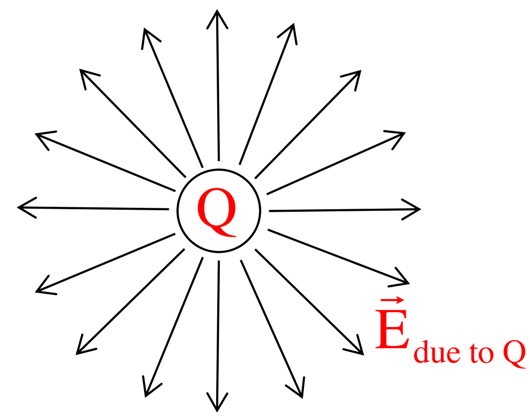
CHAPTER 25:
Electric Potentials and Energy Considerations

courtesy of
Mr. White

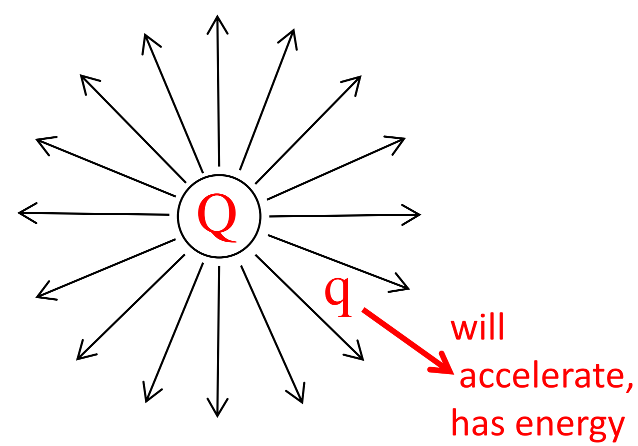
Electric Potential Fields



You should now be comfortable with the idea that a *charge configuration* will produce an electrical disturbance in its vicinity, and that knowing how much *force per unit charge* is provided to the region around the field-producing charge (whether there be a secondary charge is in the region experiencing the force or not) in the form of an *electric field* is a useful idea to entertain.

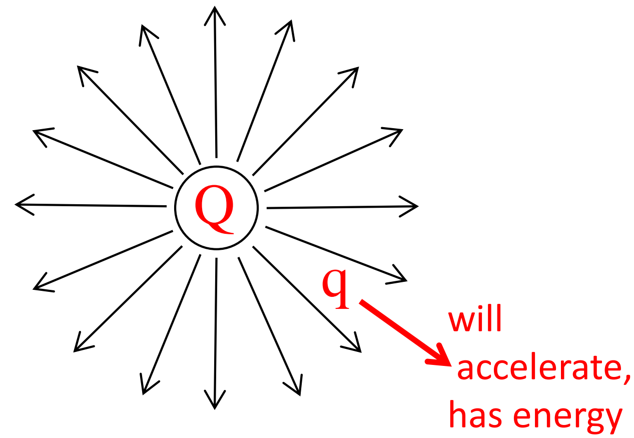
It's time to consider another related field, one associated with *energy*.

If we release a test charge *q* (or *any* charge, for that matter) in the electrical disturbance generated by our *field-producing charge*, the test charge WILL ACCELERATE.



Why will the test charge accelerate?

Because there is *POTENTIAL ENERGY* available to the test charge as it sits in the field.

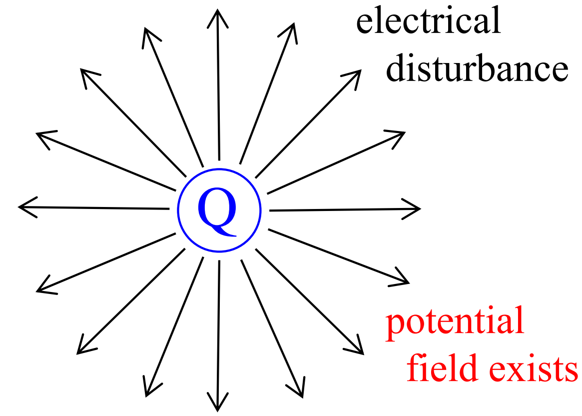


We could measure the amount of potential energy the test charge has, but that would be quite limiting (the information would be applicable only to that particular test charge).

The clever thing to do would be to mimic what we did with *electric fields*. We could measure the test charge's *potential energy* while in the field at a particular point, then divide by the size of the test charge to determine how much *POTENTIAL ENERGY PER UNIT CHARGE* is *AVAILABLE at the point* (whether the test charge is there to feel the effect or not).

This quantity, with units of *joules per coulomb* (or volts), is called the *ABSOLUTE ELECTRIC POTENTIAL* at the point of interest.

An ELECTRIC POTENTIAL FIELD, measuring the amount of *potential energy per unit charge AVAILABLE* at all points in the region of a *field-producing* charge, can be (and is) associated with *any* charge configuration.



An ELECTRIC POTENTIAL FIELD exists wherever there is charge (and, for that matter, wherever there is an *electric field*). For the potential fields to exist, there doesn't need to be present a *secondary charge* to feel the effect. And because *voltage-flds* tell us how much energy is available *PER UNIT CHARGE* at a point, the *electric potential field V* is defined as:

$$V = U/q$$

Important note: As an *absolute electric potential* is a function of the charge *q* that generates the field, a negative charge will produce a NEGATIVE *absolute electric potential* and a positive charge will produce a POSITIVE *absolute electric potential*!

Example 1: How much *potential energy* does a 2 C charge have at a point where the *absolute electric potential* is $V_1 = 3$ joules/coulomb ?

$$V_1 = \frac{U_1}{q} \quad \Rightarrow \quad U_1 = qV_1$$
$$= (2\ \text{C})(3\ \text{J/C}) = 6\ \text{J}$$

Example 2: How much *potential energy* does a -2 C charge have at a point where the *absolute electric potential* is $V_1 = 3$ joules/coulomb ?

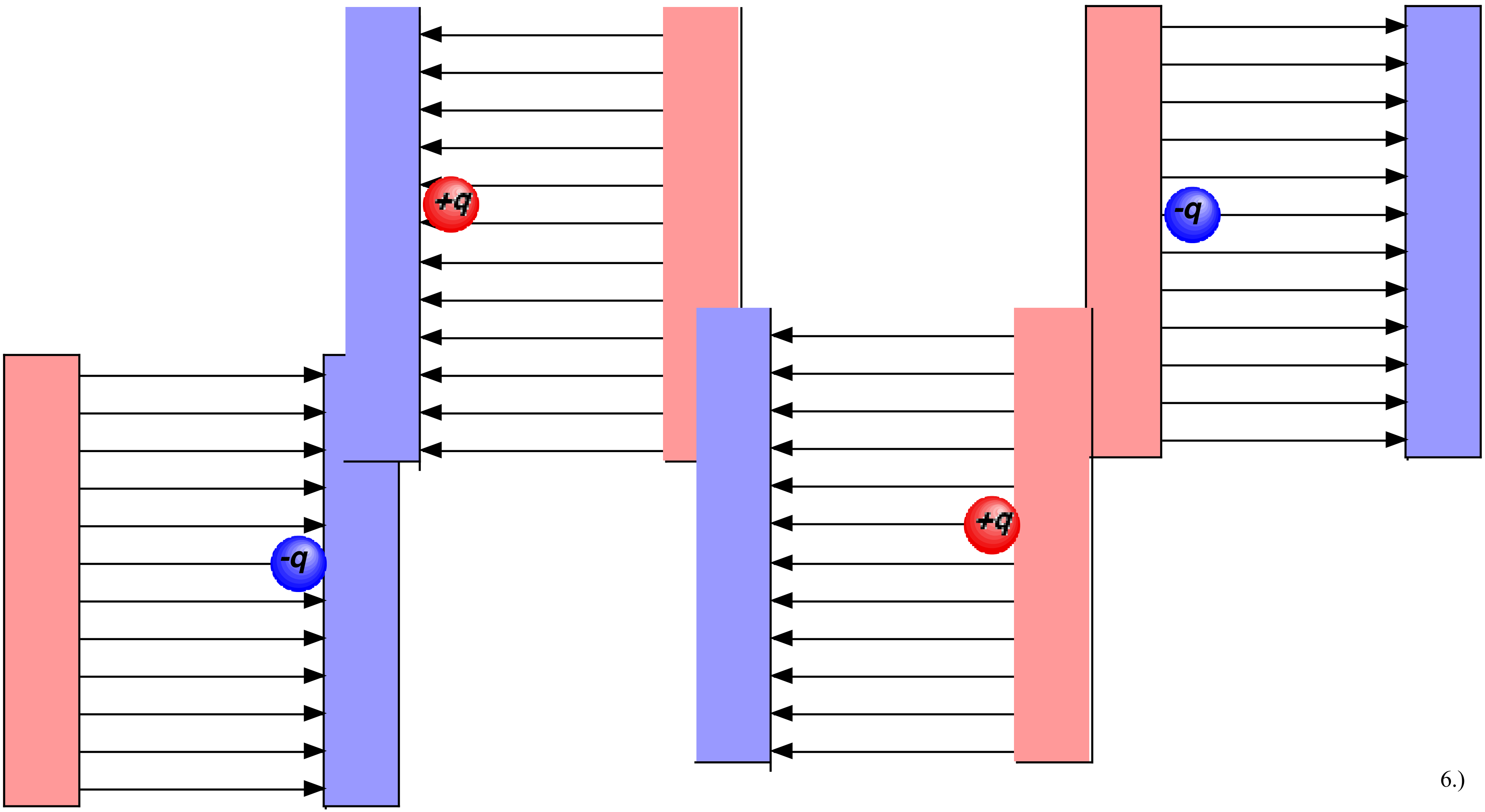
$$V_1 = \frac{U_1}{q}$$
$$\Rightarrow \quad U_1 = qV_1$$
$$= (-2\ \text{C})(3\ \text{J/C})$$
$$= -6\ \text{J}$$

Example 3: How much *potential energy* does a -2 C charge have at a point where the *absolute electric potential* is $V_1 = -3$ joules/coulomb ?

$$V_1 = \frac{U_1}{q}$$
$$\Rightarrow \quad U_1 = qV_1$$
$$= (-2\ \text{C})(-3\ \text{J/C})$$
$$= 6\ \text{J}$$

Example 4 (courtesy of Mr. White)

Is the electric charge in each of these situations in a position of *high* U_e or *low* U_e?



Work and Electric-Potential (Voltage) Fields

Note: An *absolute electric potential field* is a modified *potential energy* field.

Everything you can do with *energy considerations*, you can do with *electric potential functions*:

Just as the work done on a body moving from one point to another in a *conservative force field* equals $W = -\Delta U$, we can use the definition of absolute electric potential to write:

$$W = -\Delta U = -q\Delta V$$

$$\Rightarrow \quad \frac{W}{q} = -\Delta V$$

and

Apparently, if you know the *voltage difference* between two points, you know how much *work per unit charge* AND *potential energy per unit charge* *t*he field has available between the two points.

$$W = -\Delta U = -q\Delta V$$

$$\Rightarrow \quad \frac{\Delta U}{q} = \Delta V$$

Example 5: How much work does a field do on a moving 2 C charge if the *potential difference* between its beginning and end points is 7 volts?

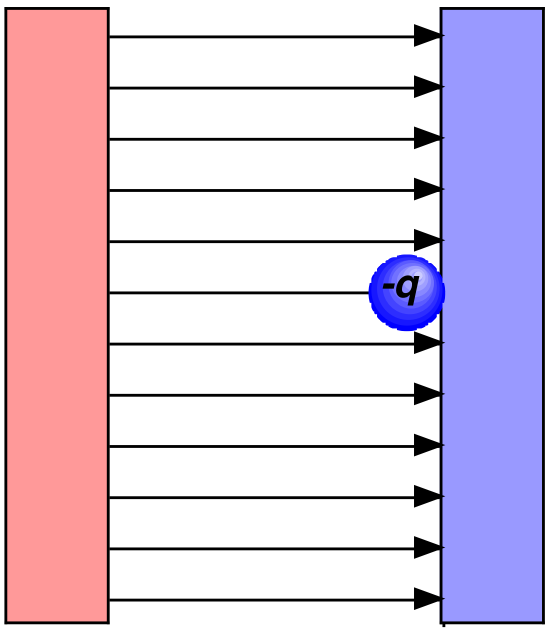
$$\frac{W}{q} = -\Delta V \quad \Rightarrow \quad W = -q\Delta V$$

$$= -(2\ \mathrm{C})(7\ \mathrm{J/C}) = -14\ \mathrm{J}$$

Example 6: (courtesy of Mr. White)

An electron (e^-) in a TV picture tube is accelerated from rest through a potential difference of 5000 V.

a.) *What is* the change in the *U* of the electron?



$$\Delta U = q\Delta V == q\left(V_2 - V_1\right)$$
$$= (-1.602x10^{-19}C)(5000V - 0)$$
$$= -8x10^{-16}J \quad \left(= -5000eV\right)^*$$

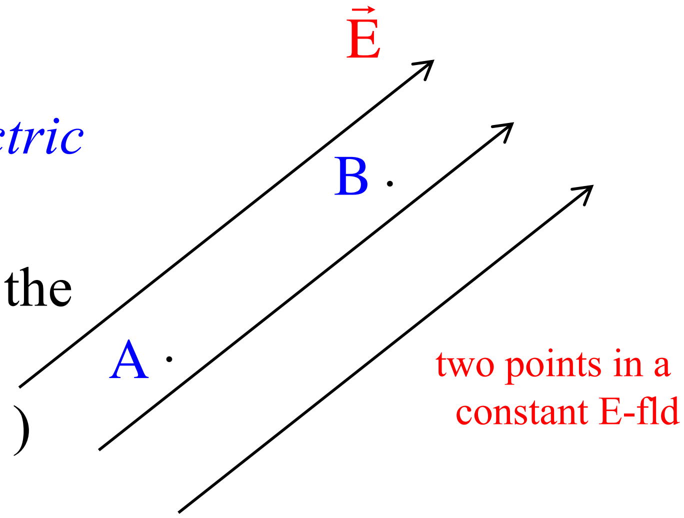
b. *What is* the final speed of the electron?

$$-\Delta U = \Delta K$$
$$-\left(-8x10^{-16}J\right) = \frac{1}{2}mv^2$$
$$\Rightarrow \quad v = \sqrt{\frac{2\left(8x10^{-16}\right)}{9.11x10^{-31}}} = 4.2x10^7 m/s$$

**Fletch's note:* An electron-volt (eV) is defined as the amount of energy an electron picks up when accelerated through a *1 volt electrical potential difference*.

Electric Potential Difference and E-flds

Assuming we are dealing with a *constant electric field* and a *straight-line path* between two points in the field, we can use the definition of work ($W = \vec{F} \bullet \vec{d}$) with the manipulated definition of the electric field ($\vec{F} = q\vec{E}$) to extend out *potential difference relationship* ($W/q = -\Delta V$) into a very interesting proposition. Specifically:



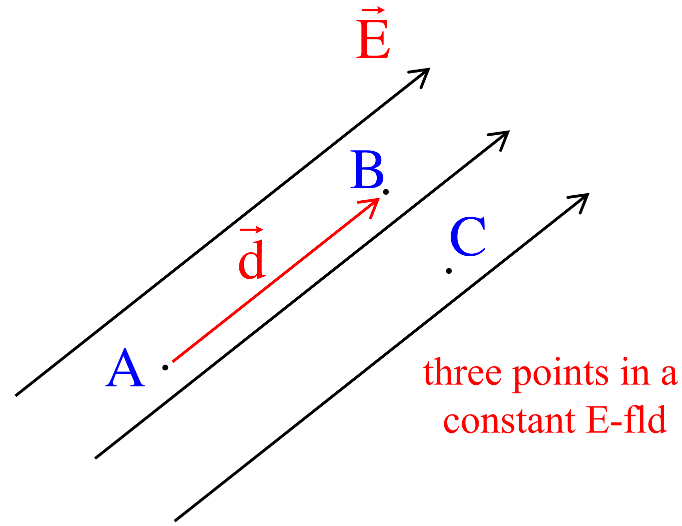
$$\frac{W_{AB}}{q} = -\Delta V_{AB} \quad \Rightarrow \quad \frac{\vec{F} \bullet \vec{d}_{AB}}{q} = -\Delta V_{AB}$$

$$\Rightarrow \quad \frac{\cancel{q}\vec{E} \bullet \vec{d}_{AB}}{\cancel{q}} = -\Delta V_{AB}$$

$$\Rightarrow \quad \vec{E} \bullet \vec{d}_{AB} = -\Delta V_{AB}$$

And what might we glean from this bit of amusement?

Observations



Example 7: Points A, B and C are identified in a constant *electric field* as shown in the sketch.

a.) Which point has the GREATER *absolute electric potential*? (That is, do *electric fields* run from *higher voltage* to *lower*, or vice versa?)

Traversing from A to B, so $\vec{d}$ points along the line of $\vec{E}$, the *dot product* in our relationship falls out as:

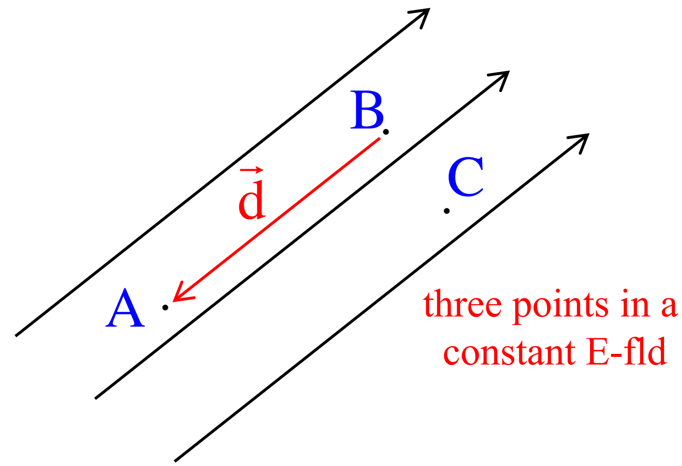
$$\vec{E} \bullet \vec{d} = -\Delta V$$

$$\Rightarrow \quad \left|\vec{E}\right|\left|\vec{d}\right| \cancelto{1}{\cos 0^\circ} = -\left(V_B - V_A\right)$$

By observation, the left-side of the equation is positive (two magnitudes multiplied together), so the right-side must also be positive. For this to be true, V_A must be larger than V_B .

