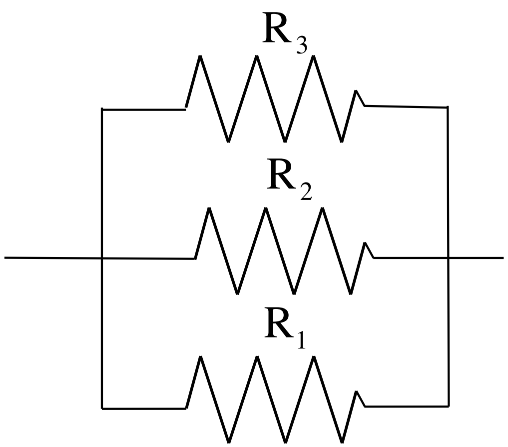


$$i_o = i_i + i_2 + i_3$$

$$\frac{\cancel{V_o}}{R_{eq}} = \frac{\cancel{V_o}}{R_1} + \frac{\cancel{V_o}}{R_2} + \frac{\cancel{V_o}}{R_3}$$

$$\Rightarrow \quad \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Characteristics of a Parallel Combinations



--*Each element* in a series combination is attached to its neighbor in *two place*.

--*Voltage* is common to each element in a parallel combination.

--*There are* nodes (junctions—places where current can slit up) internal to parallel combinations.

--*The equivalent resistance* for a parallel combination is:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots$$

--*This means* the equivalent resistance is SMALLER than the smallest resistor in the combination;

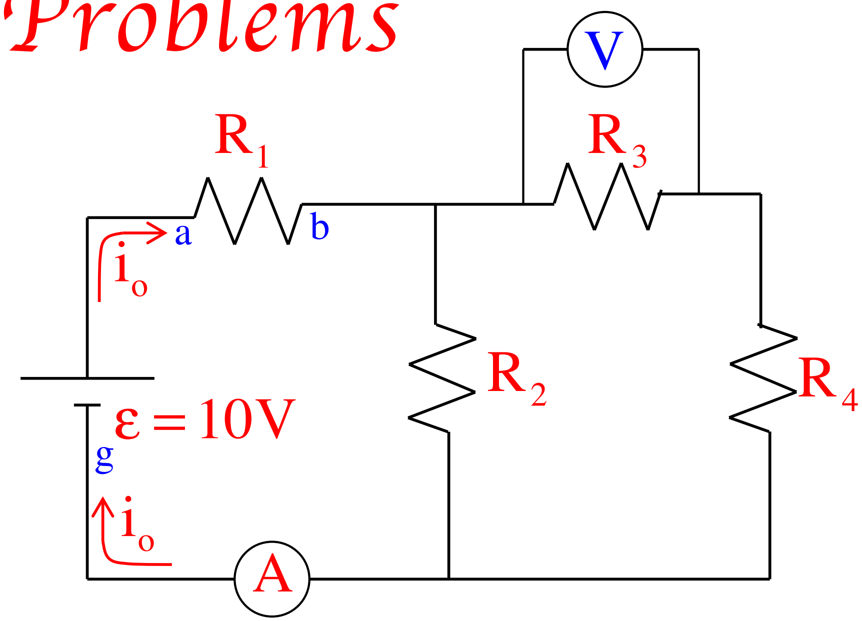
--*And,* if you add a resistor to the combination, R_{eq} will *decrease* and the current through the combination (for a given voltage) will *increase*.

Example 3: What's the equivalent resistance of three one-ohm resistors in parallel?

$$1/R_{eq} = 1/(1\Omega) + 1/(1\Omega) + 1/(1\Omega)$$

$$\Rightarrow 1/R_{eq} = 3 \quad \Rightarrow \quad R_{eq} = .333\ \Omega$$

Run and Shoot Problems



Example 4: An *ideal power supply* has EMF $\varepsilon = 10V$ powering it. Any blue letters that show up designate points on the circuit. Assume the *low voltage terminal* of the p.s. is at zero volts. Assume the resistor values are the same as the resistor subscripts.

a.) *What is* the first thing you would do if asked to work with this circuit?

Redraw the circuit without the meters. They aren't doing anything in the circuit except identifying a branch or resistor for which you want a current.

b.) *What is* the *absolute electrical potential* at *Point a*?

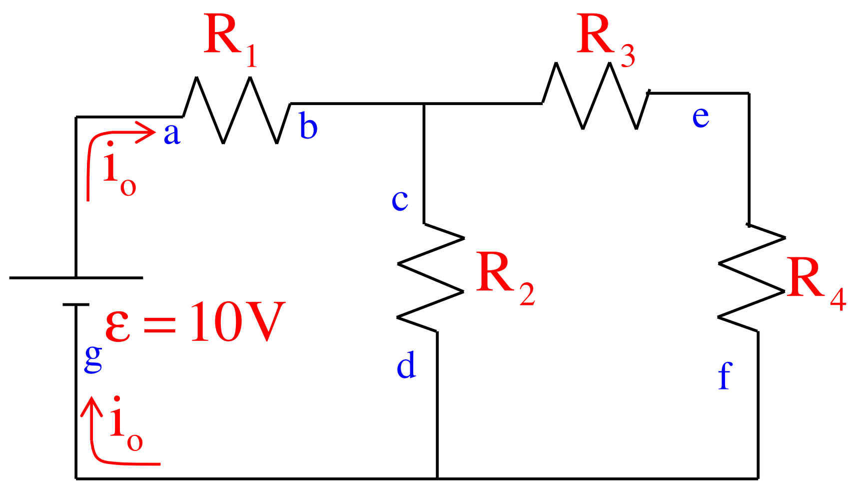
There is essentially no resistance between *Point a* and the high voltage terminal of the p.s., so their voltages are the same point being $V_a = 10v$.

c.) *What will* the ammeter read?

Some amount of current is being drawn from the power supply. Being in the same branch as *Points a*, *b* and *g*, it will be the same for all three points. It will also be the current through the ammeter. So how do we get that?

c.) *ammeter?*

Here is the circuit with the meters removed. The trick here is to find the equivalent resistance for the circuit, then use Ohm's Law.

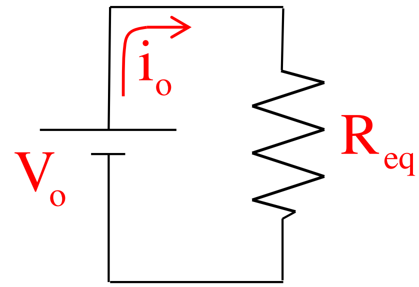


This circuit is R_1 in series with R_2 in parallel with R_3 and R_4 in series. That is:

$$R_{eq} = R_1 + \left(\frac{1}{R_2} + \frac{1}{R_3 + R_4}\right)^{-1}$$

$$= (1\ \Omega) + \left(\frac{1}{(2\ \Omega)} + \frac{1}{(3\ \Omega) + (4\ \Omega)}\right)^{-1}$$

$$= 2.56\ \Omega$$

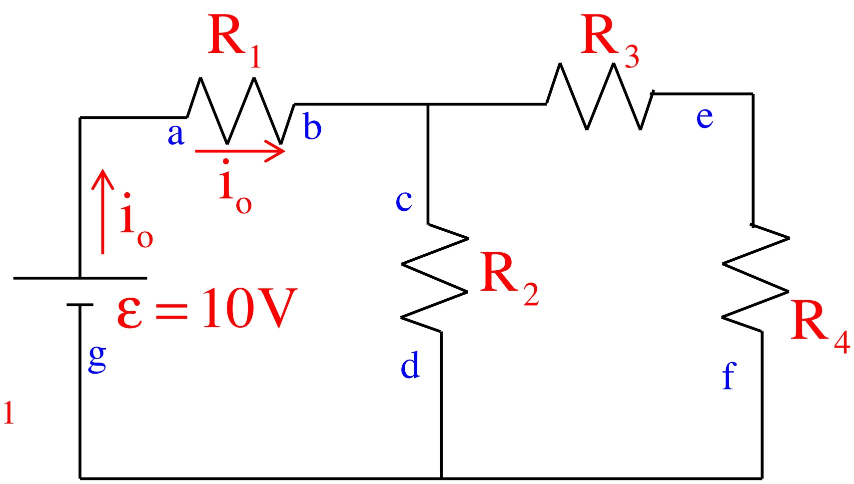


Using the R_{eq} circuit: $V_o = i_o R_{eq} \Rightarrow i_o = \frac{V_o}{R_{eq}}$

$$= \frac{(10\ V)}{(2.56\ \Omega)} = 3.9\ A$$

The ammeter will read 3.9 amps.

d.) *How much power* does R_2 dissipate?



We know the absolute electrical potential (the voltage) at *Point a* is 10 volts.

We know current goes from *high voltage* to *low voltage*, so the *voltage change* across R_1 must be a voltage DROP equal to:

$$\Delta V_1 = i_o R_1 = (3.9\ \text{A})(1\ \Omega) = 3.9\ \text{V}$$

Logic dictates that the absolute electrical potential at *Point b is*:

$$V_b = V_a - \Delta V_1 = (10\ \text{V}) - (3.9\ \text{V}) \Rightarrow V_b = 6.1\ \text{V} \quad (= V_c)$$

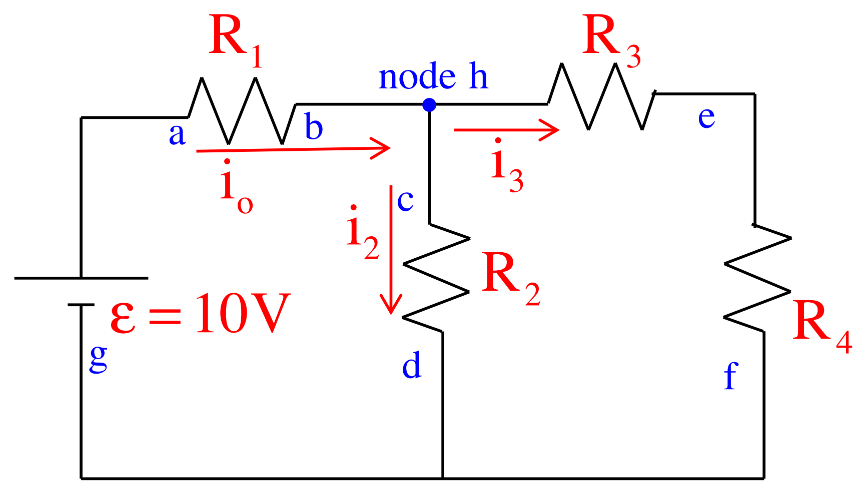
Because the absolute electrical potential at *Point d* is *zero*, the voltage across R_2 equals $V_c = 6.1$ V and:

$$V_2 = i_2 R_2 \Rightarrow (6.1\ \text{V}) = i_2 (2\ \Omega) \Rightarrow i_2 = 3.05\ \text{A}$$

The power dissipated by R_2 is, then:

$$\Rightarrow P_2 = (i_2)^2 R_2 = (3.05\ \text{A})^2 (2\ \Omega) = 18.6\ \text{W}$$

e.) What does the voltmeter read?



You should begin to see a pattern here. Every question, whether it be asking for an *ammeter reading* or *voltmeter reading* or *power calculation* or *current through an element* or *voltage cross an element*, they all require you to determine the CURRENT through the branch in which the element exists. That, in general, is what you will always be doing—trying to derive expressions for current values.

In this case, you could determine the current through the far-right branch (so you could use Ohm's Law on R_3 to get what the voltmeter would read) by using the same approach we used to get the current in the central branch in Part c (you'd just be using *Points e* and *f* instead of *Points c* and *d* in the process).

Or . . .

Look at node h

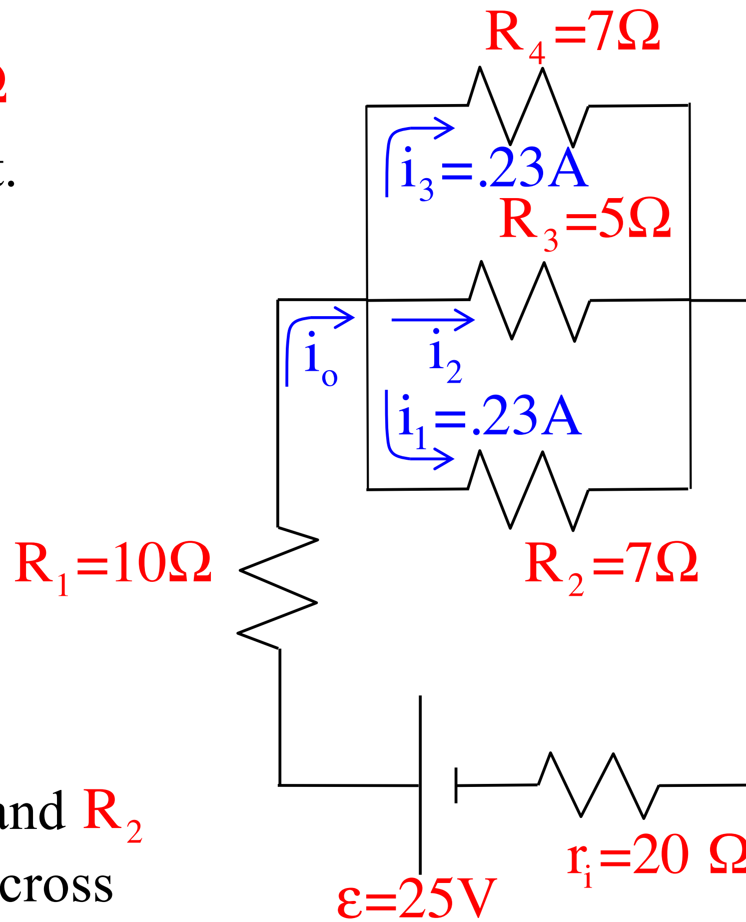
$$i_o = i_2 + i_3$$
$$\Rightarrow i_3 = i_o - i_2$$
$$= 3.9A - 3.05A$$
$$= .85A$$

So

$$V_{meter} = i_3 R_3$$
$$= (.85A)(3\ \Omega)$$
$$= 2.55\ V$$

Example 5: A power supply with 20 Ω

of internal resistance is used to power a circuit. If the current through R_4 is .23 amps, what is the current through R_1?