IMPORTANT OBSERVATION: This means that *ELECTRIC FIELDS* migrate from HIGHER *ELECTRIC POTENTIAL* to LOWER.

b.) Assume the *electric potential* at A is $V_A = 11$ volts and the *electric potential* at B is $V_B = 5$ volts. If the *distance* between the two points is 2 meters, derive an expression for the magnitude of the *electric field*.



This time, to point out how the angle works, we will traverse from B to A. Noticing that now the angle between $\vec{d}$ and $\vec{E}$ is 180°, we can write:

$$\vec{E} \bullet \vec{d} = -\Delta V$$

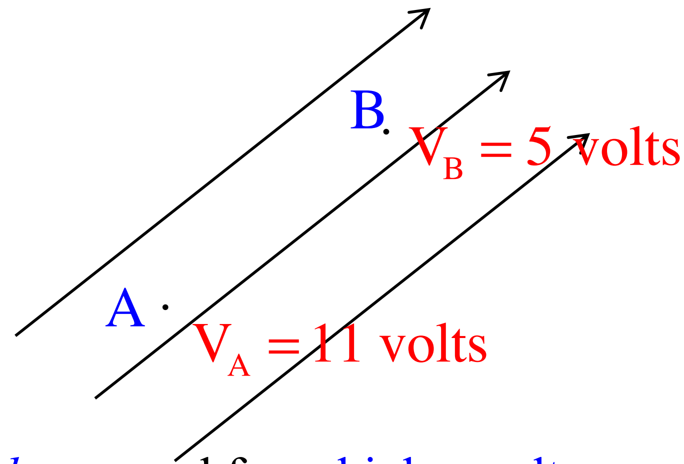
$$\Rightarrow \quad |\vec{E}| \; |\vec{d}| \; \cancelto{-1}{\cos 180^\circ} = -(V_A - V_B)$$

$$-|\vec{E}|(3\text{m}) \qquad = -(11\text{v} - 5\text{v})$$

$$\Rightarrow \quad |\vec{E}| = 2 \text{ V/m}$$

IMPORTANT SIDE POINT: The unit for *ELECTRIC FIELDS* is *newtons per coulomb*, but it is also, apparently, *volts per meter.*

c.) A positive charge Q=1C and mass *m=1 kg* moves naturally along the *E-fld* lines.



i.) Is the charge moving from *higher electrical potential to lower*, or *lower electric potential to higher*?

This has nothing to do with the charge. *Electric fields* proceed from higher voltage to lower, so it's doing the former.

ii.) Is the charge moving from higher *potential energy* to lower, or lower *potential energy* to higher?

This has EVERYTHING to do with the charge. POSITIVE CHARGES naturally move from *higher to lower voltage* along *E-fld* lines (being by definition the direction a positive charge would naturally accelerate), so it is moving from *higher* to lower potential energy.

iii.) If Q's initial velocity was 3 m/s at A, what is its velocity at B? (Note that the voltages have been put on the sketch.)

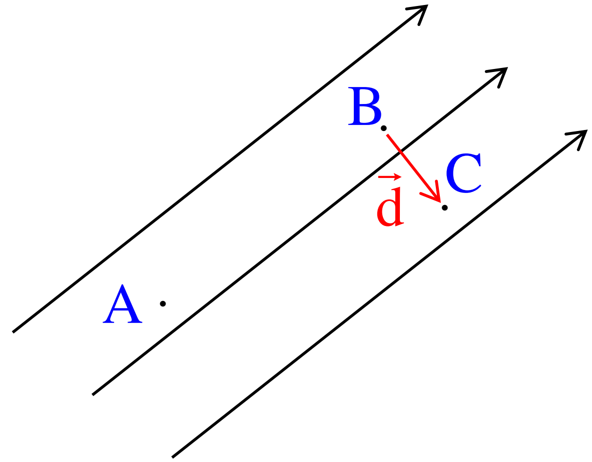
$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

$$\tfrac{1}{2} m v_A{}^2 + (qV_A) + 0 = \tfrac{1}{2} m v_B{}^2 + (qV_B)$$

$$\tfrac{1}{2}(1)(3)^2 + (1)(11) = \tfrac{1}{2}(1) v_B{}^2 + (1)(5)$$

$$\Rightarrow \quad v_B = 4.58 \text{ m/s}$$

d.) Given the *electric potential* at B is $V_B = 5 \text{ volts}$ and the *electric field,* as calculated in the previous part, is $\left|\vec{E}\right| = 2 \text{ V/m}$, what is the *voltage* (i.e., the *electric potential*) at C, assuming the distance between B and C is $.5$ meters?



This is slightly tricky. Define $\vec{d}$ on your sketch as shown. Notice that the angle between $\vec{d}$ and $\vec{E}$ is 90°. With that, we write:

$$\vec{E} \bullet \vec{d} = -\Delta V$$

$$\Rightarrow \left|\vec{E}\right| \left|\vec{d}\right| \overset{0}{\cancel{\cos 90^\circ}} = -\left(V_c - V_B\right)$$

$$\Rightarrow 0 = -\left(V_C - V_B\right)$$

$$\Rightarrow V_c = V_B = 5 \text{ V/m}$$

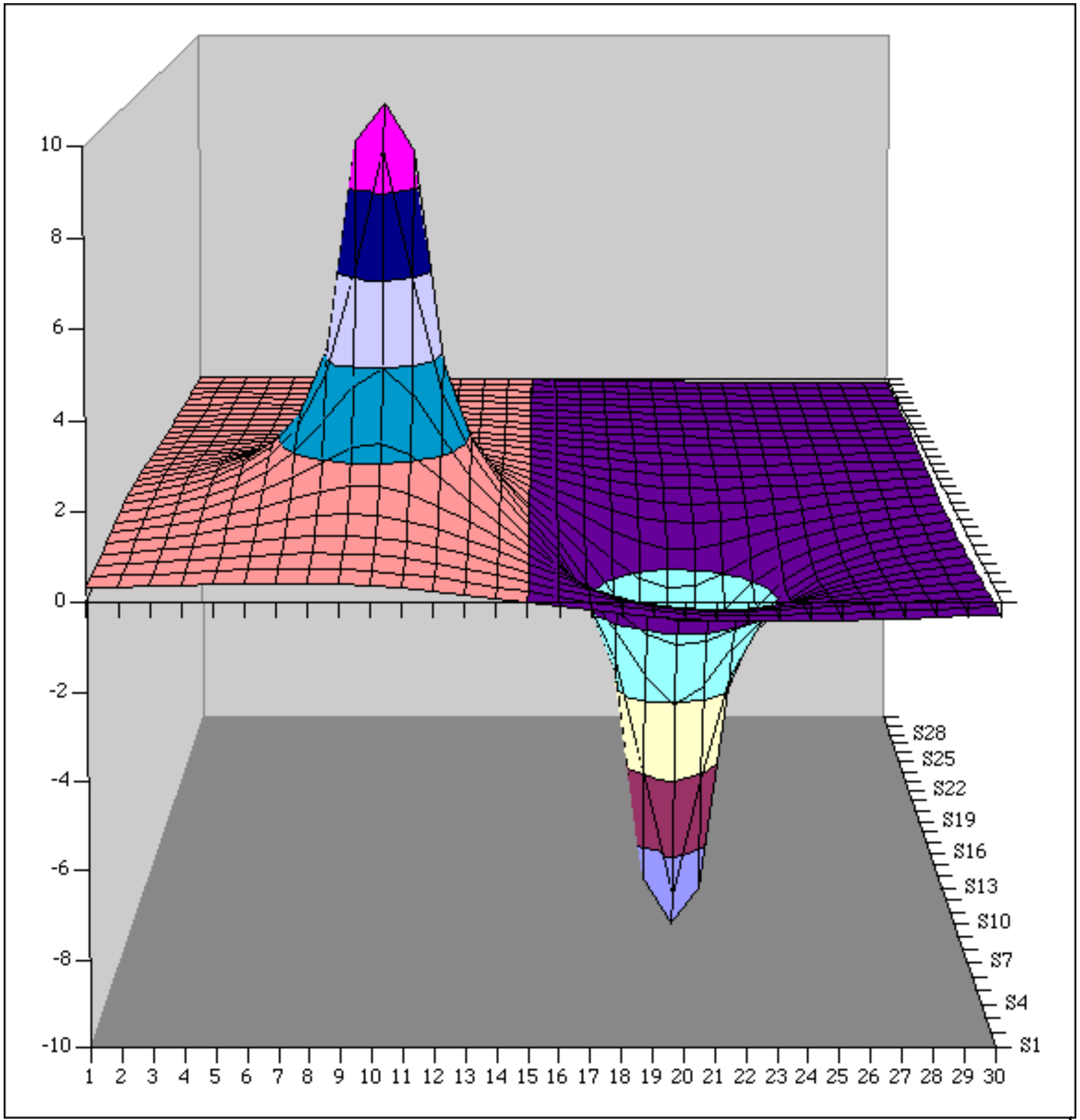
IMPORTANT POINT: An *EQUIPOTENTIAL LINE* is a line upon which every point has the same electrical potential. Points B and C are on the *5-volt equipotential line*.

IMPORTANT POINT: Equipotential lines are ALWAYS perpendicular to *electric field lines*.

Electric Potential contours (courtesy of Mr. White)

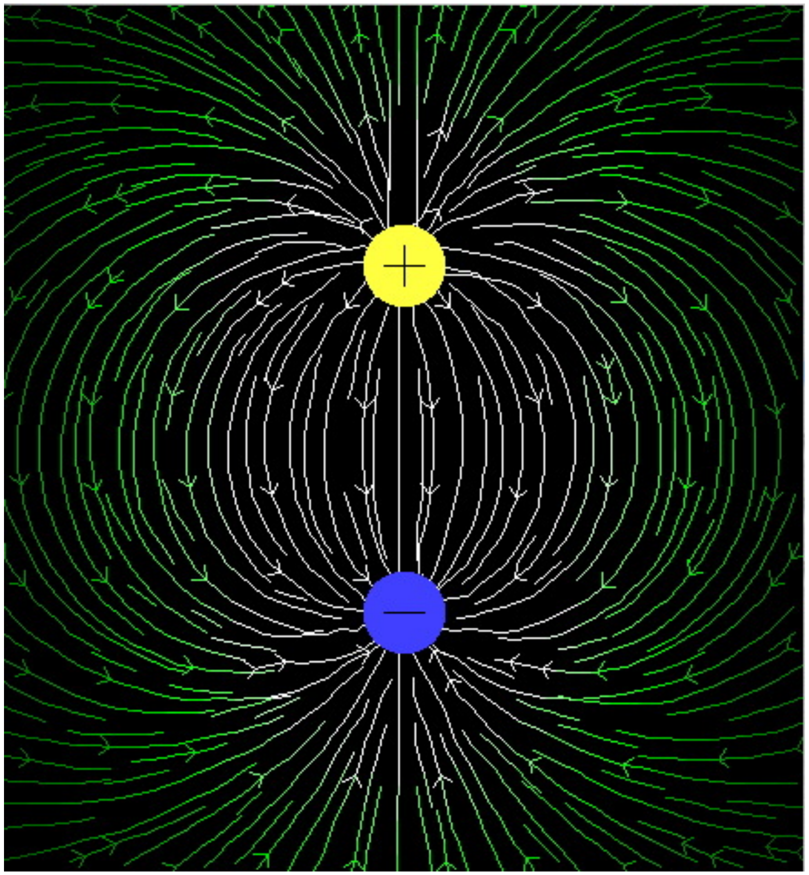
What are the potentials of each of the charges shown here? What do these potential values represent?

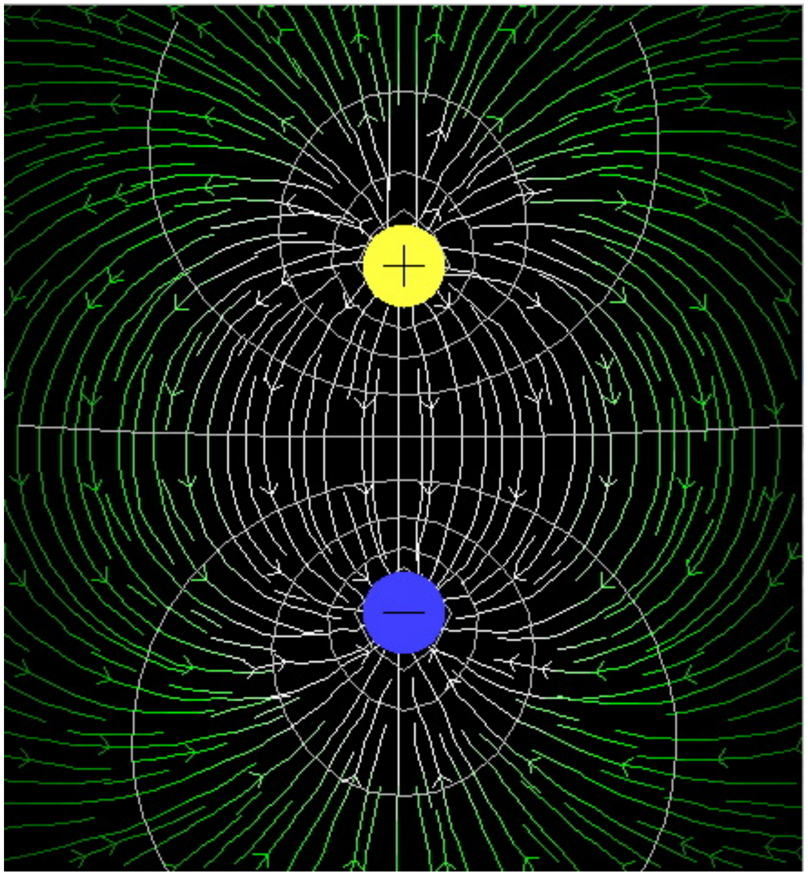
$$V_{net} = \sum_i k \frac{q_i}{r_i}$$



Equipotential Lines
(courtesy of Mr. White)

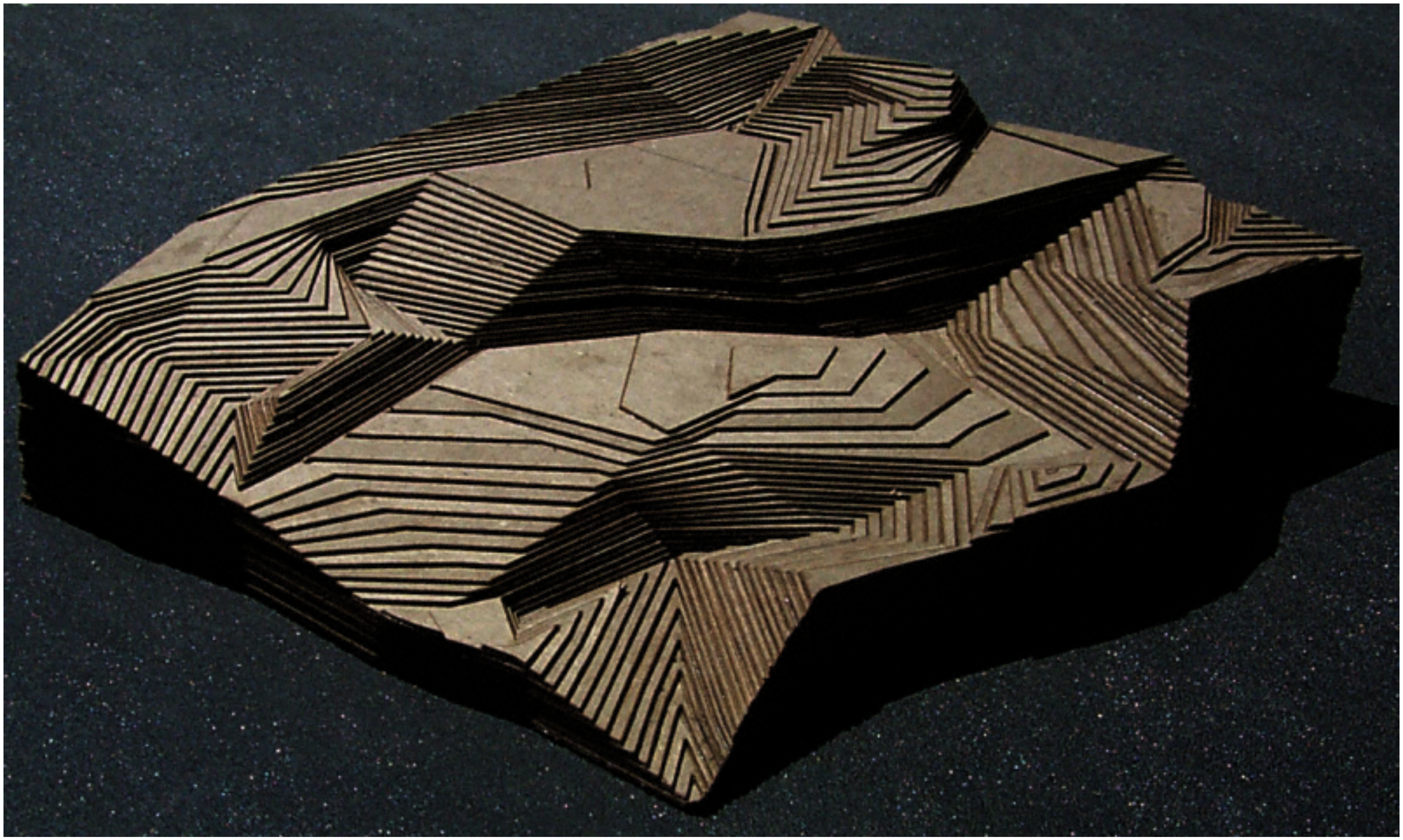
... are related to Electric Field lines. How?