Start with what is obvious.

The current through R_1 in the bottom branch will equal all the currents in the parallel combination put together, or

$$i_1 = i_2 + i_3 + i_4$$

We know the current through R_4(given) and R_2 (same size resistance with same voltage across it), so all we need is the current through R_3.

The voltage across each of the parallel resistors is the same, and equal to:

$$V_4 = i_4 R_4 = (.23\text{A})(7\ \Omega) = 1.61\text{ V}$$

The the current through R_3 is:

$$V_3 = i_3 R_3$$
$$(1.61\text{ V}) = i_3(5\ \Omega)$$
$$\Rightarrow i_3 = .32\text{ A}$$

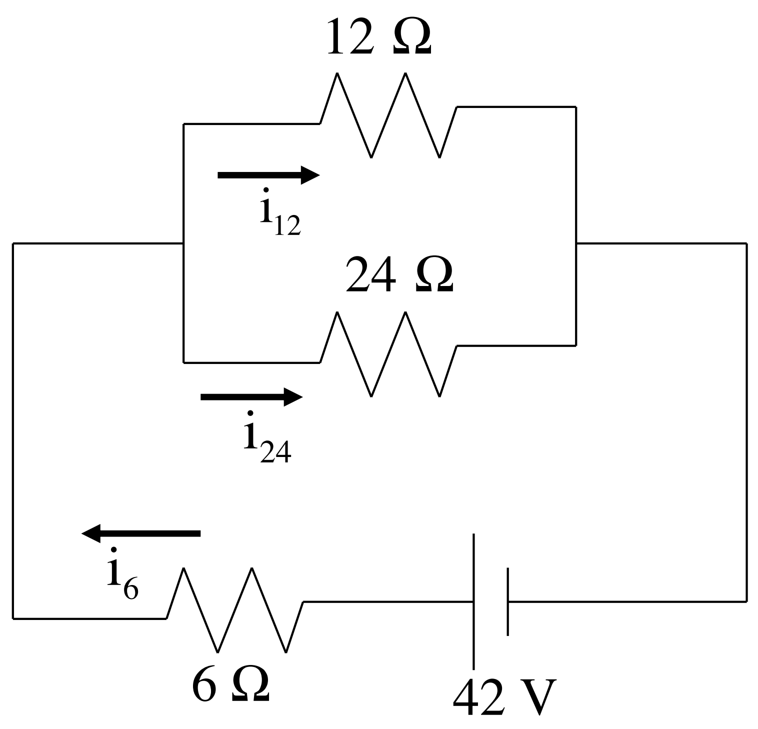
So: $i_1 = .23\text{A} + .32\text{A} + .23\text{A} = .78\text{A}$

Example 6: The current from the battery is 3 amps. How much current goes through the upper branch of the parallel combination?

This is another use-your-head question.

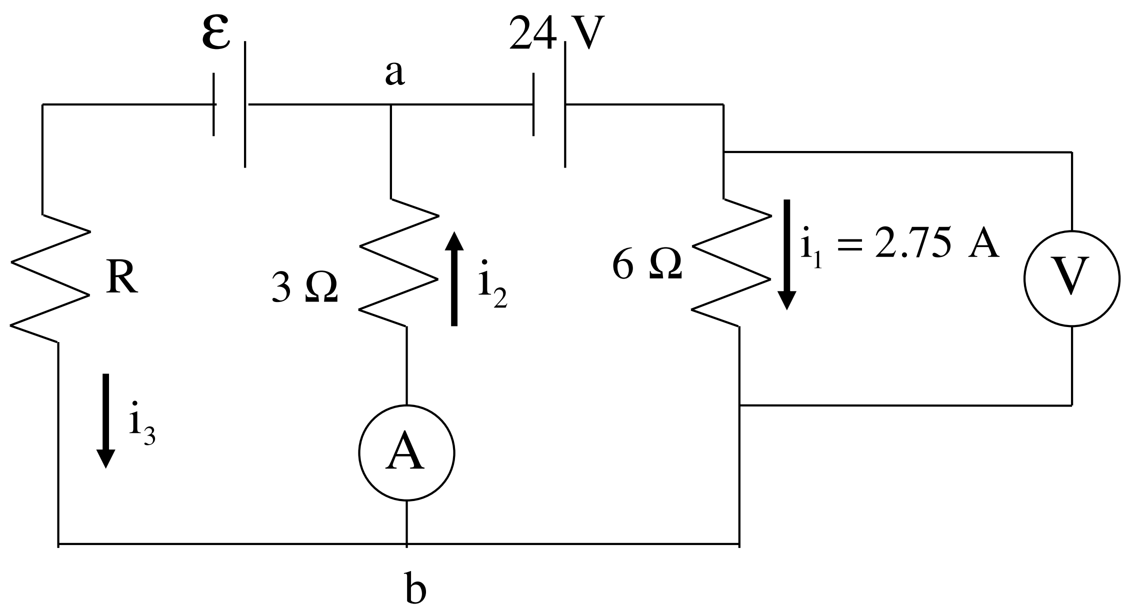
If the upper branch has half the resistance of the lower branch, it should draw twice the current.

With 3 amps coming in, that means 2 amps should pass through the upper branch.



Note: AP questions often have easy, non-mathematical, use-your-head solutions like this. That is why I'm showing you screwball problems like this. We will get into a more formal approach for analyzing circuit problems shortly.

Example 7: Consider:



a.) *What does* the voltmeter read?

$$V = i_1 R$$
$$= (2.75\,A)(6\ \Omega)$$
$$= 16.5\ V$$

b.) *What is* the voltage difference between *Points a* and *b*?

Assuming the voltage at *Points a* is zero (this is tricky as you don't know what is—you can define it anyway you want, though), the voltage changes will be due to the increase due to the battery in the right branch and the drop due to the 6 ohm resistor. That is:

$$V_{ab} = 24 - (2.75\,A)(6\ \Omega)$$
$$= 7.5\ V$$

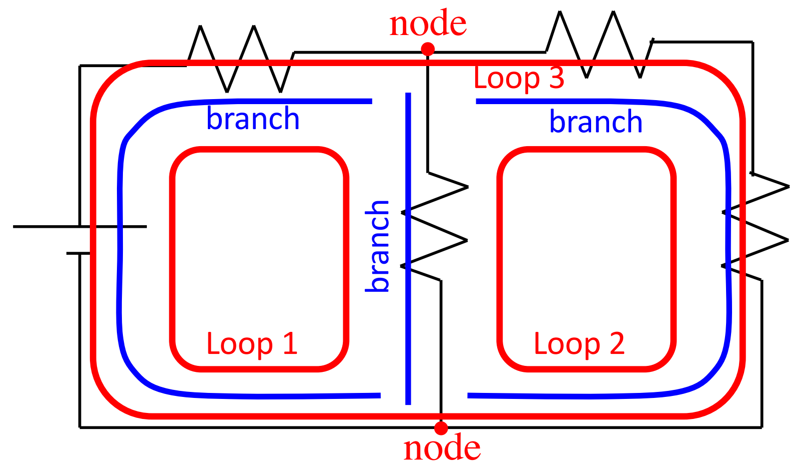
c.) *What does* the ammeter read?

This is just the *current* through the 3 ohm resistor, or:

$$V_{ab} = i_2 R_3$$
$$7.5V = i_2 (6\ \Omega)$$
$$\Rightarrow \quad i_2 = 1.25\ A$$

d.) *How is* i_1 generated? *Each battery produces* an E-fld, which permeates the entire circuit. The fields superimpose on one another, creating a net field. That net field is what motivates charge to move in each branch.

Some Definitions



A branch: A section of a circuit in which the current is the same everywhere.

--elements in series are a part of a single branch (look at sketch).

--in the circuit to the right, there are three branches.

A node: A *junction* where current can split up or be added to.

--elements in parallel have nodes internal to the combination.

--in the circuit above, there are two nodes.

A loop: Any closed path inside a circuit.

--in a circuit, loops can be traverse in a clockwise or counterclockwise direction.

--in the circuit above, there are three loops.

For Your Amusement

For the circuit to the right:

a.) How many branches are there?

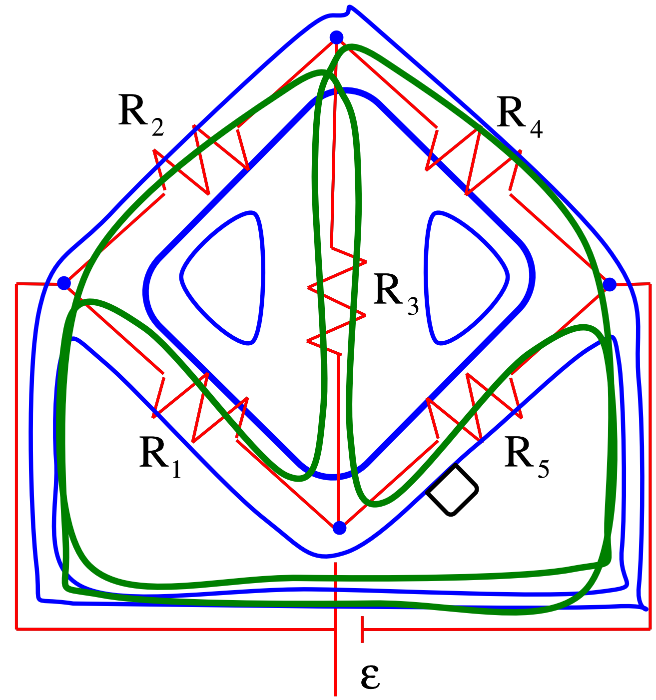
six

b.) How many nodes are there?

four

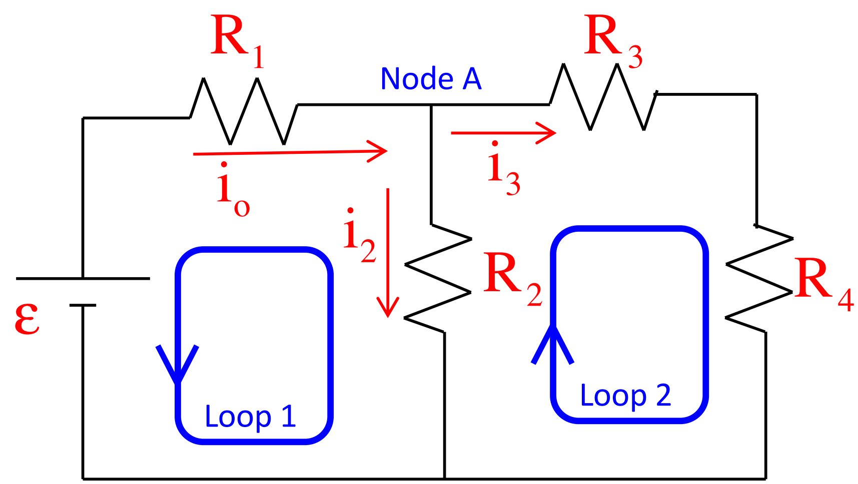
c.) How many loops are there?

seven



And that last little nubbin is supposed to be a tooth, cause this looks like a face to me!