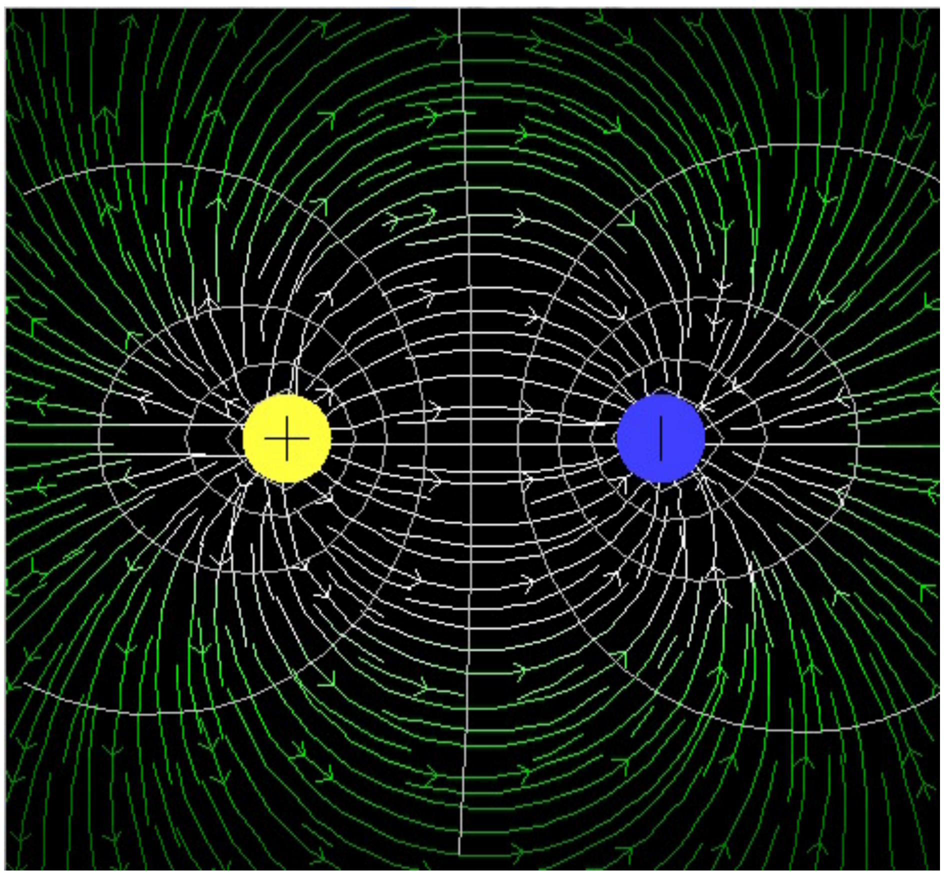
Equipotential Lines
(courtesy of Mr. White)

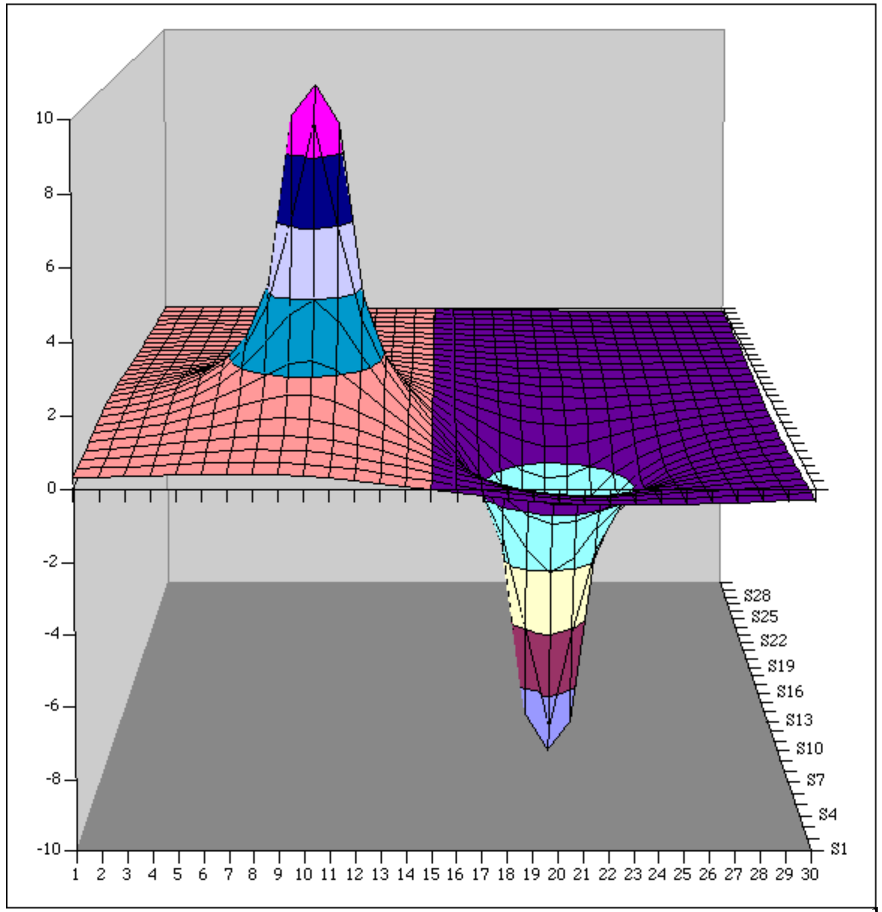
... can be visualized as a topographic map.



Equipotential Lines
(courtesy of Mr. White)

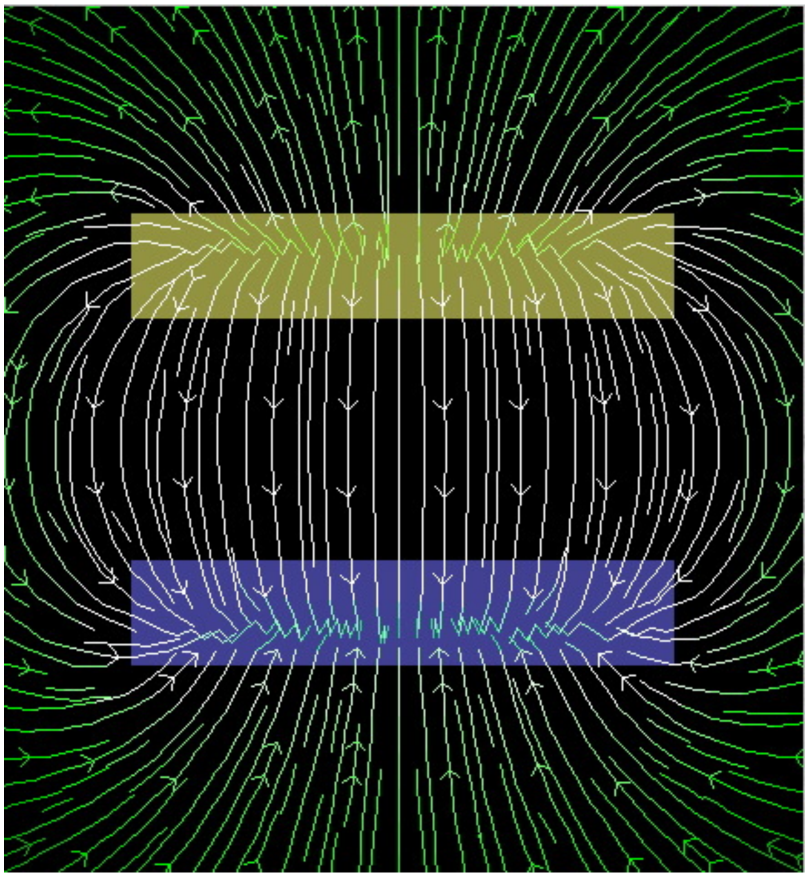
... reveal areas of higher vs. lower electric potential.

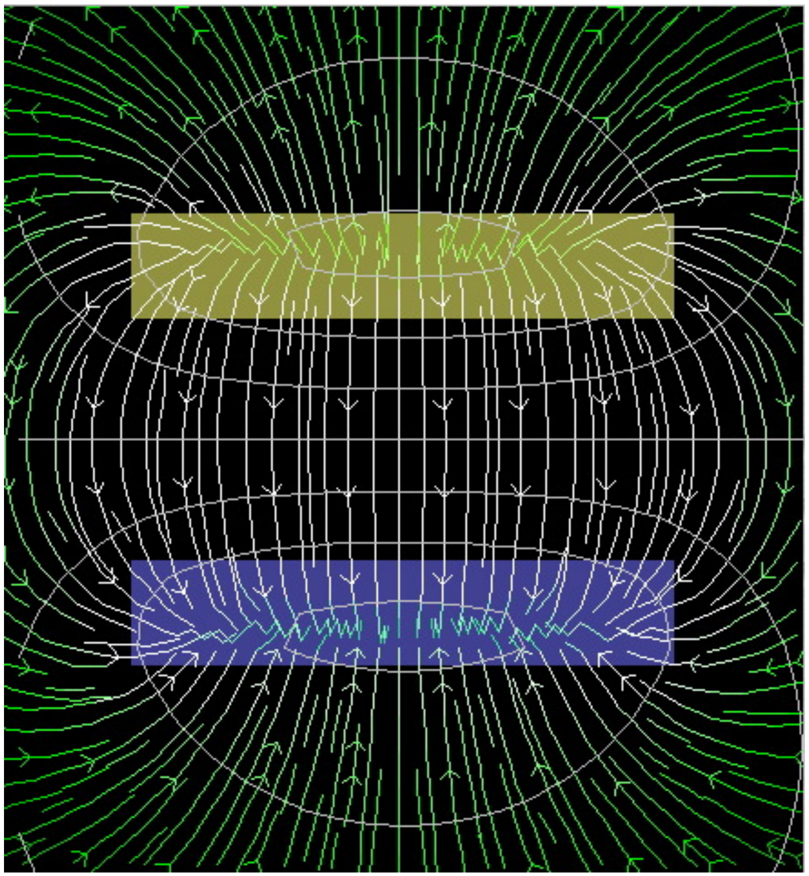




Example 8: (courtesy of Mr. White)

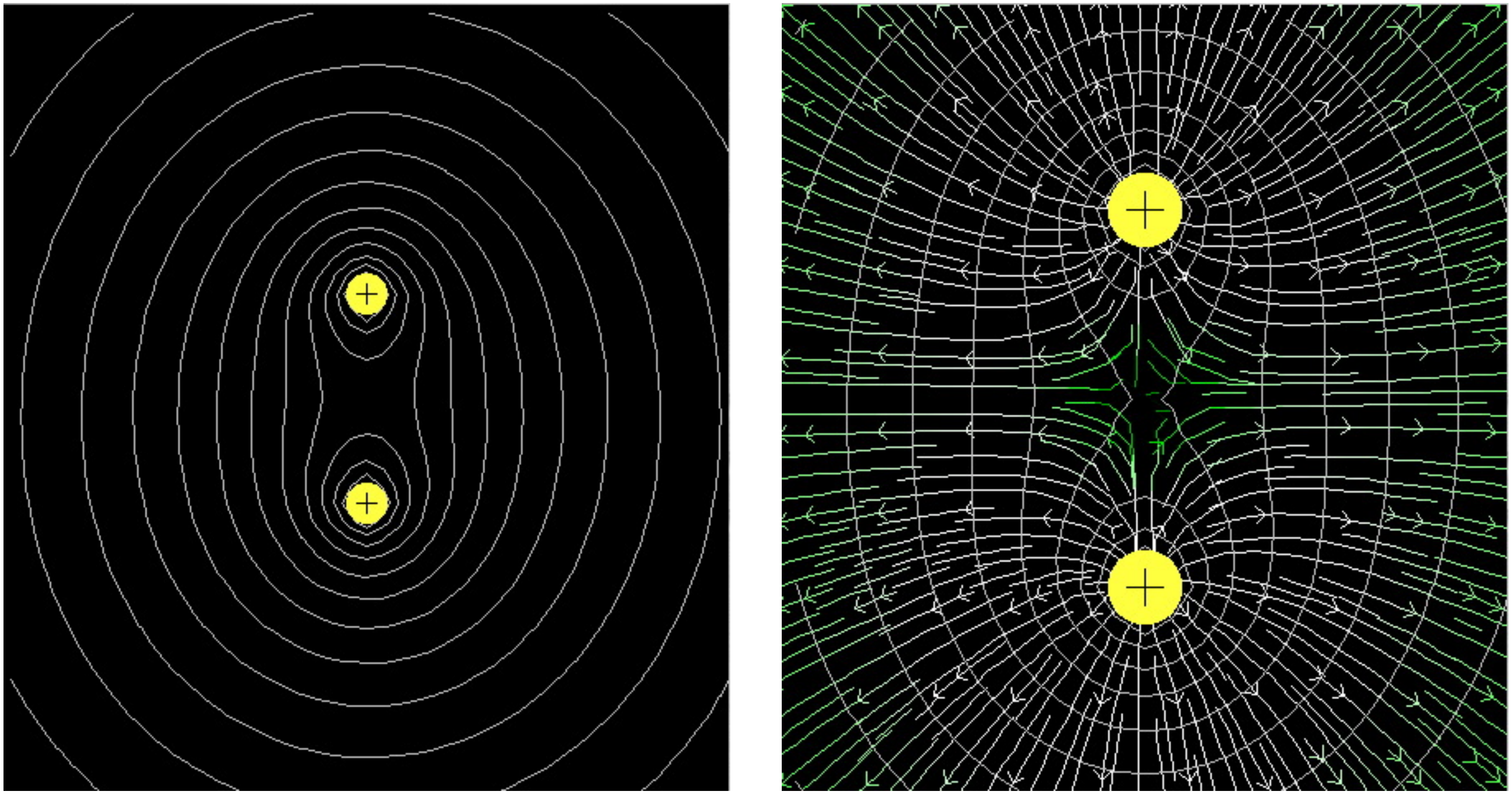
Draw appropriate equipotentials for this electric field.



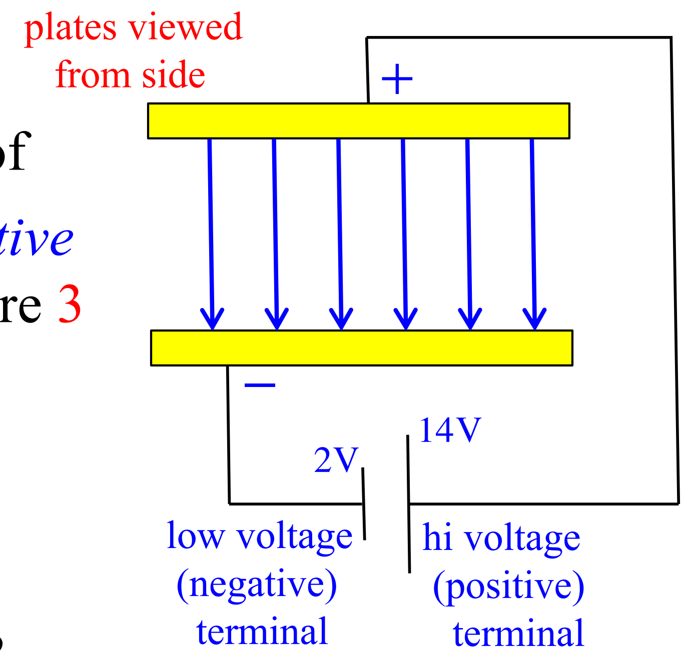


Example 9: (courtesy of Mr. White)

Draw appropriate field lines for these equipotentials.



Example 10: A battery has an *electric potential* of 14 volts at its *positive terminal* and 2 volts at it's *negative terminal*. It is connected to parallel metal plates that are 3 millimeters apart and insulated from one another.



a.) From what you know about the voltages, draw in the *electric field lines* between the plates.

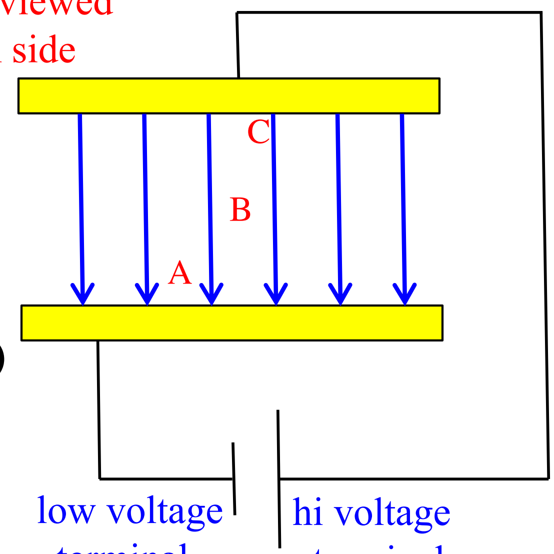
b.) How big is the *electric field* between the plates?

Traversing from the upper plate to the lower plate (i.e., from the higher voltage to the lower voltage plate ALONG THE *E-FLD* LINES, we can write:

$$
\begin{aligned}
&\vec{E}\bullet\vec{d} = -\Delta V \\
&\Rightarrow \left|\vec{E}\right| \left|\vec{d}\right| \cancelto{1}{\cos 0^\circ} = -\left(V_{-} - V_{+} \right) \\
&\Rightarrow \left|\vec{E}\right|\left(3\text{x}10^{-3}\ \text{m}\right) = -\left((2\ \text{V}) - (14\ \text{V})\right) \\
&\Rightarrow \left|\vec{E}\right| = 4000\ \text{V/m} \quad (\text{or } 4000\ \text{N/C})
\end{aligned}
$$

c.) *Points A, B and C* are identified between the plates.

A *positive charge* is placed successively at each point:



i.) *At which point* will the charge experience the greatest *electric field*?

(The *E-fld* is *constant* between the plates—all points are the same.)

ii.) *At which point* will the charge experience the greatest *electric potential*?

(The *voltage* closest the positive plate will be highest, which is C.)

iii.) *At which point* will the charge experience the greatest *potential energy*?

(A charge's potential energy at a point is related to voltage as $U = qV$, so for a *positive charge*, that will be greatest at C . . . or as close to the 14 volt plate as possible.)

d.) *Now,* a *negative charge* is placed at each point:

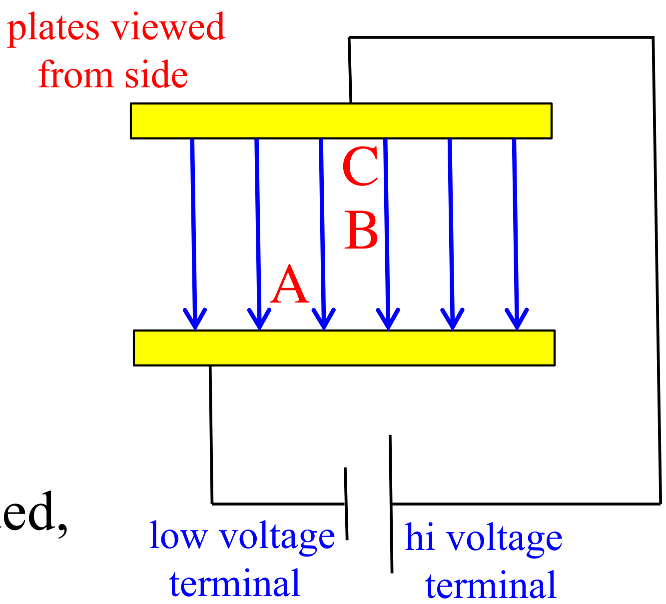
i.) *At which point* will the charge experience the greatest *electric potential*?

(*Electrical potential* has NOTHING TO DO with the charge feeling the effect: it's still C.)

ii.) *At which point* will the charge experience the greatest *potential energy*?

(Using $U = qV$, *sign included*, the greatest potential energy point for a *negative charge* is A. This makes sense if you think about it. A -1C charge on the *negative plate* (pt A)would be -2 joules whereas on the *positive plate* (pt C) it would be -14 joules. A is bigger (closer to zero)! Also, where, if you let a *negative charge* go, would it pick up the most kinetic energy? Certainly not if it was next to the positive plate. Definitely next to the negative plate at A!).

e.) An electron $(e = 1.6x10^{-19}C,\ m = 9.1x10^{31}kg)$ accelerates between the plates. How fast is it moving if it started from rest?



Note that the electron (charge $-e$) would accelerate from the negative the positive plate, and that the *potential energy* of a charge sitting at a point whose potential is V is $U = qV$ with the charge's sign included, we can write:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

$$0 + \left((-e)V_-\right) + 0 = \tfrac{1}{2}mv^2 + \left((-e)V_+\right)$$

$$\Rightarrow \left(-1.6x10^{-19}C\right)(2\ V) = \tfrac{1}{2}\left(9.1x10^{-31}\ kg\right)v_B{}^2 + \left(-1.6x10^{-19}C\right)(14\ V)$$

$$\Rightarrow \quad v = 2.1x10^6\ m/s$$

Deriving an Electric Potential From an Electric Field

We know from our experience with energy considerations that if we want the *potential energy function* that goes with a *conservative force field*, we can derive it using:

$$U(r) - U(\cancel{\text{zero pt}}^{\,0}) = -\int_{\text{zero pt}}^{r} \vec{F} \bullet d\vec{r}$$

If the force is the consequence of a charge in an *electric field*, we could divide everything by the size of the charge *q* feeling the effect, and have:

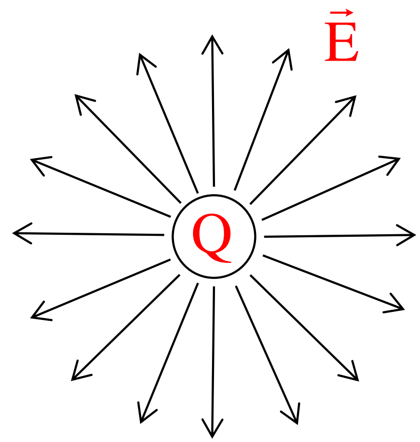
$$\frac{U(r)}{q} - \frac{U(\text{zero pt})}{q} = -\int_{\text{zero pt}}^{r} \frac{\vec{F} \bullet d\vec{r}}{q}$$

$$\Rightarrow \quad V(r) - V(\cancel{\text{zero pt}}^{\,0}) = -\int_{\text{zero pt}}^{r} \vec{E} \bullet d\vec{r}$$

A Specific Case--The Electric Potential Generated by a POINT CHARGE

Example 11: Derive a general expression for the *electric potential* generated by a point charge *Q*?

Setting the zero point for the *electric potential* to be where the *electric field* is zero (i.e., at infinity), and using the *electric field function* for a point charge as $\vec{E} = k\,Q/r^2\,\hat{r}$, we can write:



$$V(r) - V(\infty) = -\int_{\infty}^{r} \vec{E} \bullet d\vec{r}$$

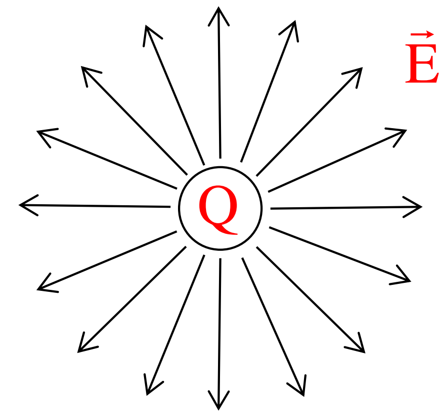
$$= -\int_{r=\infty}^{r} \left(k\frac{Q}{r^2}\hat{r} \right) \bullet d\vec{r}$$

$$= -\int_{r=\infty}^{r} \left(k\frac{Q}{r^2} \right) dr\left(\cos 0^{\circ}\right)$$

$$= -kQ\left(-\frac{1}{r} \right)\Bigg|_{r=\infty}^{r}$$

$$\Rightarrow \quad V(r)_{\text{pt chg}} = \left(\frac{1}{4\pi\varepsilon_o} \right)\frac{Q}{r}$$

Things to notice:



This function is good ONLY for *point charges*.

This is a SCALAR quantity ('cause *electric potentials* are NOT vectors).

Being a scale, a group of point charges will produce a net *electric potential* at a point that is simply the sum of the individual electric potentials. There is NO NEED to do anything with components . . . because again, voltages aren't vectors! Mathematically, this can be stated as:

$$V_{\text{net at point 1}} = \sum V_i$$
$$= \sum \left(k \frac{q_i}{r_i} \right)$$

Positive charges generate positive *electric potentials* while negative charges generate negative *electric potentials*, so the SIGN of the charge needs to be included in the use of $V(r)_{\text{pt chg}} = k\, Q/r$.

Any electric field generated by a *static charge configuration* will be conservative in nature, and can have an *electric potential function* derived for it using:

$$V(r) - V(\infty) = -\int_{\infty}^{r} \vec{E} \bullet d\vec{r}$$

So How Are Electric Fields and Electric Potentials Related?

Remember back to the Energy chapter when we related a conservative force function to its potential energy function. We found that the *spatial rate of change of potential energy* equals the *force* associated with the potential energy field, or $\vec{F} = -\left(\frac{dU}{dx}\right)\hat{i}$. There is an electrical analogue to this.

That is, the differential consequence of:

$$V(r) - V(\text{zero pt}) = -\int_{\text{zero pt}}^{r} \vec{E} \bullet d\vec{r}$$

is

$$dV = -\vec{E} \bullet d\vec{r}$$

But if that is true, it must also be true that: $\vec{E} = -\frac{dV}{dr}\hat{r}$

Except in *Cartesian coordinates* (assuming *E* is in the x-direction), $\vec{E} = -\frac{dV}{dx}\hat{i}$

which can be expanded into multiple dimensions using the *del operator* as:

$$\vec{E} = -\vec{\nabla}V = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)$$

The Electric Field Generated by a POINT CHARGE as calculated from the Electric Potential

Using the del operator in *polar spherical coordinates . . .*

$$\vec{E} = -\vec{\nabla}V$$

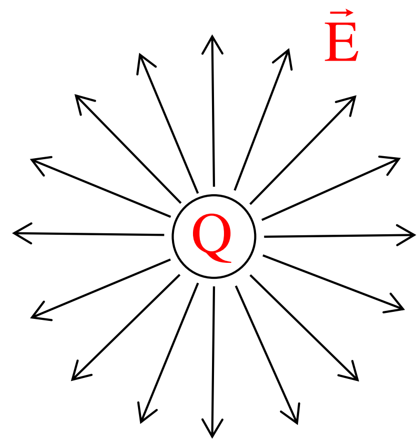
$$= -\left(\frac{\partial V}{\partial r}\hat{r} + \ \ldots\ \right)$$

$$= -\frac{\partial\left(k\,{}^{Q}\!/_{r}\right)}{\partial r}\hat{r}$$

$$= -\left(kQ\frac{\partial (r)^{-1}}{\partial r}\right)\hat{r}$$

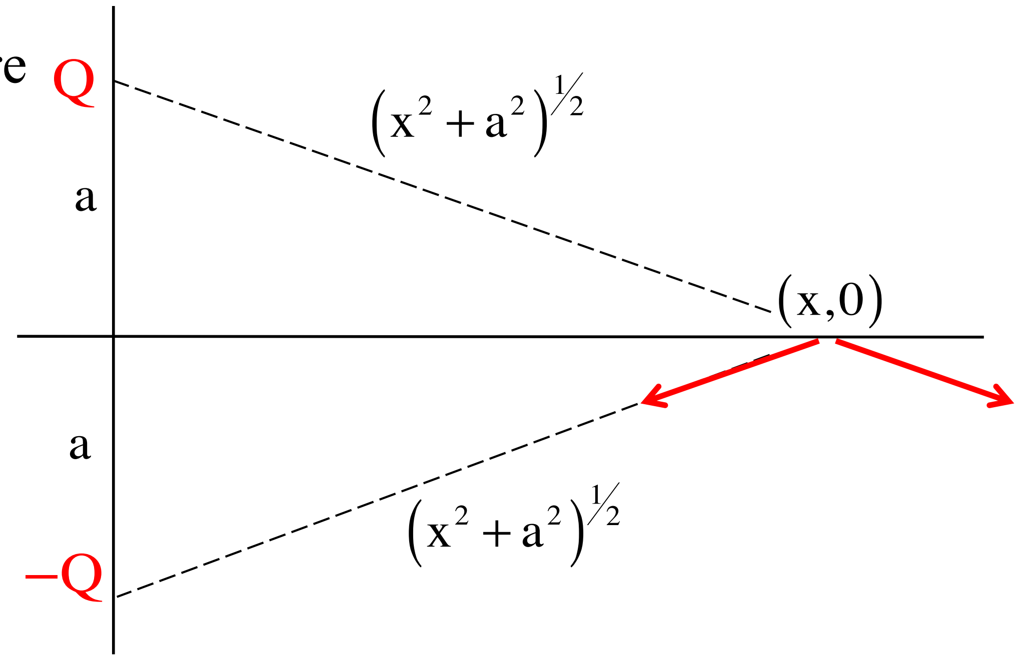
$$= -\left(kQ\left(-r^{-2}\right)\right)\hat{r}$$

$$= \left(\frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}\right)\hat{r}$$



We'll come back to this shortly. . .

Example 12: Assume the charges are equal and opposite, and are placed symmetrically as shown.



a.) Is there an *electric field* at (x,0). If so, in what direction is it?

There will be an *E-fld* at (x,0). By inspection, its x-components will add to zero leaving it with only y-components.

b.) Is there an *absolute electric potential* at (x,0). If so, in what direction is it?

TRICK QUESTION—*electric potentials* don't have directions as they are scalars. As for magnitude:

$$V_{total} = V_Q + V_{-Q}$$

$$= \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{Q}{\left(x^2+a^2\right)^{1/2}} + \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{-Q}{\left(x^2+a^2\right)^{1/2}}$$

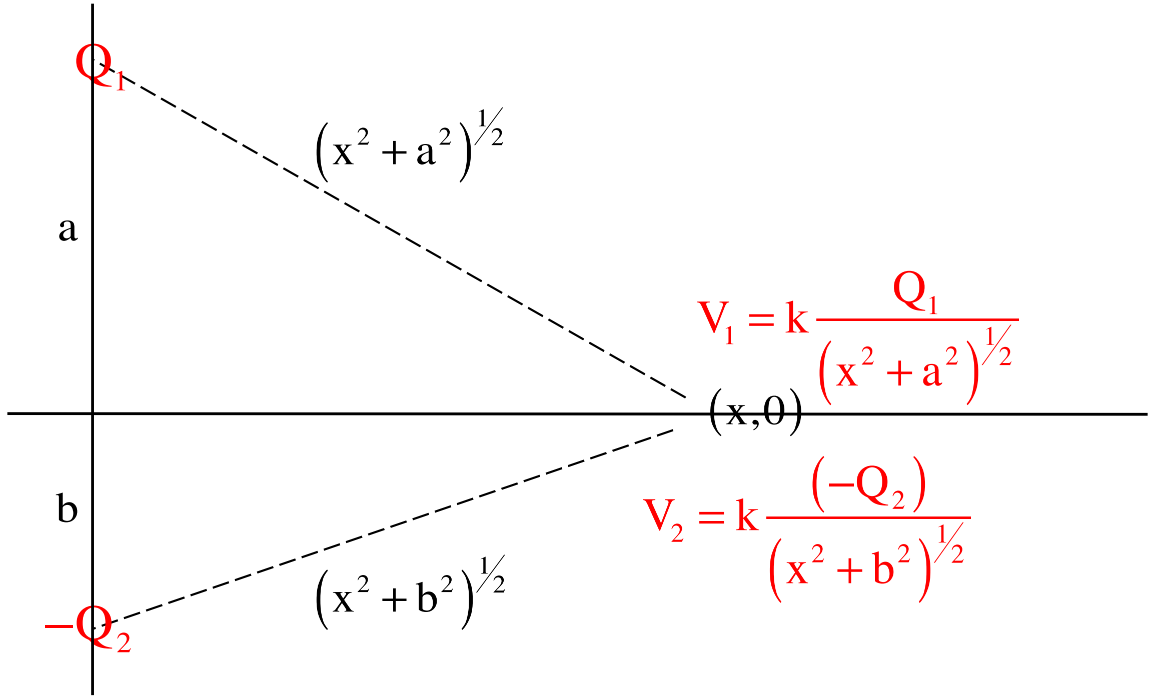
$$= 0$$

c.) Does this make sense?

Yes, if you understand how *E-flds* and *voltage flds* are related to one another.

c.) How would the problem change if the charges were no longer the same and their positions no longer symmetric?

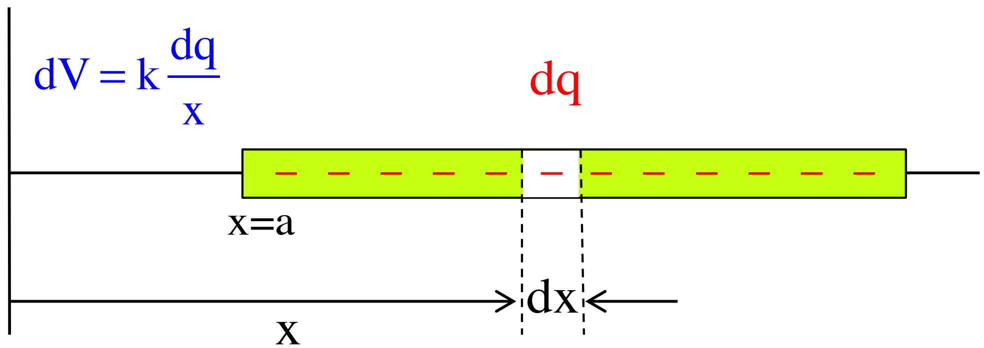
The math shows it all.



$$V_{total} = \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{Q_1}{(x^2+a^2)^{1/2}} - \left(\frac{1}{4\pi\varepsilon_o}\right)\frac{Q_2}{(x^2+b^2)^{1/2}}$$

Example 13 (A non-AP problem):

Derive an expression for the electric potential at the origin due to a rod with charge *-Q* uniformly distributed over its length *L*.



This extended charge distribution is something you've already seen. The solving technique is exactly as was before. Define the differential electric potential at the origin due to a differential bit of charge, then sum that differential electrical potential over the entire rod. You'll again need to define a *linear charge density* function $\lambda = -Q/L$

and

note that $dq = \lambda dx$.