Kirchoff's Laws—the Formal Approach



With the definitions under your belt, Kirchoff's Laws are simple (and you've been inadvertently using them in the seat-of-the-pants evaluations). They are:

Kirchoff's First Law: The sum of the currents into a node equals the sum of the currents out of a node. Mathematically, this is written as: $\sum i_{\text{into node}} = \sum i_{\text{out of node}}$

Example from the circuit's Node A: $i_o = i_2 + i_3$

Kirchoff's Second Law: The sum of the *voltage changes* around a closed path (a loop) equals ZERO. Mathematically, this is written as: $\sum \Delta V = 0$

Examples: starting at Node A:

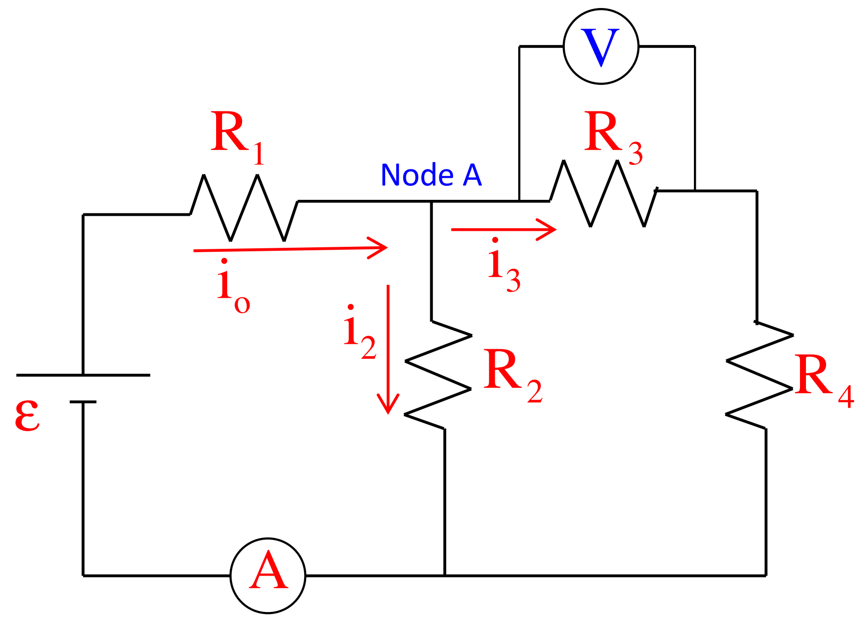
Loop 1 traversing counterclockwise: $R_1 i_o - \varepsilon + R_2 i_2 = 0$

Loop 2 traversing clockwise: $-R_3 i_3 - R_4 i_3 + R_2 i_2 = 0$

Note: Current moves from hi to lo voltage, so traversing against the current through a resistor produces a ΔV that is positive; traversing *with* current makes it negative.

Kirchoff's Laws—Using the Approach

Example 8: Determine the *meter reading* in the circuit to the right using Kirchoff's Laws. Assume the power supply is ideal with an EMF of 10 volts, and assume the resistor values are the same as their subscripts (this is essentially *Example 4*).



Step 0: Remove the meters.

Step 1: Define one *current* for each branch.

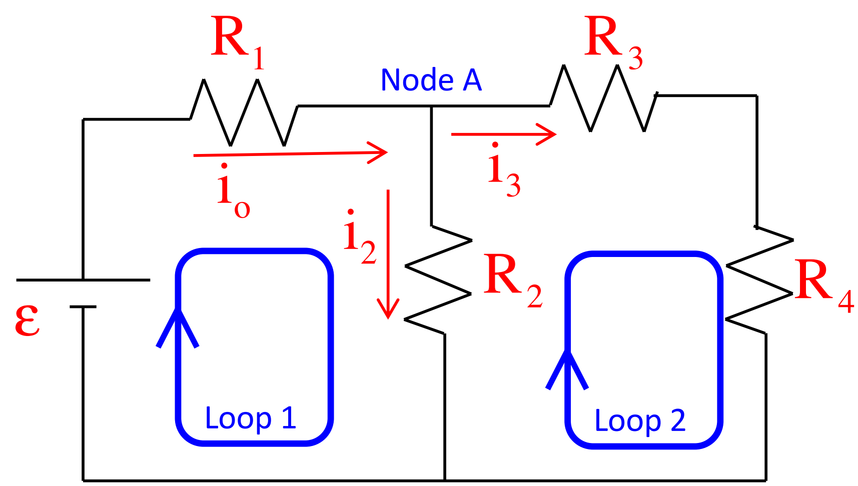
Step 2: Write out *node equations* for as many nodes as you can (see note below). Be sure to identify which node you are working with. For this problem:

Node A:

$$i_o = i_2 + i_3$$

Important note: If you had written out the node equation for the node at the bottom, you would have gotten $i_2 + i_3 = i_o$. This is the same equation as above. There will always be *fewer independent node equations* than actual nodes in a circuit. In this case, there were two nodes and only one independent node equation.

Additional note: You have three branches and three unknown currents, which means you will need *three equations to solve*. You have one *node equation,* which means you will need two more equations, presumably from your *loops*. Kindly note: there are three loops in this circuit, but you can only get TWO INDEPENDENT LOOP EQUATIONS from them. Any two of those equations will do, and any two will produce the third, which means that if you try to do this problem using nothing but loop equations, you'll end up with mush. (Try it if you don't believe me!)



Step 3. Identify and label the *loops* you will use. Use an arrow in each to show the direction you intend to traverse that loop.

Note 1: If there is a power supply in the loop, I prefer to start at the low voltage terminal and proceed through the supply. That way, the voltage change through the supply will be positive. With that in mind:

Loop 1:

$$\varepsilon - R_1 i_o - R_2 i_2 = 0$$

Loop 2:

$$R_2 i_2 - R_3 i_3 - R_4 i_3 = 0$$

Note 2: Put resistance terms first as they'll usually be assumed known whereas currents will not be.

Solving 3 Equations with 3 Unknowns

We have three equations and three unknowns. The ammeter is in the branch whose current is i_o. So how to solve for i_o? There are three approaches.

Our equations:

$$\varepsilon - R_1 i_o - R_2 i_2 = 0 \qquad \text{(equ. A)} \qquad\qquad R_2 i_2 - R_3 i_3 - R_4 i_3 = 0 \qquad \text{(equ. B)}$$

$$i_o = i_2 + i_3 \qquad \text{(equ. C)}$$

Putting in the numbers to make life easier:

$$10 - i_o - 2i_2 = 0 \qquad \text{(equ. A)} \qquad\qquad 2i_2 - 3i_3 - 4i_3 = 0 \qquad \text{(equ. B)}$$

$$\Rightarrow \quad 2i_2 - 7i_3 = 0$$

$$i_o = i_2 + i_3 \qquad \text{(equ. C)}$$

Approach 1—Brute force algebra:

I'll lay this out on the next page, just to convince you it's not the way to go.

Like I said, NASTY!

$$i_o = i_2 + i_3 \qquad \text{(equ. C)}$$

as $2i_2 - 7i_3 = 0 \qquad \text{(equ. B)}$

$$\Rightarrow \quad i_3 = \frac{2}{7} i_2$$

$$\Rightarrow \quad i_o = i_2 + i_3$$

$$= i_2 + \frac{2}{7} i_2 = \frac{9}{7} i_2$$

but $10 - i_o - 2i_2 = 0 \qquad \text{(equ. A)}$

$$\Rightarrow \quad i_2 = \frac{10 - i_o}{2} = \frac{10}{2} - \frac{1}{2} i_o$$

so $i_o = \frac{9}{7} i_2 = \frac{9}{7}\left(\frac{10}{2} - \frac{1}{2} i_0\right)$

$$\Rightarrow \quad i_o = \frac{90}{14} - \frac{9}{14} i_0$$

$$\Rightarrow \quad 14 i_o = 90 - 9 i_0$$

$$\Rightarrow \quad i_o = 90/23$$

$$\Rightarrow \quad i_o = 3.91\ \text{A}$$

Approaches 2 and 3: Matrices:

--Begin by rewriting each equation so their i_o term is in the first column, its i_2 term is in the second column, etc., and its voltage term (if there is one) is on the right side of the equal sign.

Our equations become:

$$\varepsilon - R_1 i_o - R_2 i_2 = 0 \quad \text{becomes} \quad R_1 i_o + R_2 i_2 + 0 i_3 = \varepsilon$$

$$R_2 i_2 - R_3 i_3 - R_4 i_3 = 0 \quad \text{becomes} \quad 0 i_o + R_2 i_2 - (R_3 + R_4) i_3 = 0$$

$$i_o = i_2 + i_3 \quad \text{becomes} \quad i_o - i_2 - i_3 = 0$$

--Put the information into a matrix:

i_o column, i_2 column, i_3 column, voltage column

$$\begin{vmatrix} R_1 & R_2 & 0 \\ 0 & R_2 & -(R_3 + R_4) \\ 1 & -1 & -1 \end{vmatrix} \begin{vmatrix} i_o \\ i_2 \\ i_3 \end{vmatrix} = \begin{vmatrix} \varepsilon \\ 0 \\ 0 \end{vmatrix}$$

--Using numbers:

$$\begin{vmatrix} R_1 & R_2 & 0 \\ 0 & R_2 & -(R_3+R_4) \\ 1 & -1 & -1 \end{vmatrix} \begin{vmatrix} i_o \\ i_2 \\ i_3 \end{vmatrix} = \begin{vmatrix} \varepsilon \\ 0 \\ 0 \end{vmatrix} \quad \text{becomes} \quad \begin{vmatrix} 1 & 2 & 0 \\ 0 & 2 & -7 \\ 1 & -1 & -1 \end{vmatrix} \begin{vmatrix} i_o \\ i_2 \\ i_3 \end{vmatrix} = \begin{vmatrix} 10 \\ 0 \\ 0 \end{vmatrix}$$

--You have two options at this point, depending upon your abilities with a calculator and whether there are any variables in your relationship. The first approach is a manual evaluation of the matrices and will always work.

Noting that the left-hand 3x3 matrix is called *the determinate,* solving for, say, i_o, requires the evaluation of two matrices, one divided into the other. Specifically, the *determinate* divided into the determinate with the i_o column replaced by the voltage column (the far column to the right). That is:

$$i_o = \frac{\begin{vmatrix} 10 & 2 & 0 \\ 0 & 2 & -7 \\ 0 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 0 \\ 0 & 2 & -7 \\ 1 & -1 & -1 \end{vmatrix}}$$

--How to evaluate a matrix? Start by reproducing the first two columns at the end of the matrix.

$$i_o = \frac{\begin{vmatrix} \varepsilon & R_2 & 0 \\ 0 & R_2 & -(R_3+R_4) \\ 0 & -1 & -1 \end{vmatrix} \begin{matrix} \varepsilon & R_2 \\ 0 & R_2 \\ 0 & -1 \end{matrix}}{\begin{vmatrix} R_1 & R_2 & 0 \\ 0 & R_2 & -(R_3+R_4) \\ 1 & -1 & -1 \end{vmatrix} \begin{matrix} R_1 & R_2 \\ 0 & R_2 \\ 1 & -1 \end{matrix}}$$

--With numbers:

$$i_o = \frac{\left|\begin{matrix} 10 & 2 & 0 \\ 0 & 2 & -7 \\ 0 & -1 & -1 \end{matrix}\right|\begin{matrix} 10 & 2 \\ 0 & 2 \\ 0 & -1 \end{matrix}}{\left|\begin{matrix} 1 & 2 & 0 \\ 0 & 2 & -7 \\ 1 & -1 & -1 \end{matrix}\right|\begin{matrix} 1 & 2 \\ 0 & 2 \\ 1 & -1 \end{matrix}}$$

--The first part of the execution is shown below:

$$i_o = \frac{\left|\begin{array}{ccc|cc} 10 & 2 & 0 & 10 & 2 \\ 0 & 2 & -7 & 0 & 2 \\ 0 & -1 & -1 & 0 & -1 \end{array}\right|}{\left|\begin{array}{ccc|cc} 1 & 2 & 0 & 1 & 2 \\ 0 & 2 & -7 & 0 & 2 \\ 1 & -1 & -1 & 1 & -1 \end{array}\right|}$$

$$i_o = \frac{\left|\begin{array}{ccc|cc} 10 & 2 & 0 & 10 & 2 \\ 0 & 2 & -7 & 0 & 2 \\ 0 & -1 & -1 & 0 & -1 \end{array}\right|}{\left|\begin{array}{ccc|cc} 1 & 2 & 0 & 1 & 2 \\ 0 & 2 & -7 & 0 & 2 \\ 1 & -1 & -1 & 1 & -1 \end{array}\right|} = \frac{(10)\left[(2)(-1)-(-7)(-1)\right]+\ldots}{\text{etc.}}$$

--The second part:

$$i_o = \frac{\left|\begin{array}{ccc|cc} 10 & 2 & 0 & 10 & 2 \\ 0 & 2 & -7 & 0 & 2 \\ 0 & -1 & -1 & 0 & -1 \end{array}\right|}{\left|\begin{array}{ccc|cc} 1 & 2 & 0 & 1 & 2 \\ 0 & 2 & =7 & 0 & 2 \\ 1 & -1 & -1 & 1 & -1 \end{array}\right|} = \frac{(10)\left[(2)(-1)-(-7)(-1)\right]+(2)\left[(-7)(0)-(0)(-1)\right]+\ldots}{\text{etc.}}$$

--Once you get the hang of the pattern, you can do these in your head without writing much of anything down:

$$i_o = \frac{\left|\begin{array}{ccc|cc} 10 & 2 & 0 & 10 & 2 \\ 0 & 2 & -7 & 0 & 2 \\ 0 & -1 & -1 & 0 & -1 \end{array}\right|}{\left|\begin{array}{ccc|cc} 1 & 2 & 0 & 1 & 2 \\ 0 & 2 & -7 & 0 & 2 \\ 1 & -1 & -1 & 1 & -1 \end{array}\right|} = \frac{(10)\big[(-2)-(7)\big]+0+0}{1\big[-2-(7)\big]+2\big[(-7)-0\big]+0} = \frac{-90}{-23} = 3.91\text{ A}$$

--The other alternative has to do with matrix manipulation on a calculator. Specifically, if you multiply everything by the *inverse determinate*, you end up with a 1x3 matrix whose elements are the solution for the three unknowns.