With that, we can write:

$$V = \int dV = \int_{x=a}^{a+L} \frac{1}{4\pi\varepsilon_o}\frac{dq}{x}$$

$$= \int_{x=a}^{a+L} \frac{1}{4\pi\varepsilon_o}\frac{(\lambda dx)}{x} = \frac{\left(-Q/L\right)}{4\pi\varepsilon_o}\int_{x=a}^{a+L}\frac{dx}{x}$$

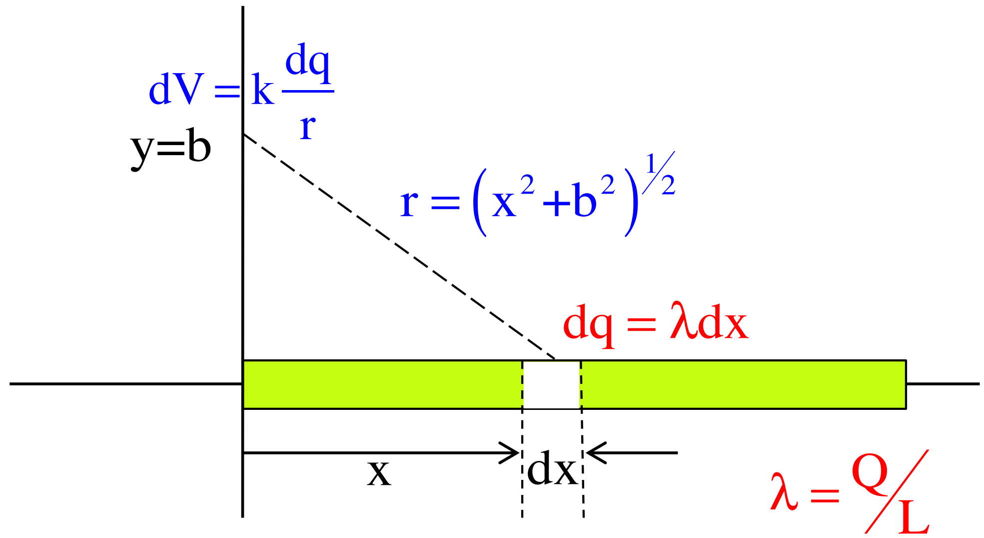
$$= \frac{\left(-Q/L\right)}{4\pi\varepsilon_o}\left(\ln x\Big|_{x=a}^{a+L}\right) = \frac{-Q}{4\pi\varepsilon_o L}\left[\left(\ln(a+L)\right)-\left(\ln a\right)\right]$$

$$= \frac{-Q}{4\pi\varepsilon_o L}\ln\left(\frac{a+L}{a}\right)$$

Example 14 (a non-AP question):

Derive an expression for the electric potential at an arbitrary point $y = b$ on the y-axis due to a rod with charge Q uniformly distributed over its length L.

SO NICE, no components!



$$V = \int dV$$

$$= \frac{1}{4\pi\varepsilon_o}\int_{x=0}^{L}\frac{dq}{r}$$

$$= \frac{1}{4\pi\varepsilon_o}\int_{x=0}^{L}\frac{(\lambda dx)}{(x^2+b^2)^{1/2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_o}\int_{x=0}^{L}\frac{1}{(x^2+b^2)^{1/2}}dx$$

(whatever that is--what's important is the set-up)

Example 15: A ring situated in the x-z plane (as shown) has *–Q's* worth of charge on it.

a.) What is the direction of the *E-fld* at (x,0)?

From observation, it's $-\hat{i}$.

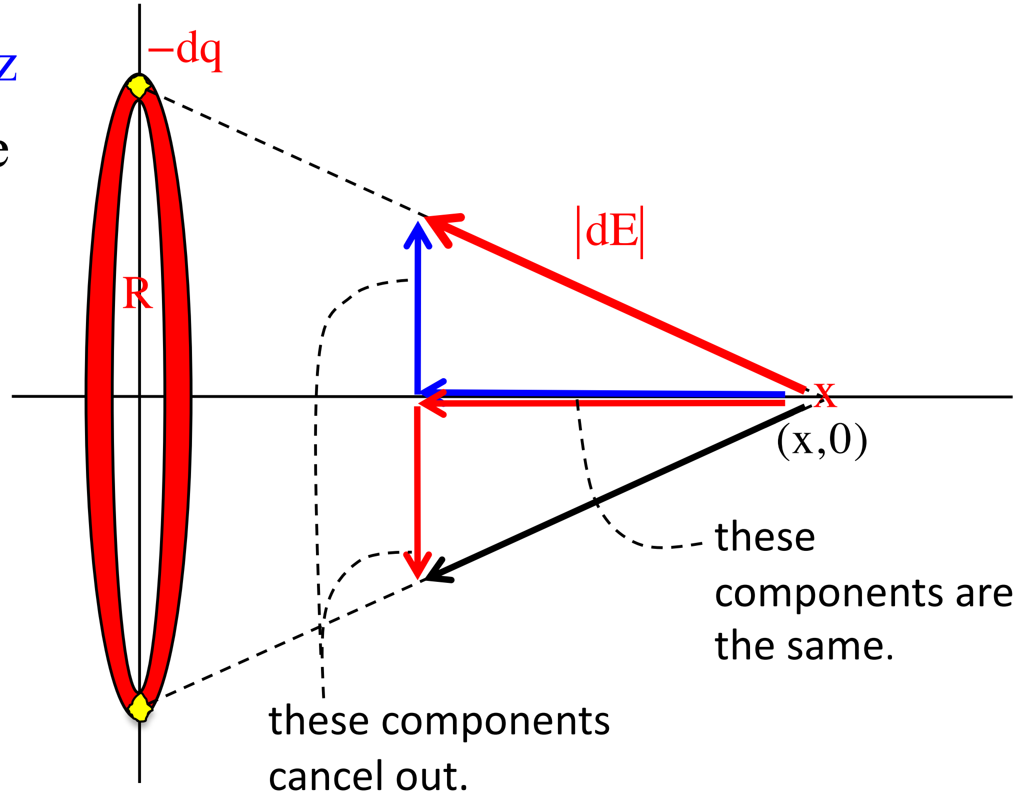
b.) Derive an expression or *V* at (x,0)?

$$V = \int dV$$

$$= \frac{1}{4\pi\varepsilon_o}\int \frac{dq}{\left(x^2+R^2\right)^{1/2}}$$

$$= \frac{1}{4\pi\varepsilon_o\left(x^2+R^2\right)^{1/2}}\int dq$$

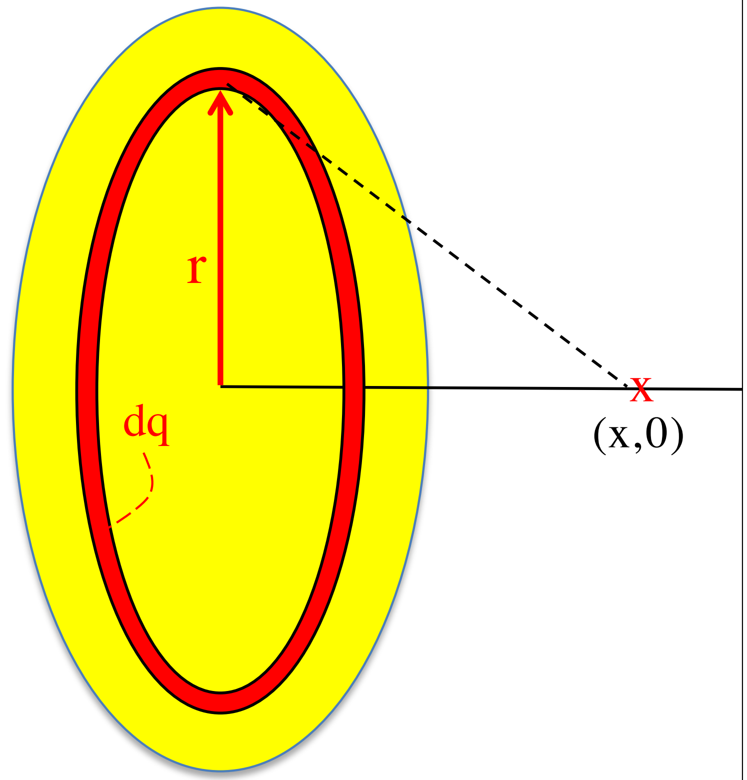
$$= \frac{-Q}{4\pi\varepsilon_o\left(x^2+R^2\right)^{1/2}}$$



c.) Do the results from Parts a and b make sense together?

sure

Example 17 (not): Consider yourself lucky. A lab I had my students do for years (minus the data part) follows: Given a charged copper disk:



a.) *Use* *Gauss's Law* to derive an expression for the *electric field* due to the disk very, very close to the disk's central axis (i.e., at a coordinate (x,0) such that x << R).

b.) *Use* the *differential-charges-dq-approach* to derive an expression for the *electric field* generated by the disk at some point down the x-axis (x,0).

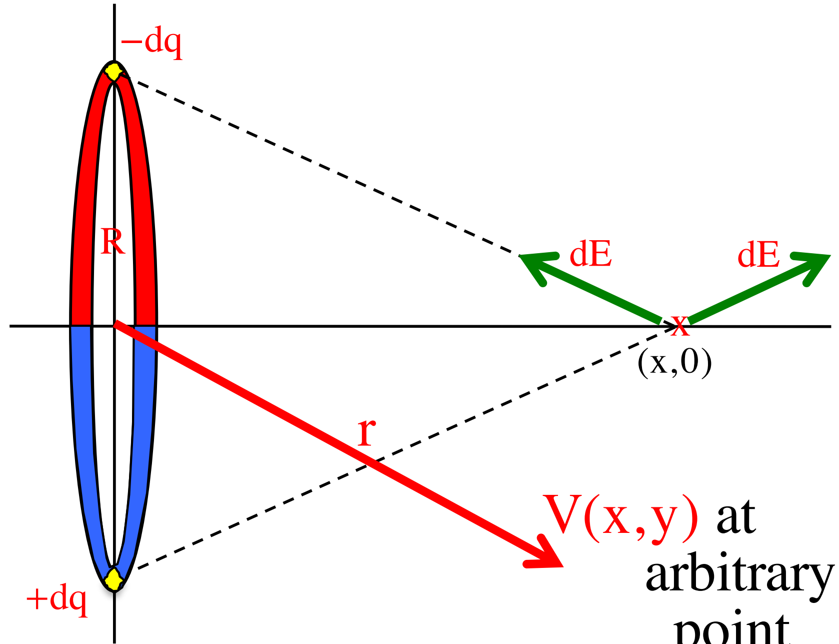
c.) *Using* the relationship derived in *Part b*, assume that x<<R in that relationship and see if that *E* matches up with the *Gauss's Law* expression.

d.) *Using dq , derive an expression* for the *electric potential* at (x,0).

e.) *Though not completely kosher, due to the* vagaries of the symmetry, you could use the *del operator* on the *electric potential function* you derived in *Part d* to derive an expression for the electric field at (x,0). Do so and see if it matches your expression from Part a.

Called THE LAB FROM HELL, students lost a lot of sleep but learned a LOT.

Example 18: In the previous "lab" example identified as *Example 17*, it was stated in *Part e* that "although it is not kosher, you could use the del operator to derive an expression for the electric field at (x,0)." What was up with that?



The question is asking us to use

$$\vec{E} = -\frac{dV}{dx}\hat{i}$$

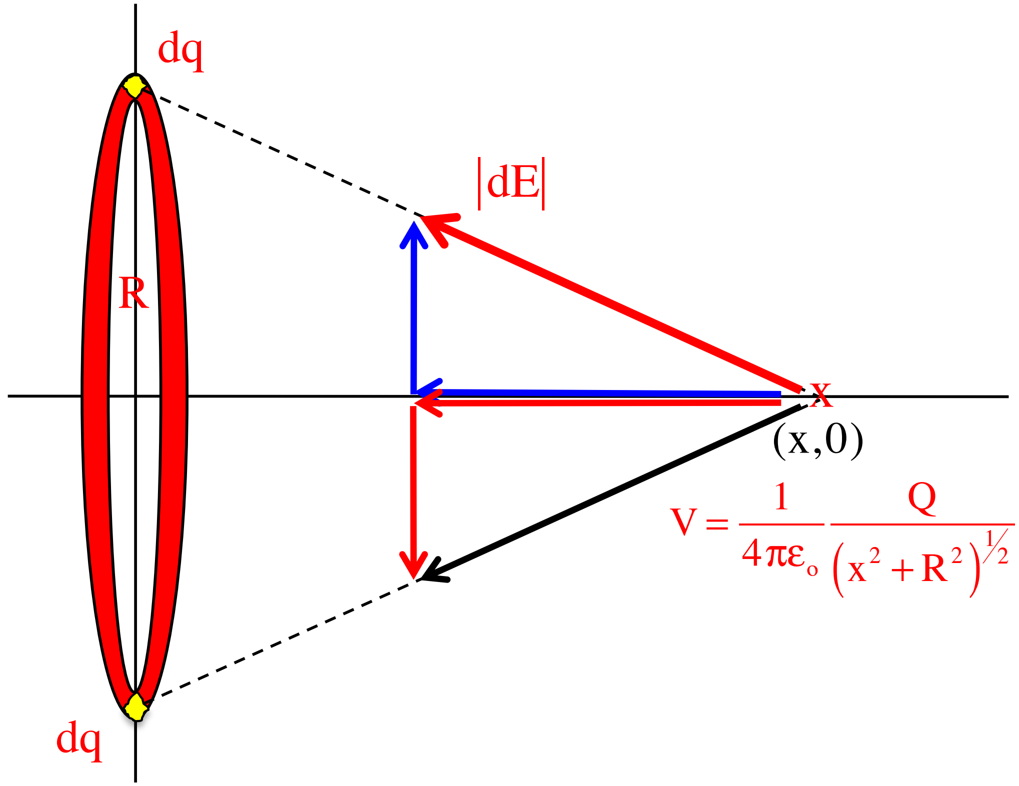
(or the del operator equivalent) to determine *E*. Why is this spooky?

Consider a hoop. What would happen if the upper half of the hoop was *negatively charged* while the lower half had an equal amount of *positive charge*?

There would be a net electric field at (x,0) in the y-direction, but the electric potential *V* at (x,0) would be ZERO. So how could you use $\vec{E} = -\left(dV/dx\right)\hat{i}$?

To make things work, you need a general expression for V(r), an expression for the electric potential *at an arbitrary point*, to use with $\vec{E} = -\vec{\nabla}V$

Due to the symmetry, a single-charge hoop has an *electric field* that is down the *x-axis*, and it has a *V-function* that is a function of *x*. As such, using $\vec{E} = -\frac{dV}{dx}\hat{i}$



yields

$$\vec{E} = -\frac{dV}{dx}\hat{i}$$

$$= \frac{d\left(\frac{1}{4\pi\varepsilon_o}\frac{Q}{(x^2+R^2)^{1/2}}\right)}{dx}\hat{i}$$

$$= \frac{Q}{4\pi\varepsilon_o}\frac{d\left((x^2+R^2)^{-1/2}\right)}{dx}\hat{i} = \frac{Q}{4\pi\varepsilon_o}\left(-\frac{1}{2}\right)(2x)(x^2+R^2)^{-3/2}\hat{i}$$

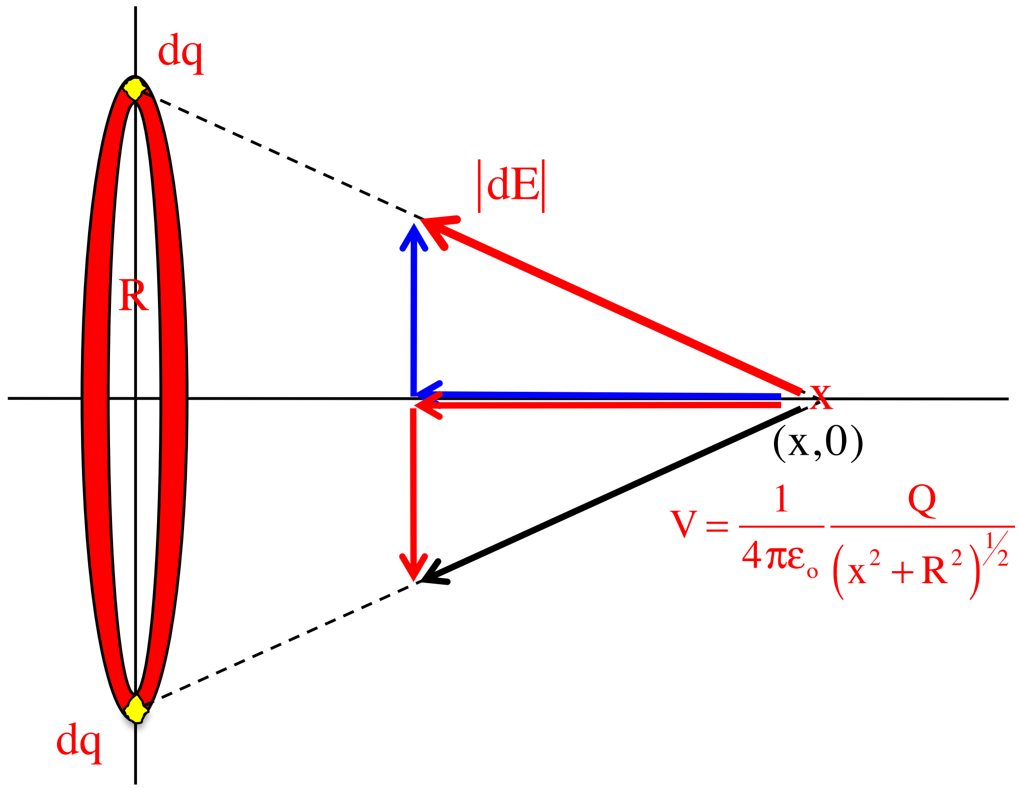
$$= \frac{Qx}{4\pi\varepsilon_o(x^2+R^2)^{3/2}}\hat{i}$$

which matches the derived E-fld expression from before.

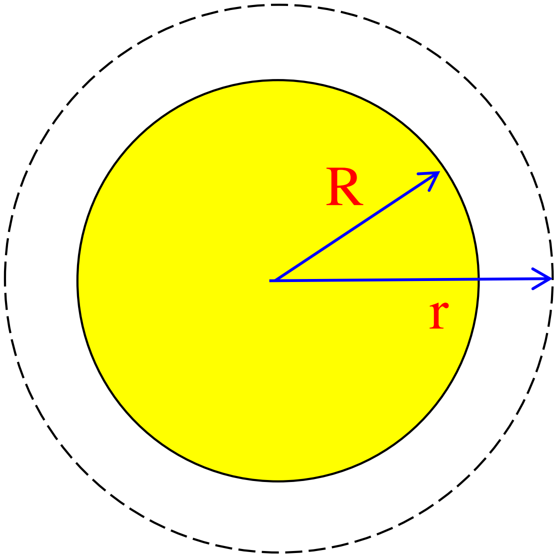
The point here is that you can use

$$\vec{E} = -\frac{dV}{dx}\hat{i}$$

even if we don't have a general expression for *V(r)*. The key is in whether the charge involved is all the same kind (i.e., either all positive or all negative). If that be the case, you can fudge some (not kosher, but it gives you the right answer).



Example 19: A solid, charged, conducting sphere of radius *R* and *Q's* worth of charge on it.



a.) Derive a general algebraic expression for the *electric potential field* for *r > R*.

If we are going to use $V(r)-V(\text{zero pt})=-\int_{\text{zero pt}}^{r}\vec{E}\bullet d\vec{r}$ we need to know the *E-fld* function for *r*>*R*. *Gauss's Law* was born to dispatch that problem. *Starting with* a *Gauss's surface,* we can write:

$$\int_S \vec{E}\bullet d\vec{A}=\frac{q_{\text{cnclosed}}}{\varepsilon_o}$$

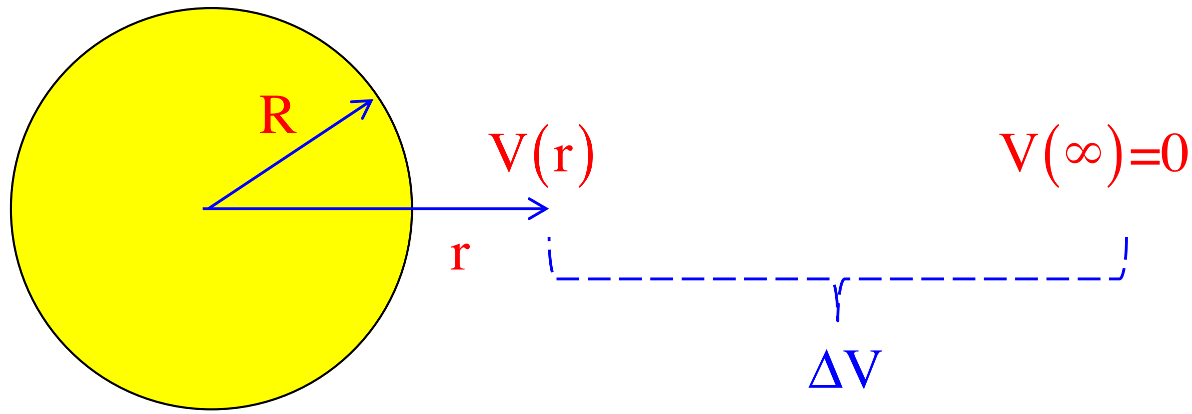
$$\Rightarrow \quad \int_S E\,dA\cos 0^o=\frac{Q}{\varepsilon_o}$$

$$\Rightarrow \quad E\int_S dA=\frac{Q}{\varepsilon_o}$$

$$\Rightarrow \quad E\left(4\pi r^2\right)=\frac{Q}{\varepsilon_o}$$

$$\Rightarrow \quad E=\frac{Q}{4\pi\varepsilon_o r^2}$$

Although it isn't stated, problems like assume the *electric potential* will be zero where the electric field is zero, which in this case is at infinity. So identifying ΔV :



$$V(r) - \cancelto{0}{V(\infty)} = -\int_{\infty}^{r} \vec{E} \bullet d\vec{r}$$

$$= -\int_{\infty}^{r} \left(\frac{Q}{4\pi\varepsilon_o} \frac{1}{r^2} \right) \bullet (dr\ \vec{r})$$

$$= -\int_{\infty}^{r} \left(\frac{Q}{4\pi\varepsilon_o} \frac{1}{r^2} \right) dr \cos 0^o = -\frac{Q}{4\pi\varepsilon_o} \int_{\infty}^{r} \left(\frac{1}{r^2} \right) dr$$

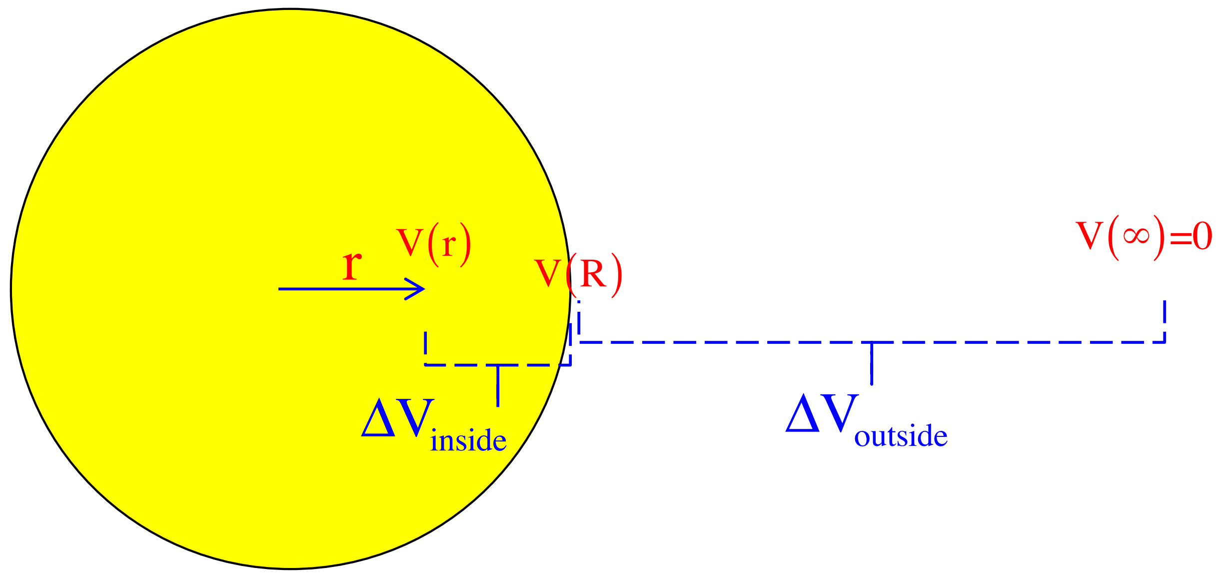
$$= -\frac{Q}{4\pi\varepsilon_o} \left(-\frac{1}{r} \right) \Bigg|_{r=\infty}^{r} = \frac{Q}{4\pi\varepsilon_o} \left(\frac{1}{r} - \cancelto{0}{\frac{1}{\infty}} \right)$$

$$\Rightarrow \quad V(r) = \frac{Q}{4\pi\varepsilon_o r}$$

b.) *Derive a* general algebraic expression for the *electric potential field* for $r < R$.

This is where it gets a bit tricky. Why? Because the electric field *inside the sphere* and *outside the sphere* are different, so we can't use one giant ΔV in $\Delta V = -\int_{\text{zero pt}}^{r} \vec{E} \bullet d\vec{r}$.

Also, because electric potentials AREN'T inverse square functions, you can't ignore the charge interior to the point of interest like you did with Gauss's Law!



How to deal with this? Derive for each discrete region ΔV using each region's known electric field function, then add them all up. That is:

$$\Delta V_{\text{inside}} + \Delta V_{\text{outside}} = \left[V(r) - V(R)\right] + \left[V(R) - \cancelto{0}{V(\infty)}\right]$$
$$= V(r)$$

To execute these changes, we need electric fields (as each $\Delta V = -\int \vec{E} \bullet d\vec{r}$ over its limits), so normally we'd be back to *Gauss's Law*.

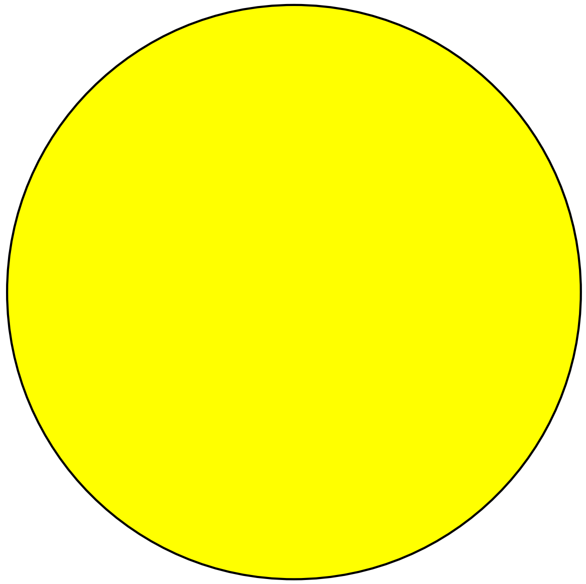
In this problem, though, we have the electric field for the outside region, and because the sphere is a conductor, we know the electric field *inside* is zero, so we can write:

$$
\begin{aligned}
V(r) &= \quad \Delta V_{\text{outside}} \quad + \quad \Delta V_{\text{inside}} \\
&= \left[V(R) - V(\infty)\right] + \left[V(r) - V(R)\right] \\
&= -\int_{\infty}^{R} \vec{E}_{\text{outside}} \bullet d\vec{r} \cancelto{0}{- \int_{R}^{r} \vec{E}_{\text{inside}} \bullet d\vec{r}} \\
&= -\int_{\infty}^{R} \left(\frac{Q}{4\pi\varepsilon_o} \frac{1}{r^2} \hat{r} \right) \bullet \left(dr\ \vec{r} \right) + 0
\end{aligned}
$$

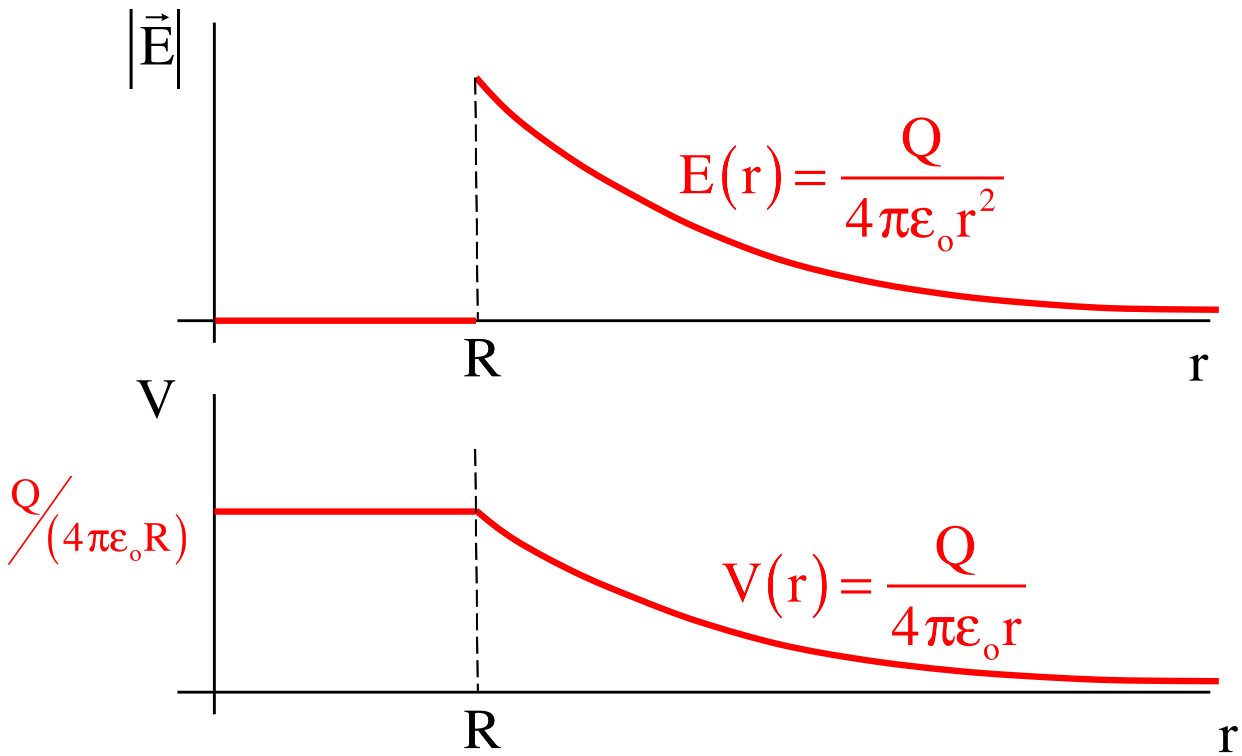
$$
\Rightarrow \quad V(r) = \frac{Q}{4\pi\varepsilon_o R} \quad \text{for } r > R
$$

Note: If the inner sphere had been an insulator, we would have had to use *Gauss's Law* to determine the electric field function in that region, and that second integral would not have been zero. We'll try this in the next problem. In the meantime . . .

c.) Sketch the graph for: *E-fld* vs *position* AND the *electric potential field* vs. *position*.

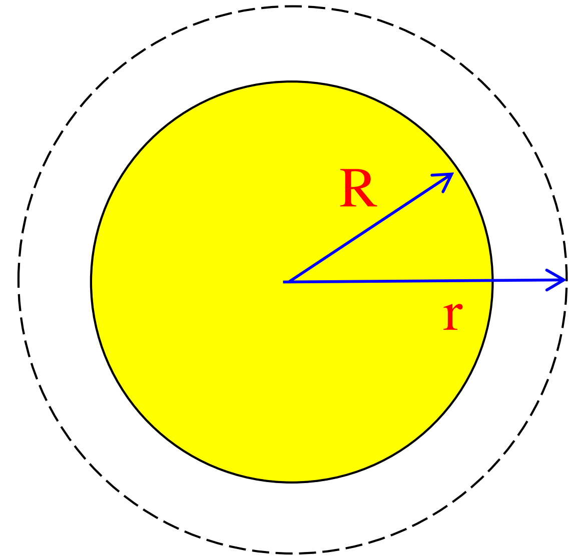


Notice that whereas the *E-fld* functions is *discontinuous*, the *V-fld function* is *CONTINUOUS*!



Example 20: A solid insulating sphere of radius *R* has *Q's* worth of charge shot uniformly throughout.

a.) Derive a general algebraic expression for the *electric potential field* for *r > R*.



As before, Gauss's Law yields:

$$\int_S \vec{E} \bullet d\vec{A} = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{Q}{4\pi\varepsilon_o r^2}$$

and the electric potential function yields:

$$V(r) - V(\infty) = -\int_\infty^r \vec{E} \bullet d\vec{r}$$

$$= -\int_\infty^r \left(\frac{Q}{4\pi\varepsilon_o} \frac{1}{r^2} \right) \bullet (dr \ \vec{r})$$

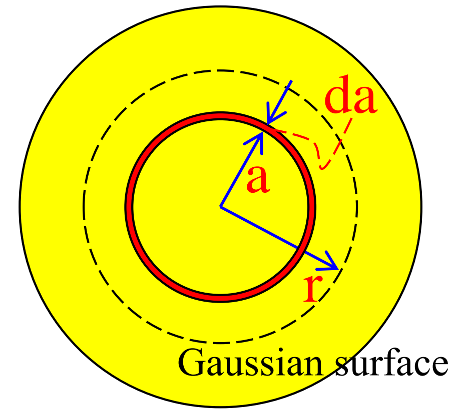
just as before . . .

$$\Rightarrow \quad V(r) = \frac{Q}{4\pi\varepsilon_o r}$$

b.) for r<R:

We know the *E-fld* outside.
For the *E-fld* inside:

$$\int_S \vec{E} \bullet d\vec{A} = \frac{\int_{a=0}^{r} \rho\, dV}{\varepsilon_o}$$



where the *volume charge density* is: $\rho = \dfrac{Q}{\left(\frac{4}{3}\pi R^3\right)}$

and the spherical shell's differential volume $dV = \left(4\pi a^2\right) da$, so:

$$\int_S \vec{E} \bullet d\vec{A} = \frac{\int_{a=0}^{r} \rho\, dV}{\varepsilon_o}$$

$$\Rightarrow \quad E\left(4\pi r^2\right) = \frac{\rho \int_{a=0}^{r} \left[4\pi a^2 da\right]}{\varepsilon_o} = \rho \frac{\left(\frac{4}{3}\pi r^3\right)}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{1}{4\pi r^2}\left(\frac{Q}{\left(\cancel{4/3}\right)\cancel{\pi} R^3}\right)\frac{\left(\cancel{4/3}\,\cancel{\pi} r^3\right)}{\varepsilon_o} \qquad \Rightarrow \quad E = \frac{Q}{4\pi\varepsilon_o R^3} r$$