$$\underbrace{\left|\begin{array}{ccc} D & E & T \end{array}\right|^{-1}\left|\begin{array}{ccc} D & E & T \end{array}\right|}_{=1}\left|\begin{array}{c} i_o \\ i_2 \\ i_3 \end{array}\right| = \left|\begin{array}{c} V_o \\ V_2 \\ V_3 \end{array}\right|\left|\begin{array}{ccc} D & E & T \end{array}\right|^{-1}$$

$$\Rightarrow \left|\begin{array}{c} i_o \\ i_2 \\ i_3 \end{array}\right| = \left|\begin{array}{c} V_o \\ V_2 \\ V_3 \end{array}\right|\left|\begin{array}{ccc} D & E & T \end{array}\right|^{-1}$$

--the alternate alternate is to have your calculator execute an rref (reduce row echalon format) operation. The following is courtesy of Mr. White.

$$\begin{aligned} i_o + 2i_2 + 0i_3 &= 10 \\ 0i_o + 2i_2 - 7i_3 &= 0 \\ i_o - i_2 - i_3 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{matrix} 1 & 2 & 0 & 10 \\ 0 & 2 & -7 & 0 \\ 1 & -1 & -1 & 0 \end{matrix}$$

Using your calculator:

a. Math -> Matrix -> Edit -> A (for name of matrix) . . . note that some calculators just have a "matrix" key you can use (versus starting with "math")

b. 3 [Enter] 4 [Enter] (this gives you a 3x4 matrix)

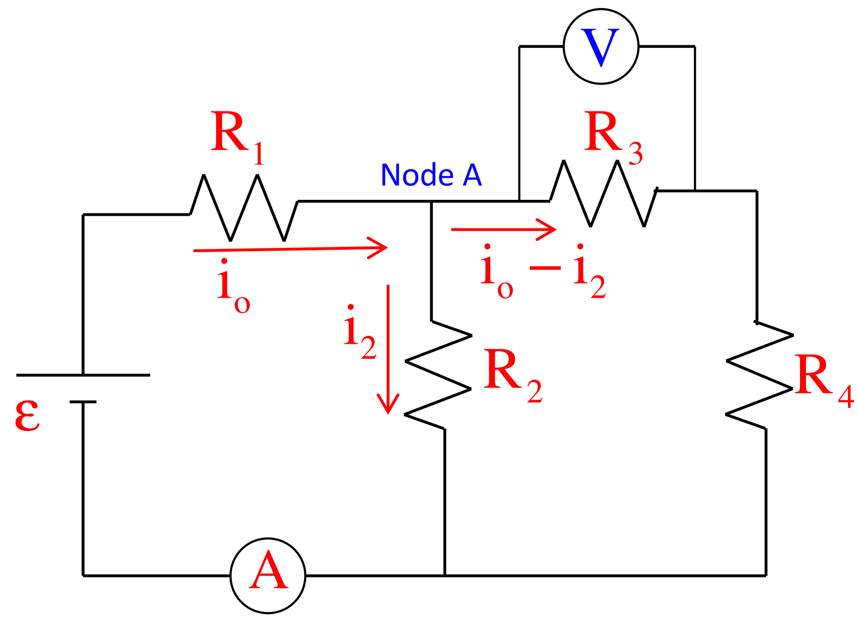
c. Enter coefficients and values into Matrix; exit, then go back to "matrix" and:

d. In "math," use "rref" A (reduced row echelon form)

e. You'll end up with 1's and the last row will give you the current values.

$$\begin{bmatrix} 1 & 2 & 0 & 10 \\ 0 & 2 & -7 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 3.91 \\ 0 & 1 & 0 & 3.04 \\ 0 & 0 & 1 & .87 \end{bmatrix} \begin{matrix} \Rightarrow i_o = 3.91 \text{ A} \\ \Rightarrow i_2 = 3.04 \text{ A} \\ \Rightarrow i_3 = .87 \text{ A} \end{matrix}$$

Example 9: Example 8 using a clever shortcut. Again the power supply is ideal with an EMF of 10 volts, and assume the resistor values are the same as their subscripts.



Step 0: Remove the meters.

Step 1: Define one *current* for each branch.

And here is the clever move.

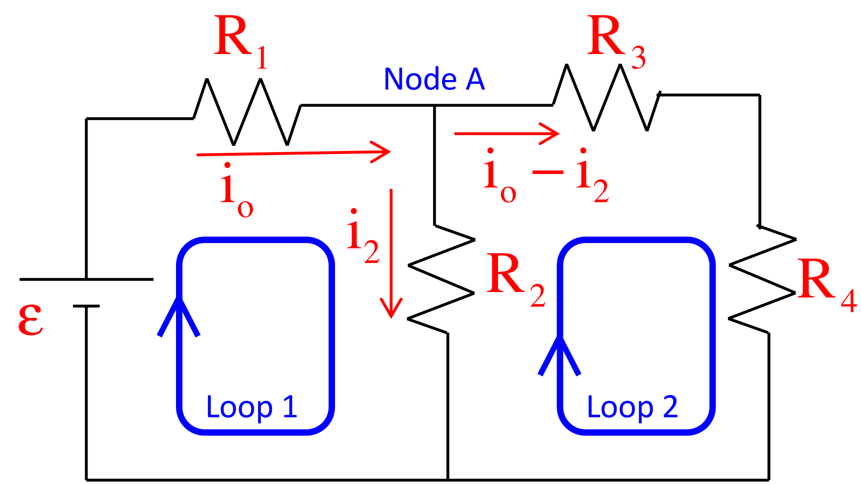
Think about it. If current i_o comes into node A, and current i_2 goes out of node A and through R_2, how much current must go through R_3? Must be $i_o - i_2$. So why not just call it that (instead of i_3)? Doing so *eliminates one unknown*, which makes the solving a lot easier.

Consequence: You only need to write two loop equations (you've already used the node information in defining the currents).

Loop 1:

$$\varepsilon - R_1 i_o - R_2 i_2 = 0$$

$$\Rightarrow \quad i_o + 2i_2 = 10$$



Loop 2:

$$R_2 i_2 - R_3(i_o - i_2) - R_4(i_o - i_2) = 0$$

$$\Rightarrow \quad 2i_2 - 3(i_o - i_2) - 4(i_o - i_2) = 0$$

$$\Rightarrow \quad -7i_o + 9i_2 = 0$$

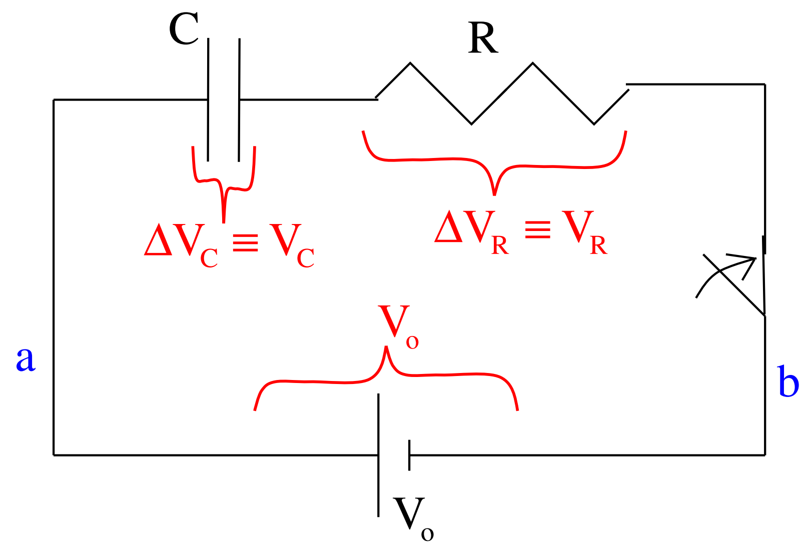
Solving:

$$\Rightarrow \quad i_o = \frac{\begin{vmatrix} 10 & 2 \\ 0 & 9 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ -7 & 9 \end{vmatrix}} = \frac{90 - 0}{9 - (-14)}$$

$$= 3.91\text{A}$$

Capacitors—Charging Characteristics

Example 10: Consider a resistor, an uncharged capacitor, a switch and a power supply all hooked in series. Note also that when the switch is thrown, the voltage across "a" and "b" is equal to both the *battery voltage* and the *sum of voltages across the resistor and capacitor*. That is:

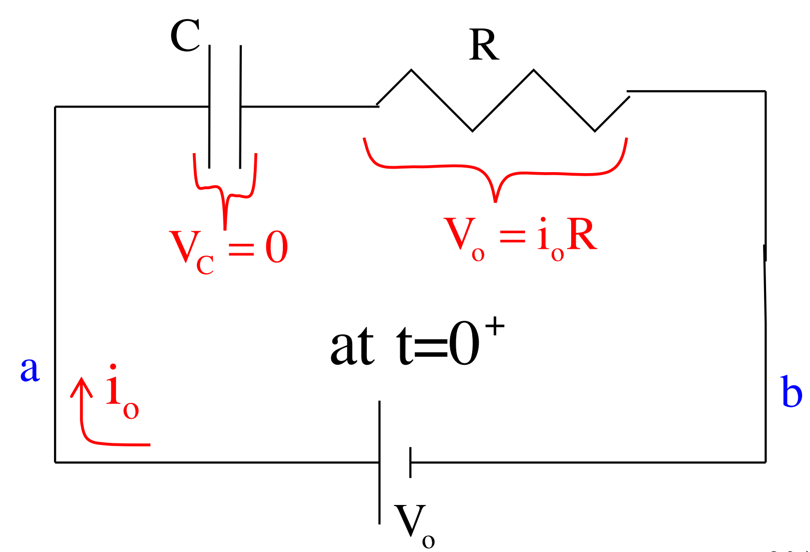


$$V_o = V_C + V_R$$

a.) At t = 0, the switch is closed. What initially happens in the circuit?

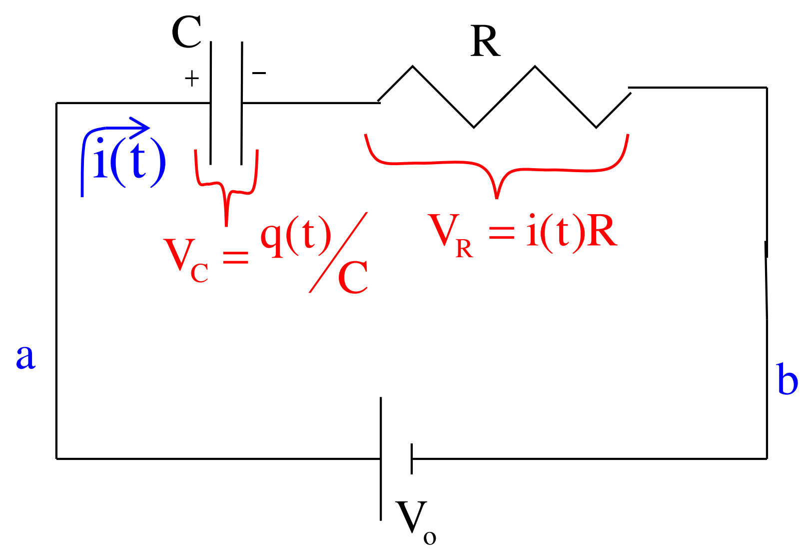
As the cap initially has no charge on its plates, it will provide no resistance to charge flow. That means *no voltage drop across the capacitor* with all the voltage drop happen across the resistor . . . which means:

$$V_o = \cancel{V_C}^{\,0} + V_R$$
$$= i_o R$$
$$\Rightarrow \quad i_o = \frac{V_o}{R}$$



b.) What happens as time proceeds?

As the cap begins to charge, some of the voltage drop happens across the resistor and some across the capacitor leaving us with a Kirchoff expression of:



$$V_o - \frac{q_{plates}}{C} - iR = 0$$

$$\Rightarrow \quad \frac{dq}{dt} + \frac{q_{plates}}{RC} = \frac{V_o}{R}$$

The problem? There are two different types of *q* in this expression. One refers to the amount of charge on one capacitor plate. The other refers to charge flowing through the circuit (current is defined as the *time rate of charge flow*).