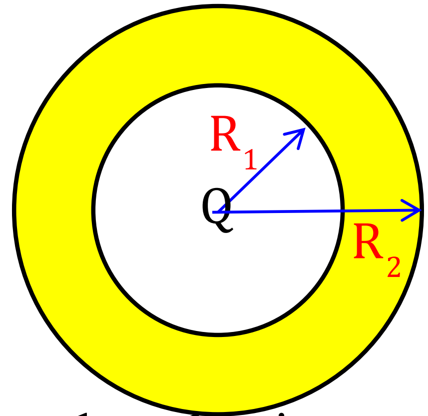
So the electric potential is:

$$
\begin{aligned}
V(r) &= \Delta V_{outside} + \Delta V_{inside} \\
&= \left[V(R) - V(R\infty)\right] + \left[V(r) - V(R)\right] \\
&= -\int_{\infty}^{R} \vec{E}_{outside} \bullet d\vec{r} - \int_{R}^{r} \vec{E}_{inside} \bullet d\vec{r} \\
&= -\int_{\infty}^{R} \left(\frac{Q}{4\pi\varepsilon_o} \frac{1}{r^2}\, \hat{r} \right) \bullet (dr\ \vec{r}) - \int_{r=R}^{r} \left(\frac{Q}{4\pi\varepsilon_o R^3}\, r\ \hat{r} \right) \bullet (dr\ \vec{r}) \\
&= -\frac{Q}{4\pi\varepsilon_o} \int_{\infty}^{R} \left(\frac{1}{r^2}\, \hat{r} \right) \bullet (dr\ \vec{r}) - \frac{Q}{4\pi\varepsilon_o R^3} \int_{r=R}^{r} (r\ \hat{r}) \bullet (dr\ \vec{r})
\end{aligned}
$$

$$
\Rightarrow \quad V(r) = -\frac{Q}{4\pi\varepsilon_o} \left(-\frac{1}{r} \right) \Bigg|_{r=\infty}^{R} - \frac{Q}{4\pi\varepsilon_o R^3} \left(\frac{r^2}{2} \right) \Bigg|_{r=R}^{r}
$$

$$
= \frac{Q}{4\pi\varepsilon_o R} - \frac{Q}{8\pi\varepsilon_o R^3}\left(r^2 - R^2\right) \qquad \text{for } r < R
$$

Example 20: A point charge at the center of a thick, uncharged conducting spherical shell of inside radius R_1 and outside radius R_2. Determine the electrical potential for:



a.) $r > R_2$

The first thing to note is that when you are asked to “determine” a value, that is not the same as “derive” a value. As long as you justify what you do, you can be as clever as you want. In this case, the observation to be made is that outside R_2 (especially from a distance), this charge configuration just looks like a point charge (even with the charge redistributing itself on the conductor). We know the electric potential function for a point charge. It’s

$$V(r) = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r}$$

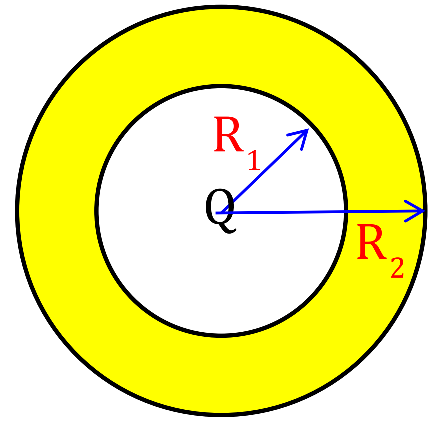
b.) $R_2 > r > R_1$

Again, being clever: You know the electric field inside a conductor is zero, so the electric potential difference between any two points must be zero. You also know that electric potential functions are continuous. Evaluating the field for $r > R_2$ right at R_2, we get the electric potential at the outside edge of the conductor, which must be the electric potential throughout the conductor. That evaluation is:

$$V(r) = \frac{1}{4\pi\varepsilon_o}\frac{Q}{R_2}$$

c.) $r < R_1$

This is where things get tricky.
The temptation is to look at hollow region in which Q resides, think, "That's a point charge with an electric field equal to:



$$\left|\vec{E}(r)\right| = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}$$

And conclude that because the electric field is that of a point charge, the electric potential must be that of a point charge, or

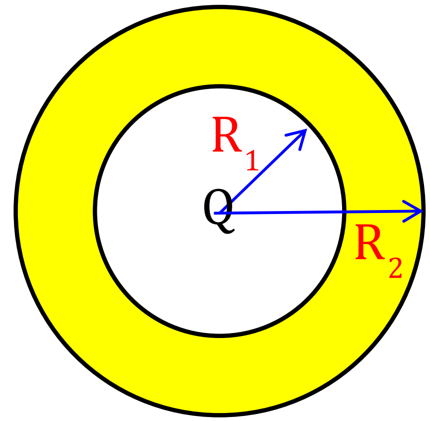
$$V(r) = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r}$$

The problem with this is that this function, evaluated at $r = R_1$ yields $V(r) = \frac{1}{4\pi\varepsilon_o}\frac{Q}{R_1}$

But this is part of the conductor, and we've already concluded that the electric potential has to be $V(r) = \frac{1}{4\pi\varepsilon_o}\frac{Q}{R_2}$ inside the conductor.

So what gives?

We know that for spherical symmetry, Gauss's Law states that an electric field is generated only by the charge *inside* the sphere upon which the point of interest resides. That is because the electric field generated by each individual points charges is an inverse square function in distance (i.e., $\left|\vec{E}(r)\right| = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}$)

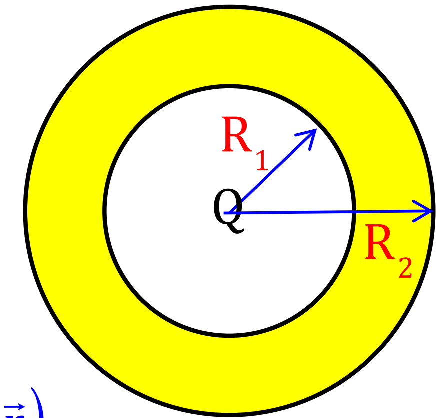


Electric potentials aren't like that. They are simply inverse distance functions , or $V(r) = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r}$

To get the electric potential function for a charge geometry that is spherical in nature, we not only have to consider the charge *inside* the sphere upon which the point of interest resides, we also have to consider the charge *outside* that sphere, also. That means:

$$\begin{aligned} V(r) &= \Delta V_{outside} + \Delta V_{in\_conductor} + \Delta V_{inside} \\ &= \left[V(R_2) - V(R\infty)\right] + \left[V(R_1) - V(R_2)\right] + \left[V(r) - V(R_1)\right] \\ &= -\int_{\infty}^{R_2} \vec{E}_{outside} \bullet d\vec{r} - \cancelto{0}{\int_{R_2}^{R_1} \vec{E}_{in\_conductor} \bullet d\vec{r}} - \int_{R_1}^{r} \vec{E}_{inside} \bullet d\vec{r} \end{aligned}$$

That is:



$$V(r) = \Delta V_{outside} + \Delta V_{in\_conductor} + \Delta V_{inside}$$

$$= \left[V(R_2) - V(R\infty)\right] + \left[V(R_1) - V(R_2)\right] + \left[V(r) - V(R_1)\right]$$

$$= -\int_{\infty}^{R_2} \vec{E}_{outside} \bullet d\vec{r} \cancelto{0}{- \int_{R_2}^{R_1} \vec{E}_{inside} \bullet d\vec{r}} - \int_{R_1}^{r} \vec{E}_{inside} \bullet d\vec{r}$$

$$= -\int_{\infty}^{R_2} \left(\frac{Q}{4\pi\varepsilon_o} \frac{1}{r^2} \hat{r} \right) \bullet (dr\ \hat{r}) - \int_{R_2}^{R_1} (0\hat{r}) \bullet (dr\ \hat{r}) - \int_{R_1}^{r} \left(\frac{Q}{4\pi\varepsilon_o r^2} \hat{r} \right) \bullet (dr\ \hat{r})$$

$$= -\frac{Q}{4\pi\varepsilon_o} \int_{\infty}^{R_2} \left(\frac{1}{r^2} \hat{r} \right) \bullet (dr\ \hat{r}) - \frac{Q}{4\pi\varepsilon_o} \int_{R_1}^{r} \left(\frac{1}{r^2} \hat{r} \right) \bullet (dr\ \hat{r})$$

$$\Rightarrow \quad V(r) = -\frac{Q}{4\pi\varepsilon_o} \left(-\frac{1}{r} \right) \Bigg|_{\infty}^{R_2} - \frac{Q}{4\pi\varepsilon_o} \left(-\frac{1}{r} \right) \Bigg|_{R_1}^{r}$$

$$= \frac{Q}{4\pi\varepsilon_o R_2} + \frac{Q}{4\pi\varepsilon_o} \left(\frac{1}{r} - \frac{1}{R_1} \right)$$

Note that for $r = R_1$, this becomes:

$$V(r = R_1) = \frac{Q}{4\pi\varepsilon_o R_2} + \frac{Q}{4\pi\varepsilon_o} \left(\frac{1}{r} - \frac{1}{R_1} \right) = \frac{Q}{4\pi\varepsilon_o R_2} + \frac{Q}{4\pi\varepsilon_o} \left(\frac{1}{R_1} - \frac{1}{R_1} \right)$$

$$= \frac{Q}{4\pi\varepsilon_o R_2}$$ as expected for a point inside the conductor!

SUMMARY—Conductors . . .

Electric Fields:

a.) Free charge on a conductor in a static setting stays on the conductor's surface.

b.) Close to the surface of a conductor, the *E-fld* is *perpendicular to the surface and* has a magnitude $E = \sigma / \varepsilon_o$.

c.) Inside a conductor, the *E-fld* is zero in a static charge situation (otherwise, electrons would migrate).

Electric Potentials:

a.) Free charge on a conductor will distribute itself so as to create a equipotential surface (the voltage will be the same at every point on the surface)..

b.) As the electric field inside a conductor is zero, the *voltage field* (the electric potential field) inside a conductor will be CONSTANT.

How do you determine the total energy in a systems of particles? (We did a problem like this back in the Gravitation section). The idea is simple:

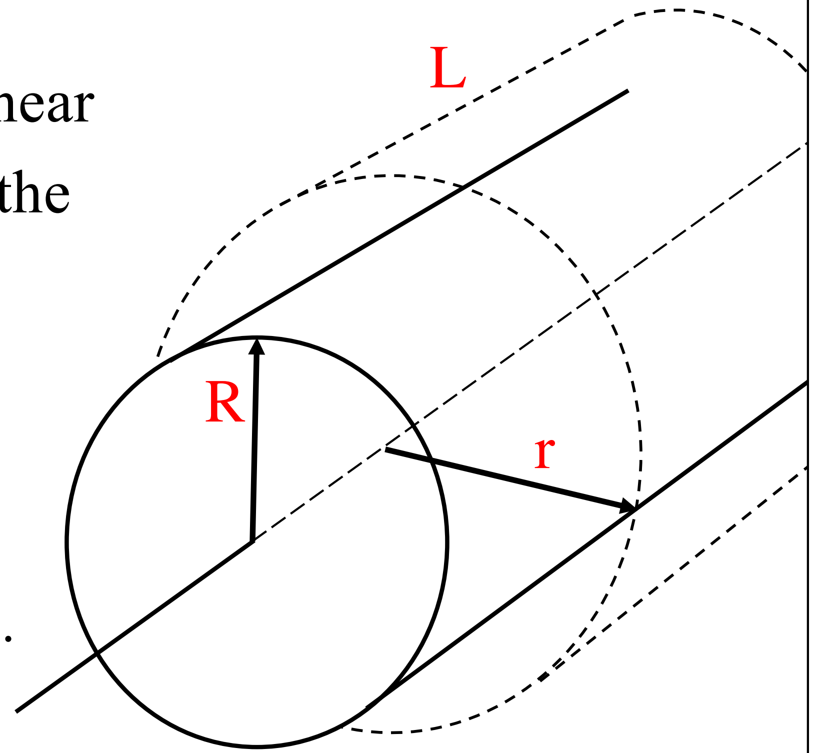
1.) It takes no energy to bring a charge q_1 in from infinity. Once in, though, it will generate an electric potential field whose point-magnitude will (because it is a point-charge) equal to $V_1 = k\,{}^{q_1}\!/_{r}$ with V = 0 at infinity.

2.) Bringing a second charge q_2 in from infinity will require work in the amount of $W = -q_2 \Delta V$. That work will go into the total energy wrapped up in the system. That new charge will produce it's own voltage field with a similar function defining it.

3.) Bringing a third charge q_3 in from infinity will require work as it deals with the fields generated by both q_1 and q_2, the the total amount of work done (and the total energy in the system) will become:

$$W_{total} = W(q_1 \text{ did on } q_2) + W(q_1 \text{ did on } q_3) + W(q_2 \text{ did on } q_3)$$

Example 5: A long wire of radius *R* has a linear charge density λ on it. What can we say about the electric potential function for:



For r>R

1.) We need the electric field function for this region so we can use the $\Delta V = -\int \vec{E} \bullet d\vec{r}$ approach to determine the electric potential outside the wire. Starting with a Gaussian surface whose outside surface area dS will be its circumference $(2\pi r)$ times its length L, and noting that the charge inside

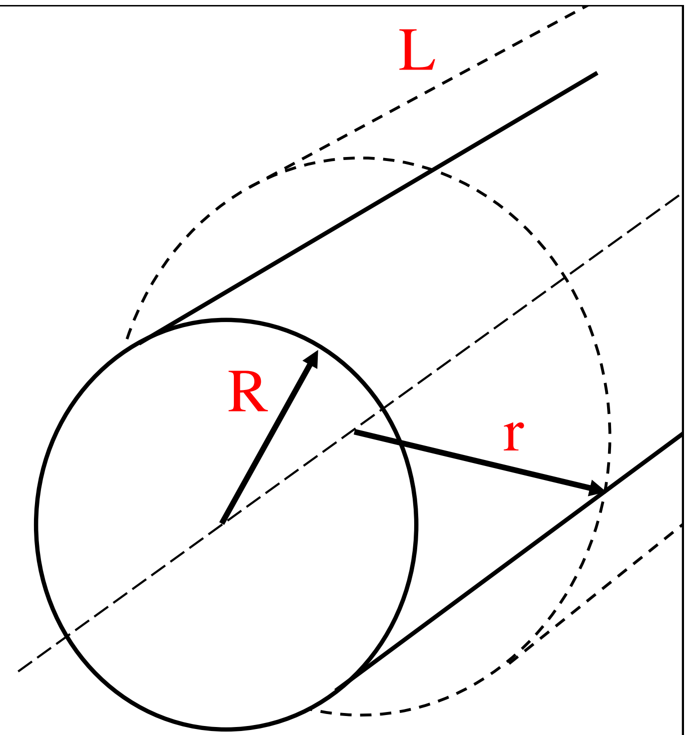
its length L, and noting that the charge inside the Gaussian surface will be equal to λ times the length of the Gaussian surface L, we can write:

$$\int_S \vec{E} \bullet d\vec{S} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \quad \left|\vec{E}\right|(2\pi r L) = \frac{\lambda L}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{\lambda}{2\pi\varepsilon_o r}$$

2.) With the electric field, we run into a problem if we try to assume the electric potential is zero at infinity. Following the math will show why:

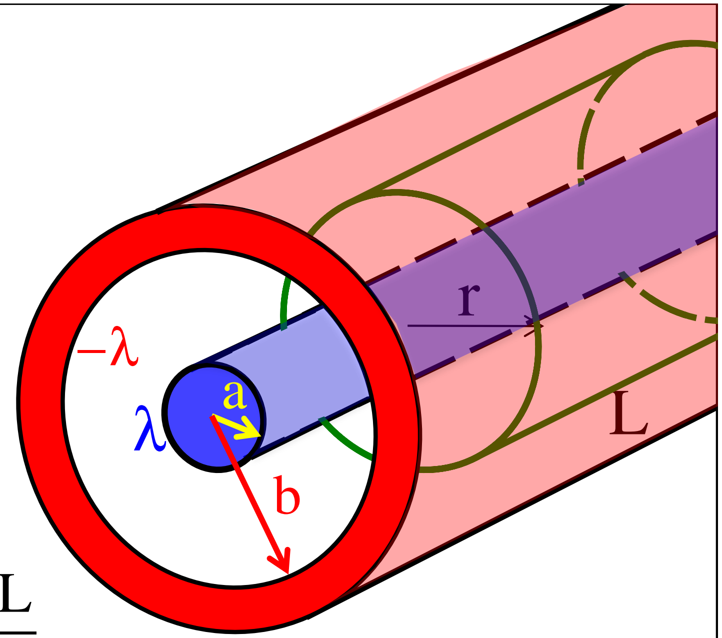


$$V(r)-V(\infty)=-\int_{r=\infty}^{r}\vec{E}\bullet d\vec{r}$$
$$=\int_{r=\infty}^{r}\left(\frac{\lambda}{2\pi\varepsilon_o r}\hat{r}\right)\bullet(dr\ \hat{r})=\frac{\lambda}{2\pi\varepsilon_o}\int_{r=\infty}^{r}\frac{1}{r}dr\cos 0^o$$
$$=\frac{\lambda}{2\pi\varepsilon_o}\ln(r)\Big|_{r=\infty}^{r}=\frac{\lambda}{2\pi\varepsilon_o}\left[\ln(r)-\ln(\infty)\right]=\frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{r}{\infty}\right)$$ yikes . . .

Bottom line: Although you could be asked to use Gauss's Law to determine the electric field generated by an insulator or conductor whose geometry is cylindrical, you will not be asked to determine the electric potential relative to infinity.

Having said that, you COULD be asked to determine the electric potential DIFFERENCE between two points in the region around or inside a cylindrically symmetric charge configuration. That is:

Let's say you have a coaxial cable in which the inside wire has a linear charge density of λ on it while the outside sheath has a charge density of $-\lambda$ on it. What is the voltage difference between the two cables?