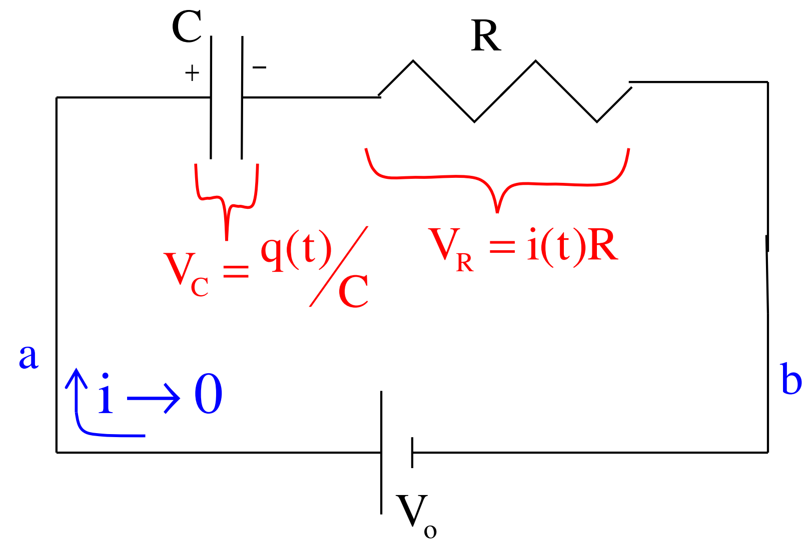
Although this won't always be the case, in this instance the rate at which charge accumulates on the cap plates will equal the rate at which charge passes by per unit time, and we can write:

$$i = dq/dt = dq_{plate}/dt$$

This means Kirchoff's Law can be written as:

$$\frac{dq}{dt} + \frac{q_{plates}}{RC} = \frac{V_o}{R}$$

$$\Rightarrow \quad \frac{dq_{plate}}{dt} + \frac{q_{plates}}{RC} = \frac{V_o}{R}$$



Note that as time proceeds toward infinity, the charge on the capacitor plates reaches maximum, all the voltage drop happens across the capacitor, current in the circuit drops to zero and there is no voltage drop across the resistor. In that case:

$$V_o = V_C + V_R \; (V_R \rightarrow 0)$$

$$= \frac{Q_{max}}{C}$$

$$\Rightarrow \quad Q_{max} = V_o C$$

Solving:

$$\frac{dq}{dt}+\left(\frac{1}{RC}\right)q=\frac{V_o}{R}$$

$$\Rightarrow \frac{dq}{dt}=\left(\frac{1}{RC}\right)(V_oC-q)=\left(\frac{1}{RC}\right)(Q_{max}-q)$$

$$\Rightarrow \frac{dq}{(q-Q_{max})}=-\frac{dt}{RC}$$

$$\Rightarrow \int_0^{q(t)}\frac{dq}{(q-Q_{max})}=-\int_{t=0}^{t}\frac{dt}{RC} \quad \Rightarrow \quad \ln\left|q-Q_{max}\right|\Big|_{q=0}^{q(t)}=-\frac{t}{RC}$$

$$\Rightarrow \ln\left|q(t)-Q_{max}\right|-\ln\left|-Q_{max}\right|=-\frac{t}{RC} \quad \Rightarrow \quad \ln(Q_{max}-q(t))-\ln(Q_{max})=-\frac{t}{RC}$$

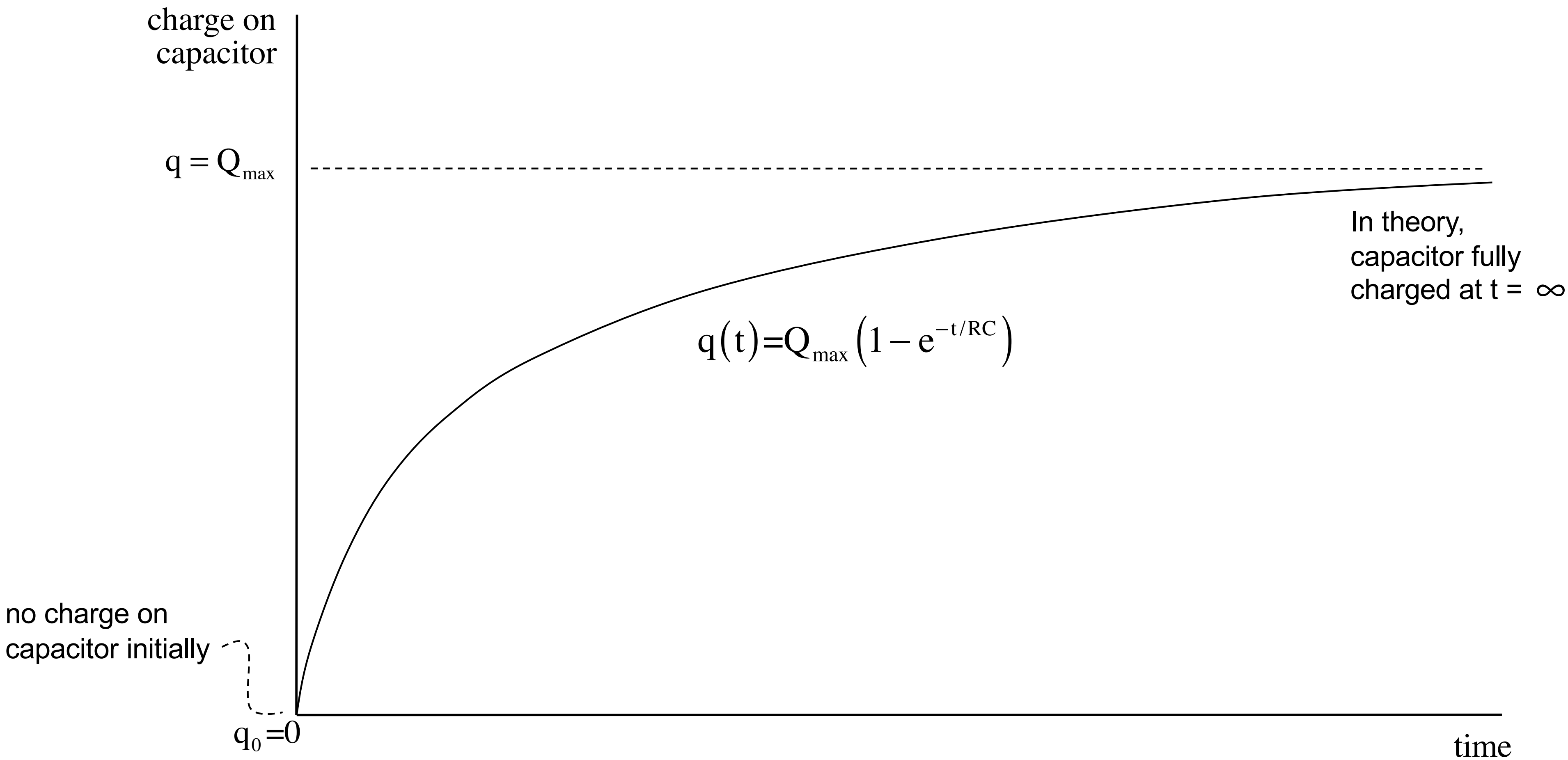
because $|a-b|=(b-a)$ if $b>a$.

$$\Rightarrow \ln\left[\frac{(Q_{max}-q(t))}{(Q_{max})}\right]=-\frac{t}{RC} \quad \Rightarrow \quad e^{\ln\frac{(Q_{max}-q(t))}{(Q_{max})}}=e^{-\frac{t}{RC}}$$

$$\Rightarrow \frac{(Q_{max}-q(t))}{(Q_{max})}=e^{-\frac{t}{RC}} \quad \Rightarrow \quad Q_{max}-q(t)=Q_{max}e^{-\frac{t}{RC}} \quad \Rightarrow \quad q(t)=Q_{max}\left(1-e^{-\frac{t}{RC}}\right)$$

Time Constant for a Capacitor

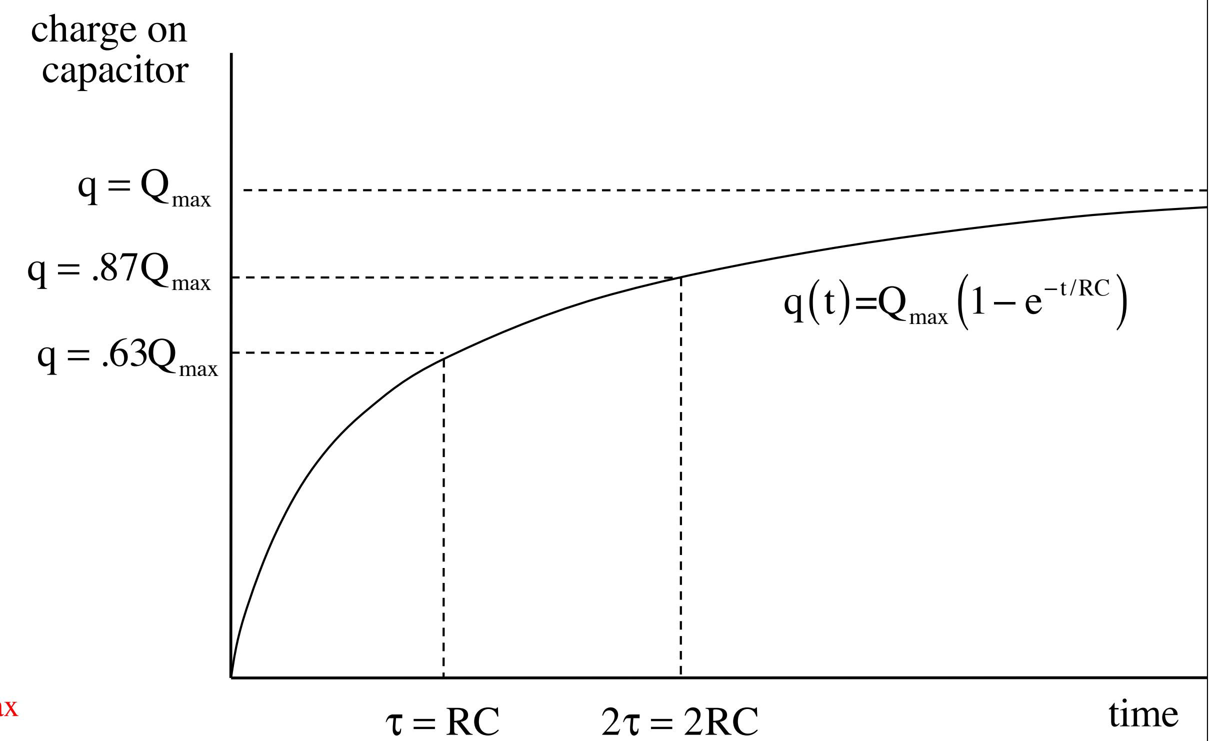
A graph of the *charging* characteristic of a charging capacitor is shown below.



It would be nice to get a feel for how fast a capacitor/resistor combination will charge or discharge.

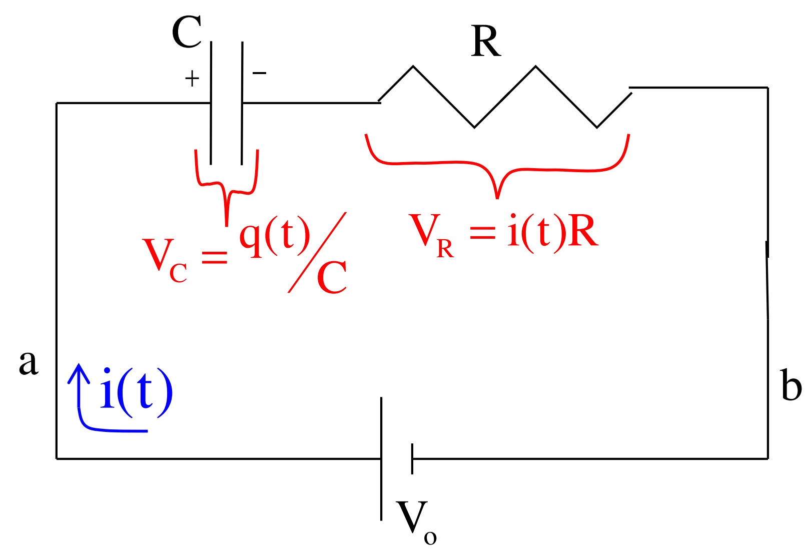
To that end, how much charge would the cap have accumulated after a time equal to *RC*?

$$q(t=RC) = Q_{max}\left(1 - e^{-\frac{RC}{RC}}\right)$$
$$= Q_{max}\left(1 - e^{-1}\right)$$
$$= Q_{max}\left(1 - \frac{1}{e}\right)$$
$$= Q_{max}(1 - .37) = .63Q_{max}$$



This time is defined as *one time constant* τ. It is the amount of time it takes the capacitor to charge to 63% of its maximum. *Two time constants* will charge it to 87% of its maximum (try the calculation if you don't believe me).

c.) *What is* the *current* as a function of time?

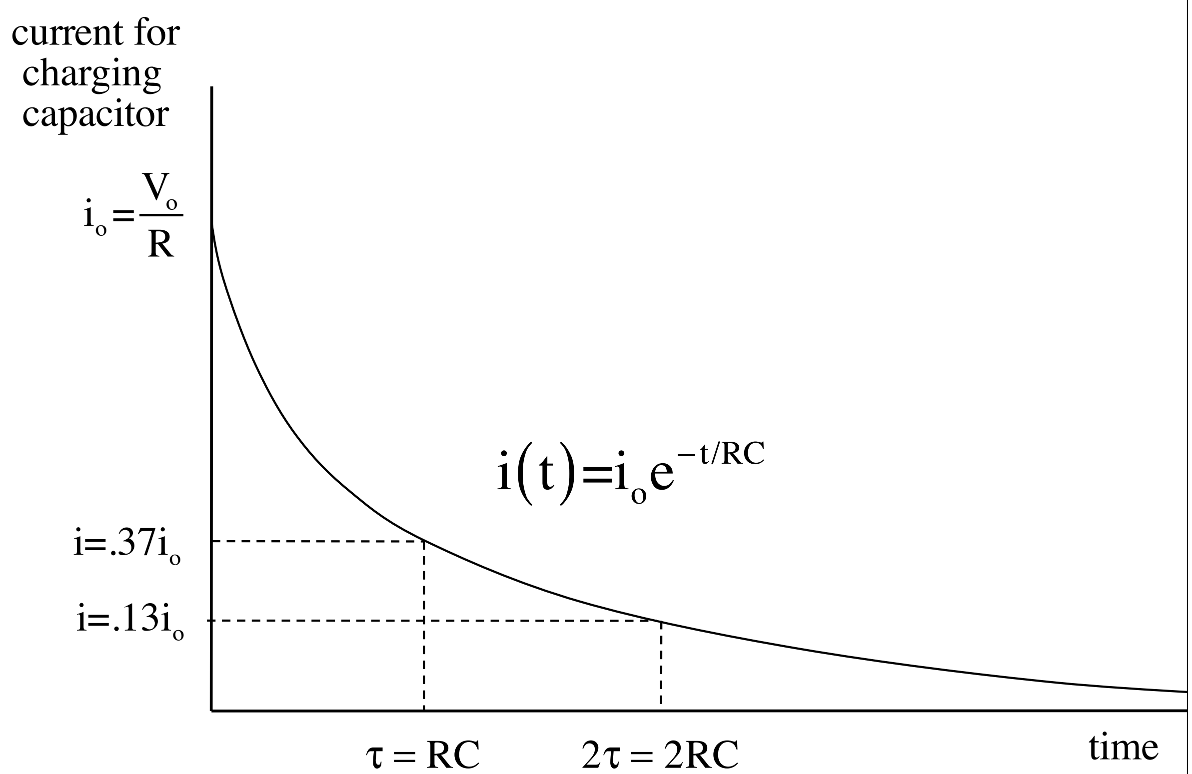


$$
\begin{aligned}
i(t) &= \frac{dq_{plate}}{dt} \\
&= \frac{d\left(Q_{max} - Q_{max}e^{-t/RC}\right)}{dt} \\
&= -Q_{max}\left(-\frac{1}{RC}\right)e^{-t/RC} \\
&= \left(\frac{1}{R}\right)\left(\frac{Q_{max}}{C}\right)e^{-t/RC} \\
&= \left(\frac{V_o}{R}\right)e^{-t/RC} \\
&= i_o e^{-t/RC}
\end{aligned}
$$

A graph of the *current* characteristics for a charging capacitor/resistor combination:

Note that after *one time constant*, the current is:

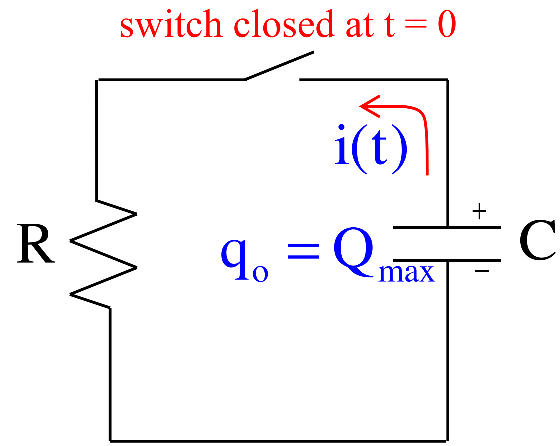
$$i(t{=}RC) = i_o e^{-\frac{RC}{RC}} = \frac{i_o}{e} = .37 i_o$$



After one time constant, the capacitor’s current will have dropped 63% and will be at 37% of its maximum. After two time constants, it will be at 13% of its maximum.

Capacitors—Discharging Characteristics

Example 11: At $t = 0$, the switch is thrown and a charged capacitor begins to discharge.



a.) How are current through the circuit and charge on the capacitor plates related?

When a capacitor is discharging, the *rate of change of charge on the plate* is negative (charge is leaving) and:

$$i = dq/dt = -\left(dq_{plate}/dt\right)$$

Using this with Kirchoff's Law (tracking along the direction of current flow) yields:

$$-iR + \frac{q_{plates}}{C} = 0$$

$$\Rightarrow \quad - \quad \frac{dq}{dt} \quad + \frac{1}{RC} q_{plates} = 0$$

$$\Rightarrow \quad -\left(-\frac{dq_{plates}}{dt}\right) + \frac{1}{RC} q_{plates} = 0$$

Solving:

$$-\left(-\frac{dq_{plate}}{dt}\right)+\left(\frac{1}{RC}\right)q_{plates}=0$$

$$\Rightarrow \quad \frac{dq_{plate}}{dt}=-\left(\frac{1}{RC}\right)q_{plates}$$

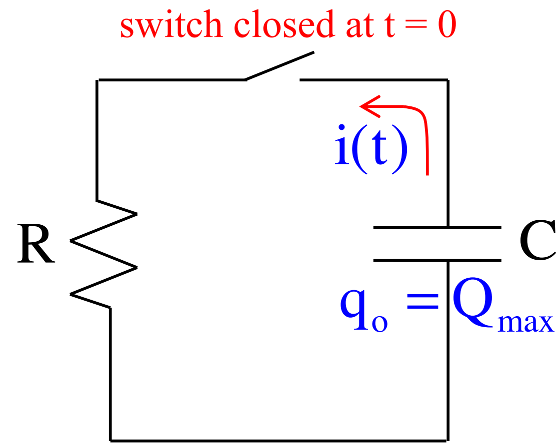
$$\Rightarrow \quad \frac{dq_{plate}}{q_{plate}}=-\left(\frac{1}{RC}\right)dt$$

$$\Rightarrow \quad \int_{Q_{max}}^{q(t)}\left(\frac{1}{q_{plate}}\right)dq_{plate}=-\left(\frac{1}{RC}\right)\int_{t=0}^{t}dt$$

$$\ln(q)\Big|_{Q_{max}}^{q(t)}=\ln[q(t))-\ln(Q_{max})]=-\frac{t}{RC}$$

$$\Rightarrow \quad \ln\left(\frac{q(t)}{Q_{max}}\right)=-\frac{t}{RC} \quad \Rightarrow \quad e^{\ln\left(\frac{q(t)}{Q_{max}}\right)}=e^{-\frac{t}{RC}}$$

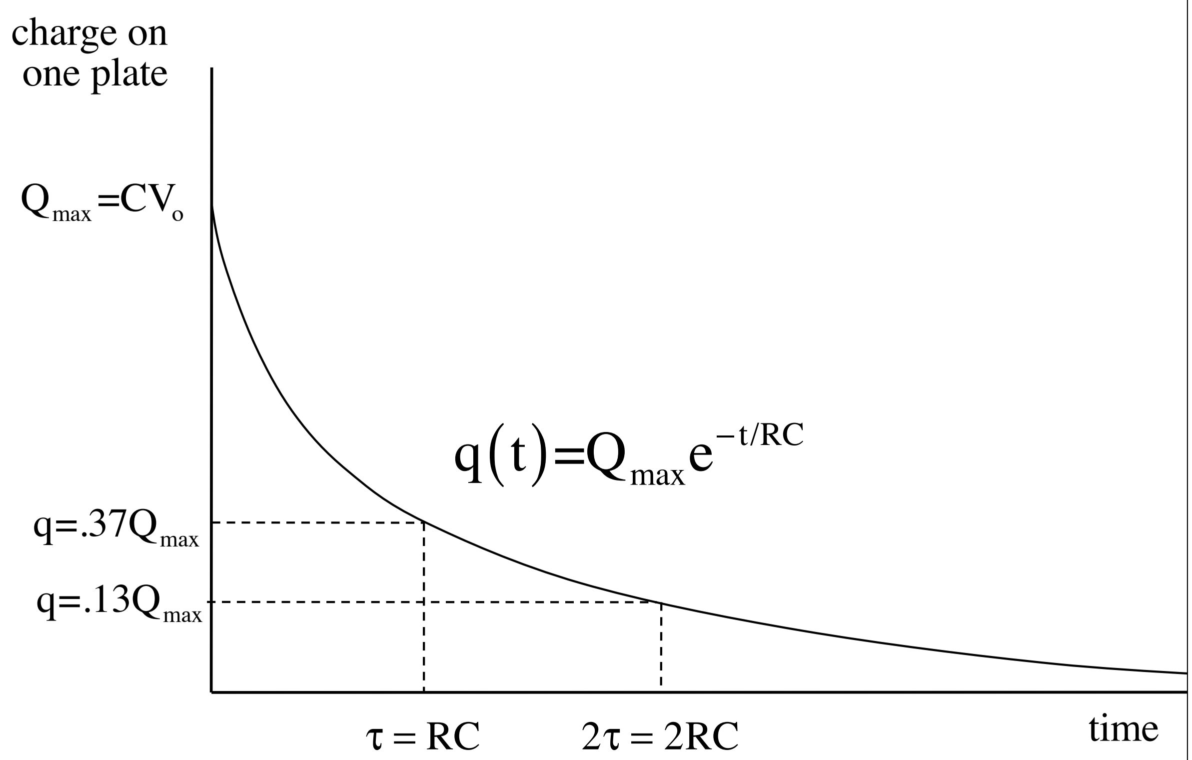
$$\Rightarrow \quad \frac{q(t)}{Q_{max}}=e^{-\frac{t}{RC}} \quad \Rightarrow \quad q(t)=Q_{max}e^{-\frac{t}{RC}}$$



A graph of the *charge on plate* characteristics for a discharging capacitor/resistor combination:

Note that after *one time constant*, the charge is:

$$q(t=RC) = Q_{max} e^{-\frac{RC}{RC}}$$
$$= \frac{Q_{max}}{e}$$
$$= .37 Q_{max}$$



After one time constant, the capacitor's charge will have dropped 67% and will be at 37% of its maximum. After two time constants, it will be at 13% of its maximum.

c.) *What is* the *current* as a function of time?

$$i(t) = -\frac{dq(t)_{plate}}{dt}$$

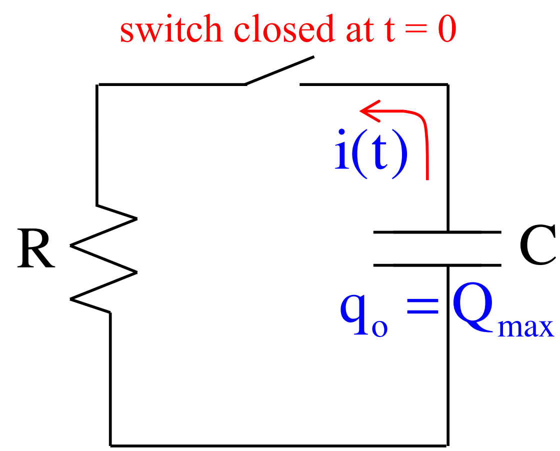
$$= -\frac{d\left(Q_{max}e^{-\frac{t}{RC}}\right)}{dt}$$

$$= -\left(-\frac{1}{RC}\right)Q_{max}e^{-\frac{t}{RC}}$$

$$= \frac{1}{R}\left(\frac{Q_{max}}{C}\right)e^{-\frac{t}{RC}}$$

$$= \frac{1}{R}(V_o)e^{-\frac{t}{RC}}$$

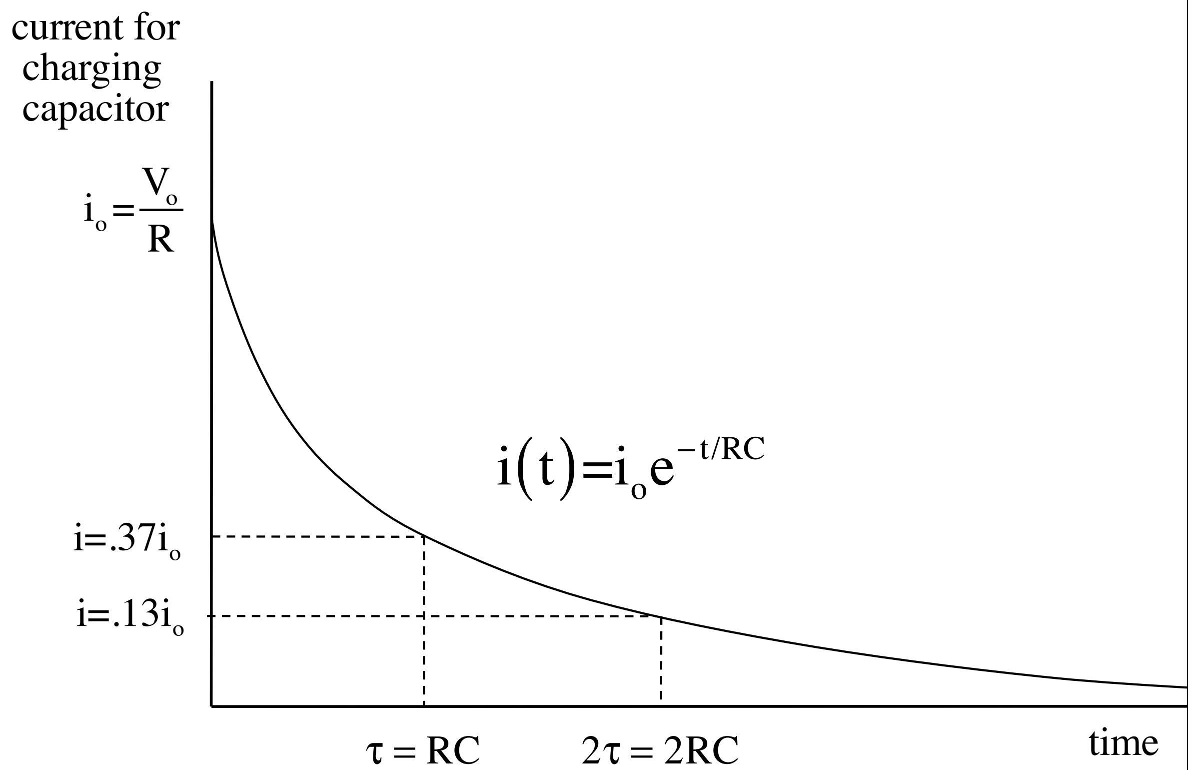
$$= i_o e^{-\frac{t}{RC}}$$



A graph of the *current* characteristics for a discharging capacitor/resistor combination:

Note that after *one time constant*, the current is:

$$i(t{=}RC) = i_o e^{-\frac{RC}{RC}}$$
$$= \frac{i_o}{e}$$
$$= .37 i_o$$

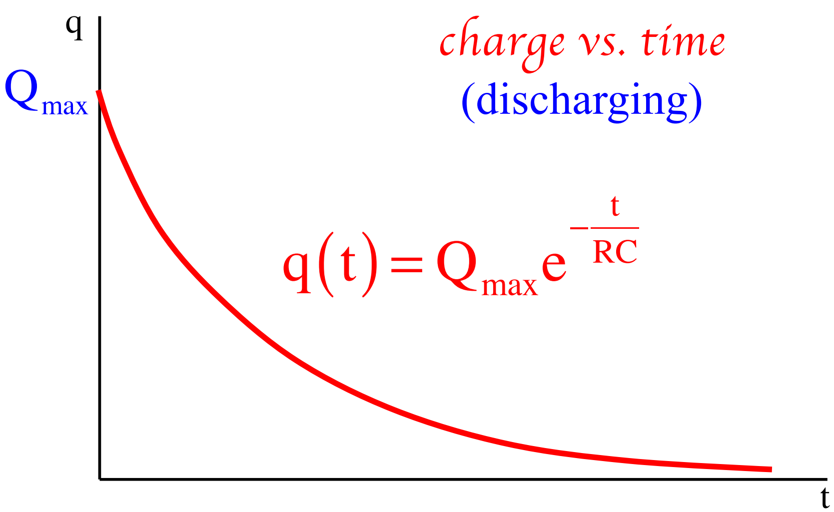


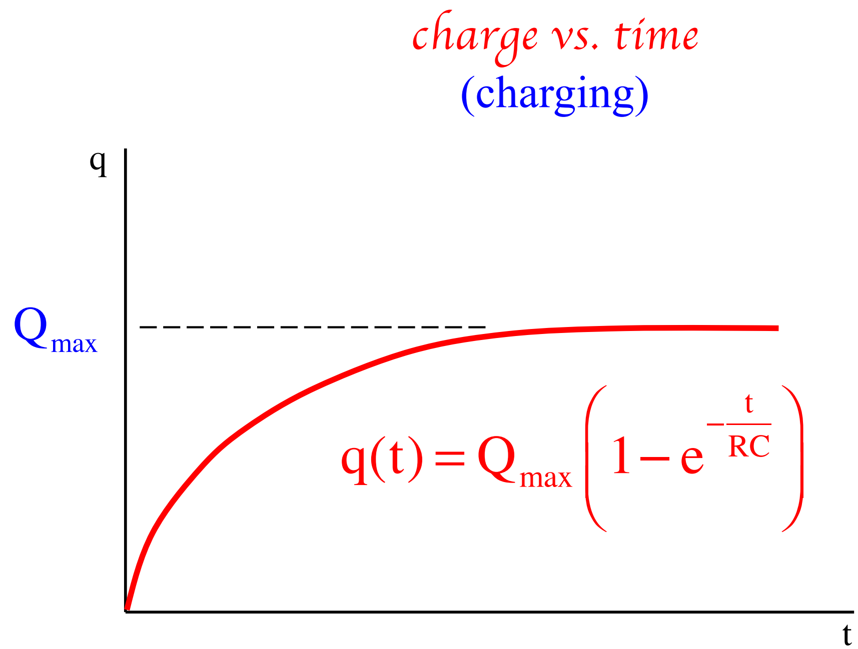
After one time constant, the capacitor's current will have dropped 67% and will be at 37% of its maximum. After two time constants, it will be at 13% of its maximum.

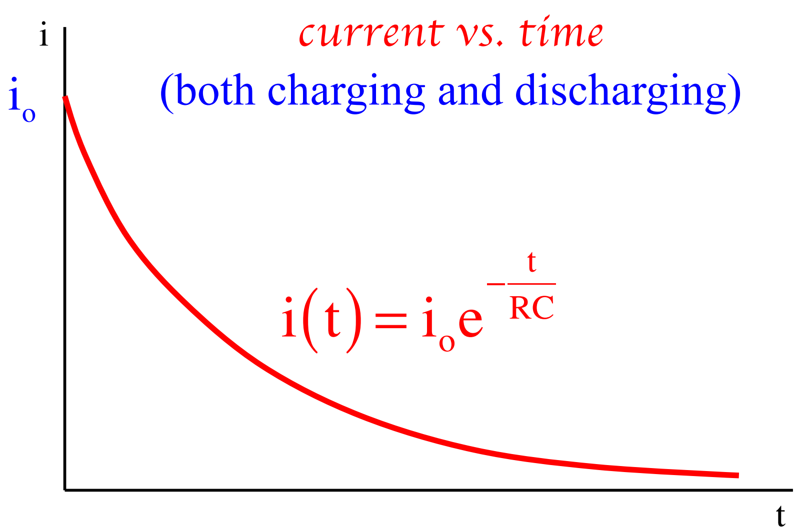
And yes, it's the same as for a charging circuit.

Summary of Graphs

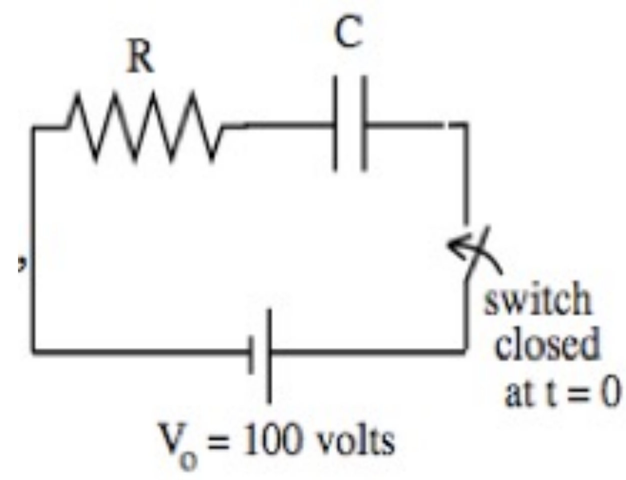
Graphs of capacitor charging and discharging characteristics.







Example 11: A one microfarad cap is in series with a 10 k-ohm resistor, a battery whose voltage is 100 volts and a switch.



a.) The capacitance value tells you something that is always true no matter what the voltage across the cap happens to be. What does it tell you?

The capacitance tells you the amount of charge one plate can hold per volt across the plates.

b.) What is the initial current in the circuit?

With no charge initial on the cap (and, hence, no voltage across the cap), we can write:

$$V_o = V_{cap} + V_{res}$$
$$(100\text{ V}) = 0 + i_o(10^4\ \Omega)$$
$$\Rightarrow\ i_o = 10^{-2}\text{ A}$$

c.) What is the circuit's current after a long time? *It will go* to zero.

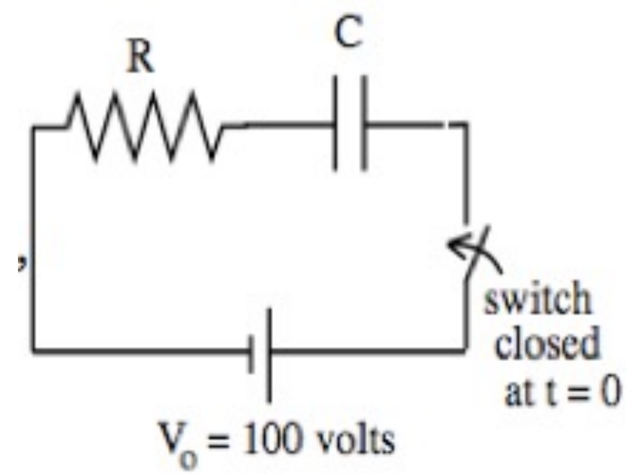
d.) How much charge will the cap hold when fully charged?

When fully charged, all the voltage drop will be across the cap (no current in the circuit, so no drop across the resistor), and

$$V_o = V_{cap} + V_{res}$$
$$(100\text{ V}) = \frac{Q_{max}}{C} + 0$$
$$(100\text{ V}) = \frac{Q_{max}}{(10^{-6}\text{ f})} + 0$$
$$\Rightarrow\ Q_{max} = 10^{-4}\text{ C}$$

e.) *How much energy* does the cap hold when fully charged?

$$U = \frac{1}{2}CV^2$$
$$= \frac{1}{2}(10^{-6}\ f)(10^{2}\ V)^2$$
$$\Rightarrow \quad U = .5x10^{-2}\ \text{joules}$$



f.) *Where is* the energy stored?

In the electric field between the plates.

g.) *You are told* the system's time constant is 10^{-2} seconds. What does that tell you?

It tells you how long it will take for the cap to charge up to or discharge by 63%.

h.) *Where is* the charge alluded to found?

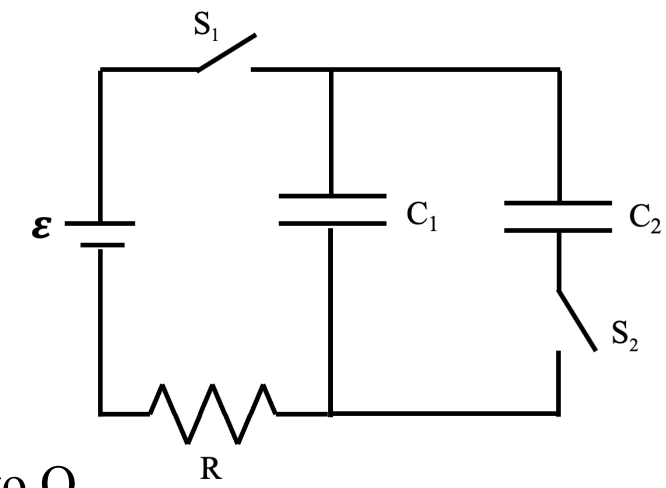
On one plate of the cap.

THINGS TO NOTICE ABOUT CAPACITORS

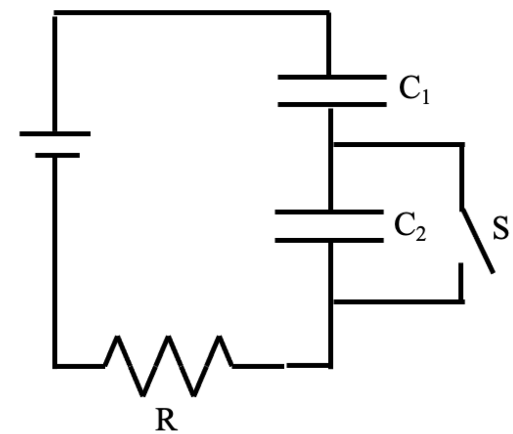
1.) Caps in series have common charge on their plates.

2.) Caps in parallel have common voltage across their plates.

3.) Close S_1 with S_2 open and C_1 charges to Q. Keep S_1 closed and close S_2. C_1 is still across the battery so it keeps Q on it, but C_2 is now also across the battery so more charge is drawn from the battery. Because the battery is still connected, *the total charge in the system is not fixed*.

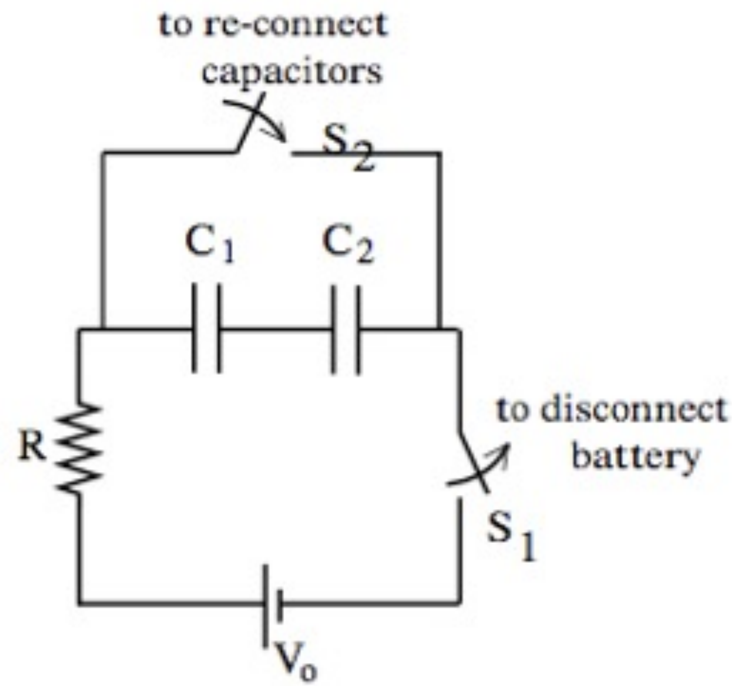


4.) Different scenario: Close S_1 with S_2 open and C_1 charges to Q. Open S_1 and C_1 is now isolated (disconnected from the battery). Close S_2 so that C_1 and C_2 are now in parallel. Because the battery is no longer in the circuit, the charge in the system is fixed. That means the charge on C_1 has to redistribute itself which it will do until the voltage across each cap is the same (and if the charge that flows from C_1 to C_2 is q, then the charge left on C_1 is (Q – q)).



5.) Second circuit: S starts out closed so C_1 charges to Q. S is opened so the caps are now in series. What changes? Because the battery voltage is now distributed between the two caps, C_1's voltage goes down but the charge on the two caps has to be the same.

Example 12: You charge up two unequal capacitors that are in series. You disconnect the battery by opening S_1 , then reconnect the two caps by closing S_2 .



a.) What is initially common to the two caps?

Caps in series have the same amount of charge on their plates.

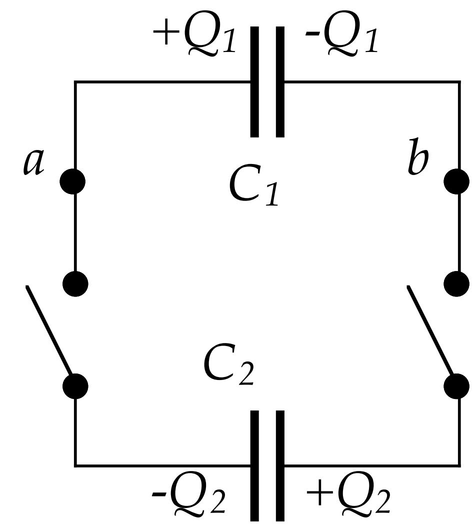
b.) When you throw both switches, how are the caps related (series or parallel)?

They are now connected in two places, which is characteristic of a parallel combination.

c.) So what should happen when both switches are thrown?

The the caps now in parallel, their voltages have to be the same so charge will have to rearrange itself necessitating current to momentarily flow.

Example 11: (courtesy of Mr. White): Examine the circuit here, where $C_1 > C_2$ and both have been charged to the same potential V.



a.) What is the potential between points a across b before the switches are closed?

Q1/C1

b.) What happens to the charges after the switches are thrown?

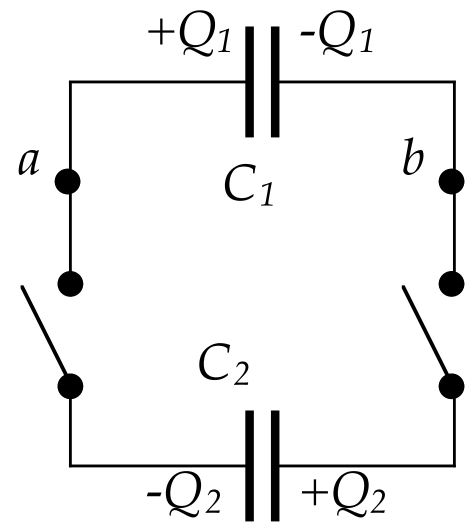
This is a screwball problem in the sense that the voltages initially have the same magnitude, but their polarities are reversed. That means that when the switches are thrown, charge will flow between the caps. Specifically:

$Q_1 > Q_2$ $(\text{as } C_1 > C_2)$, so excess Q_1 will flow until $V_1 = V_2$ again (but now with the right polarity)

c.) What is the potential across a and b a long time after the switches have been closed?

This is going to take some room:

c.) What is the potential across *a* and *b* a long time after the switches have been closed?



The total charge in the system is: $Q_1 - Q_2 = C_1V - C_2V$

Letting the new charge on C_1 be *q*, the new charge on C_2 will be: $(Q_1 - Q_2) - q = (C_1V - C_2V) - q$

With the new voltages the same, we can write:

$$V_{C1} = V_{C2}$$

$$\frac{q}{C_1} = \frac{(C_1 - C_2)V - q}{C_2}$$

$$\Rightarrow \quad qC_2 = C_1(C_1 - C_2)V - C_1q$$

$$\Rightarrow \quad qC_2 + qC_1 = C_1(C_1 - C_2)V$$

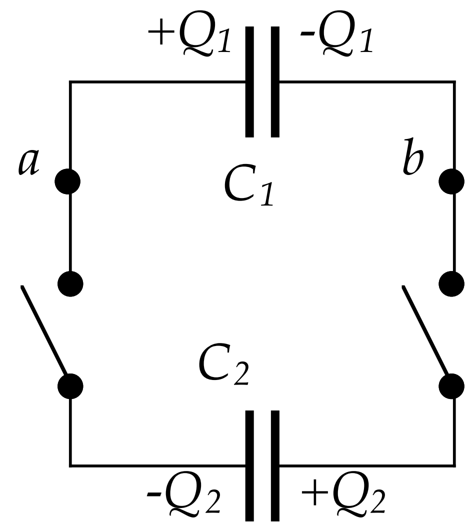
$$\Rightarrow \quad q(C_1 + C_2) = C_1(C_1 - C_2)V$$

$$\Rightarrow \quad q = \frac{C_1(C_1 - C_2)V}{(C_1 + C_2)}$$

Knowing *q*, we can write:

$$V_{C1} = \frac{q}{C_1} = \frac{\frac{\cancel{C_1}(C_1 - C_2)V}{(C_1 + C_2)}}{\cancel{C_1}} = \frac{(C_1 - C_2)V}{(C_1 + C_2)}$$

d.) What is the energy stored in the system before and after the closing of the switches?



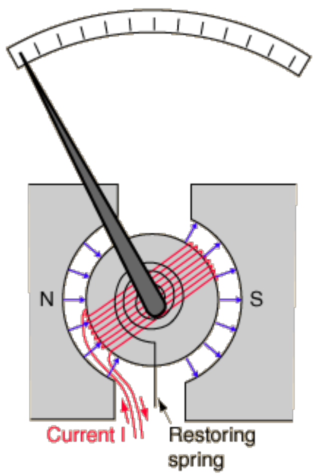
$$U_{initial} = \frac{1}{2}C_1V^2 + \frac{1}{2}C_2V^2 = \frac{1}{2}(C_1 + C_2)V^2$$

$$U_{final} = \frac{1}{2}C_{equivalent}V'^2 = \frac{1}{2}(C_1 + C_2)\left(V\frac{C_1 - C_2}{C_1 + C_2}\right)^2$$

$$U_{final} = \left(\frac{C_1 - C_2}{C_1 + C_2}\right)^2 U_{initial}$$

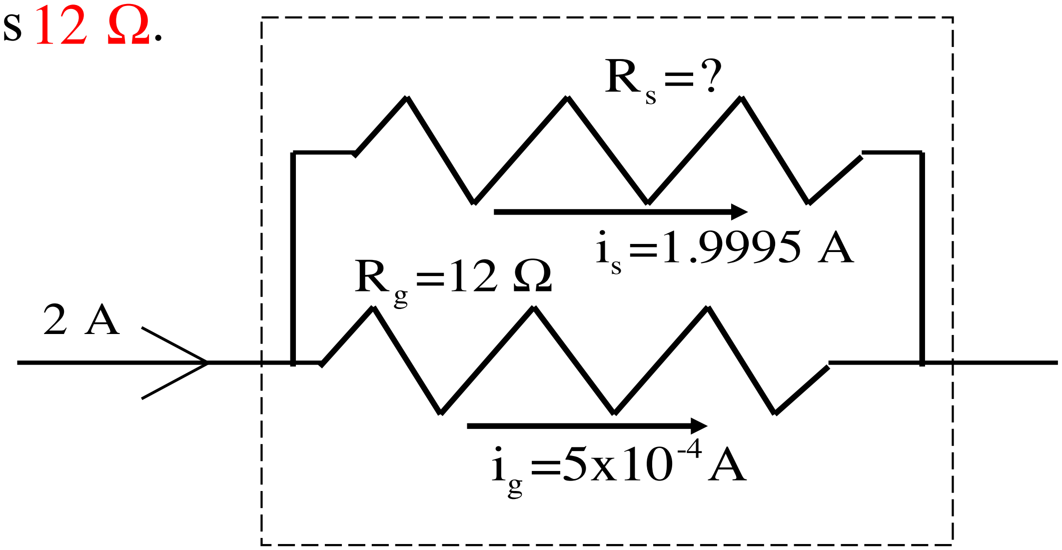
The Galvanometer

A galvanometer is an ammeter built to a very specific specification. All galvanometers swing maximum deflection when $5x10^{-4}$ amps flow through them. Although they are usually discussed in the chapter on magnetism (they are built using a coil in a magnetic field), it is possible to use a galvanometer to build an ammeter designed to handle larger currents, and to build voltmeters.



Example 12: Design a 2-amp ammeter assuming the galvanometer's resistance is 12 Ω.

For the galvanometer to go full deflection--something you want it to do when 2 amps flow into it--you need to shunt some of the current away from the galvanometer. The current that does not flow into the galvanometer flows through the shunt resistor. In this case, that will be:



2 - .0005 = 1.9995 amps.

With the shunt resistor in parallel with the galvanometer and the voltage across each being the same, we can write:

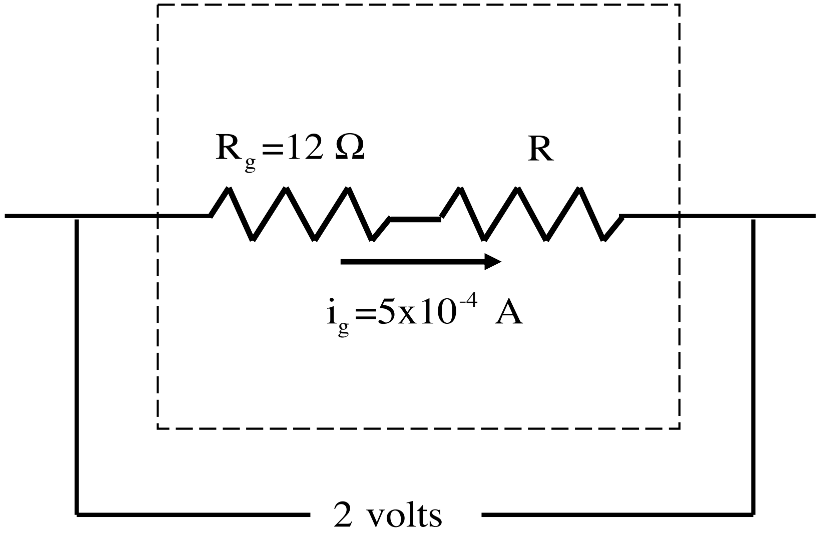
$$i_g R_g = i_s R_s$$

$$\Rightarrow \quad (5\text{x}10^{-4}\ \text{A})(12\ \Omega) = 1.9995(\text{R})$$

$$\Rightarrow \quad \text{R} \approx 3\text{x}10^{-3}\ \Omega$$

Example 13: Design a 12-volt voltmeter assuming the galvanometer's resistance is 12 Ω.

You want the galvanometer to go full deflection when 12 volts is across its terminals. Unfortunately, 12 volts will produce a HUGE current, so you need to cut the down current. You can do that by adding a resistor in series. As the current in a series combination is the same everywhere, we can write:



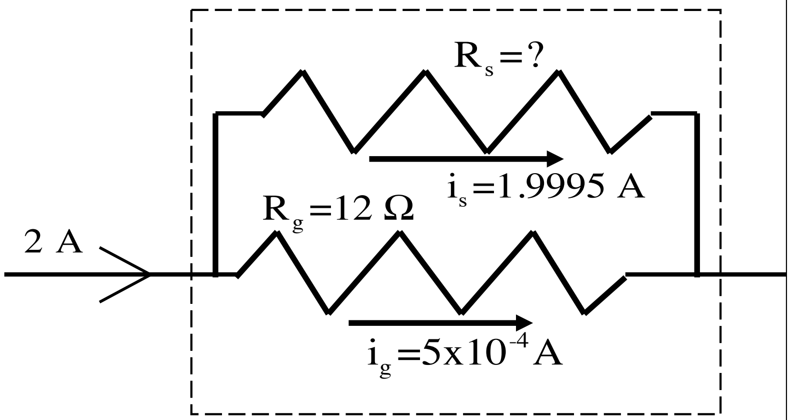
$$V = i_g R_g + i_g R$$

$$\Rightarrow \quad 2 = (5x10^{-4}\,A)(12\,\Omega) + (5x10^{-4}\,A)(R)$$

$$\Rightarrow \quad R \approx 4x10^{3}\Omega$$

Observations:

--The galvanometer-engineered ammeter consists of a 12 Ω galvanometer in parallel with (in this case) a $3x10^{-3}$ Ω resistor (that is, essentially a wire). As the equivalent resistance of a parallel combination is *smaller than the smallest resistor* in the combination, that means that the equivalent resistance of the ammeter is REALLY SMALL—exactly as expect.



--The galvanometer-engineered voltmeter consists of galvanometer and, in this case, an additional 40,000 Ω resistor in series. As the equivalent resistance of a series combination is *larger than the largest resistor* in the combination, that means that the equivalent resistance of the voltmeter is REALLY Big—again, exactly as expect.