1.) We need the E-fld between the structures. Gauss's Law to the rescue:

$$\int_S \vec{E}\bullet d\vec{S} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \quad |\vec{E}|(2\pi rL) = \frac{\lambda L}{\varepsilon_o}$$

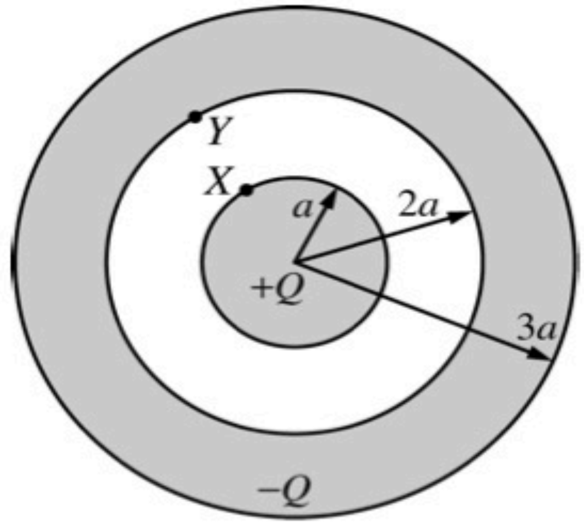
$$\Rightarrow \quad E = \frac{\lambda}{2\pi\varepsilon_o r}$$

2.) With the E-fld, and assuming we are traversing from the inside-out (i.e., with the electric field), we can write:

$$\begin{aligned}\Delta V &= -\int_{r=a}^{b} \vec{E}\bullet d\vec{r} \\ &= -\int_{r=a}^{b}\left(\frac{\lambda}{2\pi\varepsilon_o r}\hat{r}\right)\bullet(dr\ \hat{r}) = -\frac{\lambda}{2\pi\varepsilon_o}\int_{r=a}^{b}\frac{1}{r}dr\,\cancelto{1}{\cos 0^o} \\ &= -\frac{\lambda}{2\pi\varepsilon_o}\ln(r)\Big|_{r=a}^{b} = -\frac{\lambda}{2\pi\varepsilon_o}\left[\ln(b)-\ln(a)\right] = -\frac{\lambda}{2\pi\varepsilon_o}\ln\left(\frac{b}{a}\right)\end{aligned}$$

. . . Which makes sense as electric potential drops as you go from positive to negative plates.

Review slides/your notes (courtesy of Mr. White)



E&M 2.

In the figure above, a nonconducting solid sphere of radius a with charge $+Q$ uniformly distributed throughout its volume is concentric with a nonconducting spherical shell of inner radius $2a$ and outer radius $3a$ that has a charge $-Q$ uniformly distributed throughout its volume. Express all answers in terms of the given quantities and fundamental constants.

(a) Using Gauss's law, derive expressions for the magnitude of the electric field as a function of radius r in the following regions.

i. Within the solid sphere ($r < a$)

ii. Between the solid sphere and the spherical shell ($a < r < 2a$)

iii. Within the spherical shell ($2a < r < 3a$)

iv. Outside the spherical shell ($r > 3a$)

(b) What is the electric potential at the outer surface of the spherical shell ($r = 3a$)? Explain your reasoning.

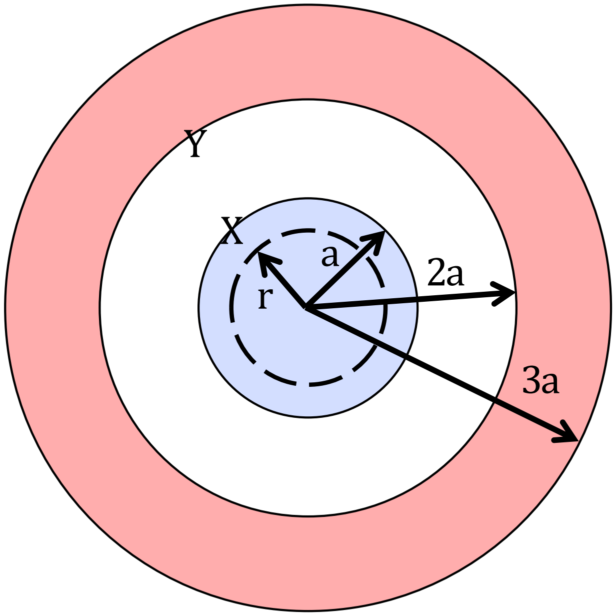
(c) Derive an expression for the electric potential difference $V_X - V_Y$ between points X and Y shown in the figure.

a.) Note that the positive charge is in blue and the negative charge is in redish:

i.) $r < a$

The charge inside the Gaussian surface is the fraction of charge inside r, or:

$$q_{encl} = \left(\frac{\left(\frac{4}{3}\right)\pi r^3}{\left(\frac{4}{3}\right)\pi a^3} \right) Q = \left(\frac{r^3}{a^3} \right) Q$$



For a Gaussian surface inside a:

$$\int_S \vec{E} \bullet d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \int_S E\,dA\cos 0^o = \frac{(\text{fraction of q inside Gaussian surface})}{\varepsilon_o}$$

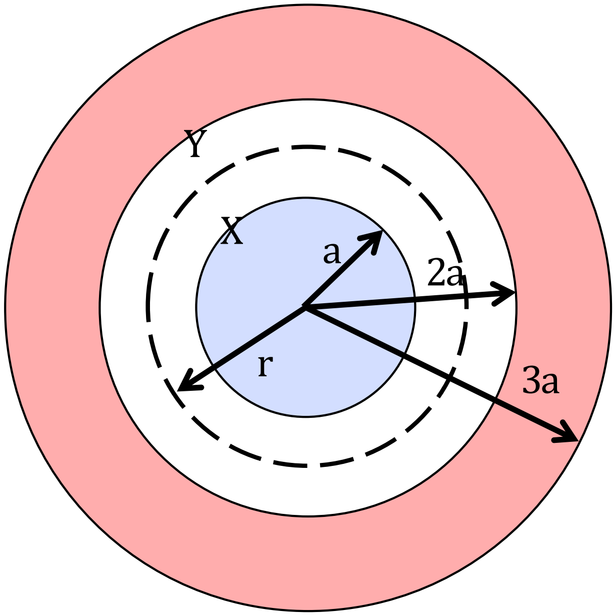
$$\Rightarrow E\int_S dA = \frac{\left(\frac{r^3}{a^3}\right)Q}{\varepsilon_o}$$

$$\Rightarrow E\left(4\pi r^2\right) = \frac{r^3 Q}{a^3 \varepsilon_o}$$

$$\Rightarrow E = \frac{Q}{4\pi\varepsilon_o a^3} r$$

a.) Use Gauss's Law to derive the electric field for:

ii.) $a < r < 2a$



For a Gaussian surface between *a* and *2a*:

$$\int_S \vec{E} \bullet d\vec{A} = \frac{q_{cnclosed}}{\varepsilon_o}$$

$$\Rightarrow \int_S E\,dA\cos 0^o = \frac{Q}{\varepsilon_o}$$

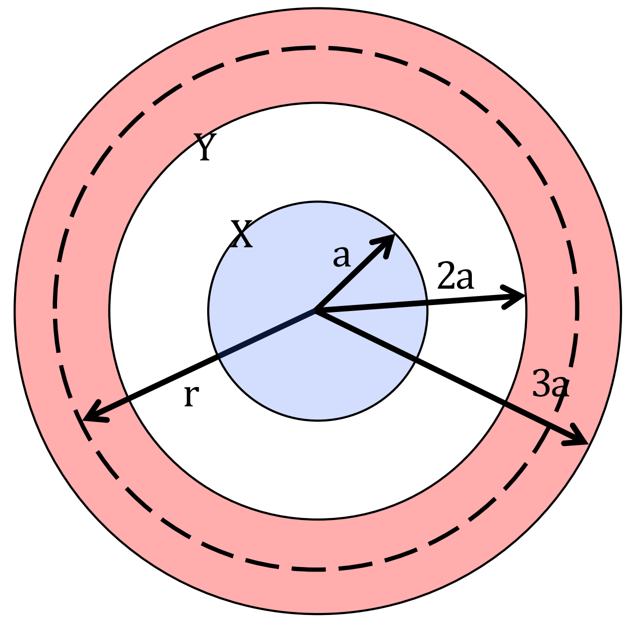
$$\Rightarrow E\int_S dA = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow E\left(4\pi r^2\right) = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow E = \frac{Q}{4\pi\varepsilon_o r^2}$$

a.) Use Gauss's Law to derive the electric field for:

iii.) $2a < r < 3a$



For a Gaussian surface between $2a$ and $3a$:

We need the volume density function (this is the easiest way to go):

$$\rho = \frac{-Q}{V} = \frac{-Q}{\left(\frac{4}{3}\pi R_3^{\ 3}\right) - \left(\frac{4}{3}\pi R_2^{\ 3}\right)}$$

$$= \frac{-3Q}{4\pi}\frac{1}{\left(R_3^{\ 3} - R_2^{\ 3}\right)} = \frac{-3Q}{4\pi}\frac{1}{\left((3a)^3 - (2a)^3\right)}$$

$$= \frac{-3Q}{4\pi}\frac{1}{19a^3}$$

With the volume of a differential spherical shell of radius c and thickness dc inside the outer insulator being:

$$dV = (\text{surface area})(\text{thickness})$$
$$= \left(4\pi c^2\right)dc$$

With the density function:

$$\int_S \vec{E}\bullet d\vec{A} = \frac{q_{cnclosed}}{\varepsilon_o}$$

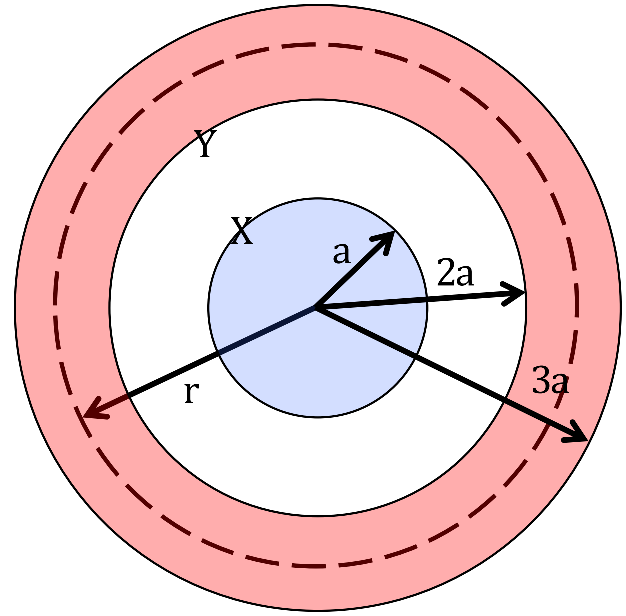
$$\Rightarrow \quad E\left(4\pi r^2\right) = \frac{Q + \int_{c=R_2}^{r} \rho dV}{\varepsilon_o}$$

$$\Rightarrow \quad E\left(4\pi r^2\right) = \frac{Q + \rho\int_{c=R_2}^{r} \rho dV}{\varepsilon_o}$$

$$\Rightarrow \quad E\left(4\pi r^2\right) = \frac{Q + \rho\int_{c=R_2}^{r} \left(4\pi c^2\ dc\right)}{\varepsilon_o}$$

$$\Rightarrow \quad E\left(4\pi r^2\right) = \frac{Q + \rho\left(4\pi\right)\left(\left.\frac{c^3}{3}\right|_{c=R_2}^{r}\right)}{\varepsilon_o} = \frac{Q + \frac{-3Q}{4\pi}\frac{1}{19a^3}\left(4\pi\right)\left(\frac{r^3}{3} - \frac{{R_2}^3}{3}\right)}{\varepsilon_o}$$

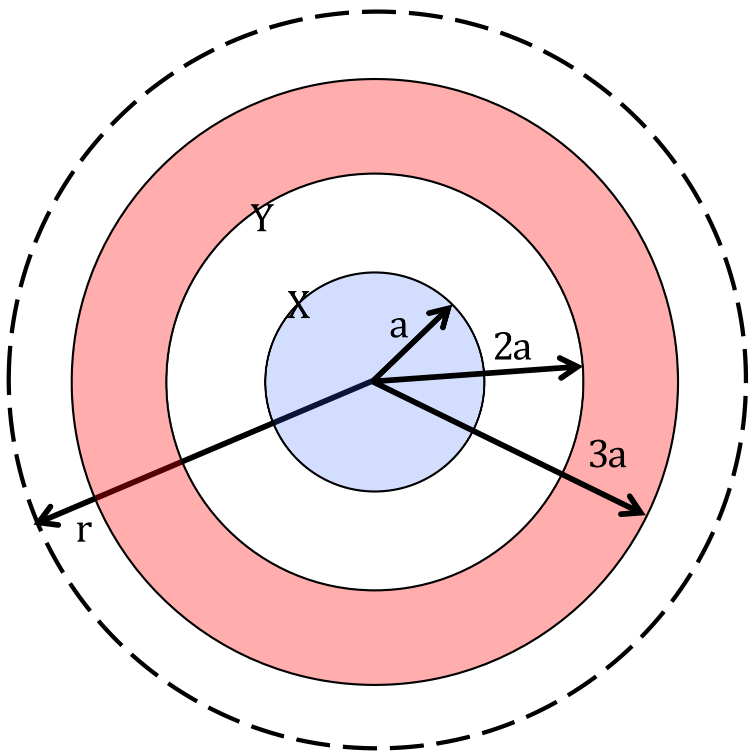
$$\Rightarrow \quad E = \frac{Q}{4\pi\varepsilon_o r^2}\left(1 - \frac{1}{19a^3}\left(r^3 - \left(2a\right)^3\right)\right)$$



Note: As a check, if we went all the way out to $r = 3a$, the E-field should be zero as there would be NO net charge enclosed in the Gaussian surface at that radius (+Q on the inside sphere and –Q on the outside sphere). Trying it, we get:

$$E = \frac{Q}{4\pi\varepsilon_o r^2}\left(1 - \frac{1}{19a^3}\left((3a)^3 - (2a)^3\right)\right)$$

$$= \frac{Q}{4\pi\varepsilon_o r^2}\left(1 - \frac{1}{19a^3}\left(19a^3\right)\right) = 0 \quad \text{(yippee)}$$

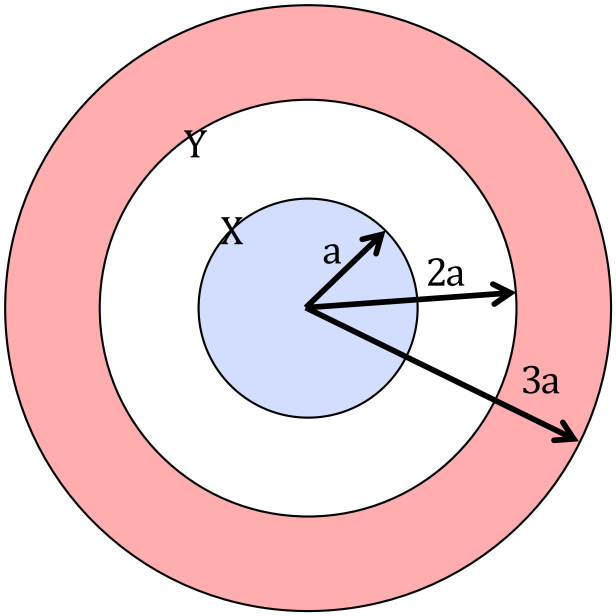


a. continued) Use Gauss's Law to derive the electric field for:

iv.) $r > 3a$

For a Gaussian surface outside of *3a*, the enclosed charge will be zero and E = 0.

b.) What is the electric potential at the outside edge of the outer shell (i.e., at $r = 3a$)? Explain your response.

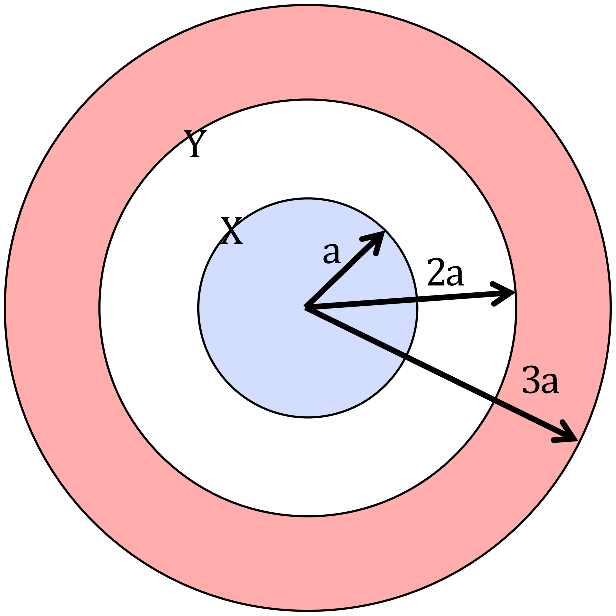


The electric potential at the outer edge is ZERO. This bit of trickery is based on the fact that the electric field is zero in the outer region, which means ΔV between any two points in the region must be zero. If the electric potential is zero at infinity, it must be zero at the sphere's edge.

c.) Derive an expression for the electric potential difference between points *X* and *Y* as shown in the sketch.

To do this, we need the electric potential function for the volume between *a* and *2a*.

This is tricky. Because we've already used Gauss's Law to determine the electric fields in the system, the temptation might be to use the extended approach to determine electrical potential differences from infinity (assuming that V = 0 at infinity) by setting up the integrals as shown below.



$$V(r) = \Delta V_{outside} + \Delta V_{R_3 \text{ to } R_2} + \Delta V_{Y \text{ to } X}$$

$$= \left[V(3a) - V(\infty)\right] + \left[V(2a) - V(3a)\right] + \left[V(r) - V(2a)\right]$$

$$= -\int_{r=\infty}^{3a} \cancelto{0}{\vec{E}_{outside}} \bullet d\vec{r} - \int_{r=3a}^{2a} \vec{E}_{R_3 \text{ to } R_1} \bullet d\vec{r} - \int_{r=2a}^{r} \vec{E}_{Y \text{ to } X} \bullet d\vec{r}$$

$$= -\int_{r=\infty}^{3a} (0\ \hat{r}) \bullet (dr\ \vec{r}) - \int_{r=3a}^{2a} \left(\frac{Q}{4\pi\varepsilon_o r^2} - \frac{Q}{4\pi\varepsilon_o r^2} \left(\frac{1}{19a^3} \right) \left(r^3 - (2a)^3 \right) \right) \bullet (dr\ \vec{r}) - \int_{r=2a}^{r} \left(\frac{Q}{4\pi\varepsilon_o r^2} \hat{r} \right) \bullet (dr\ \vec{r})$$

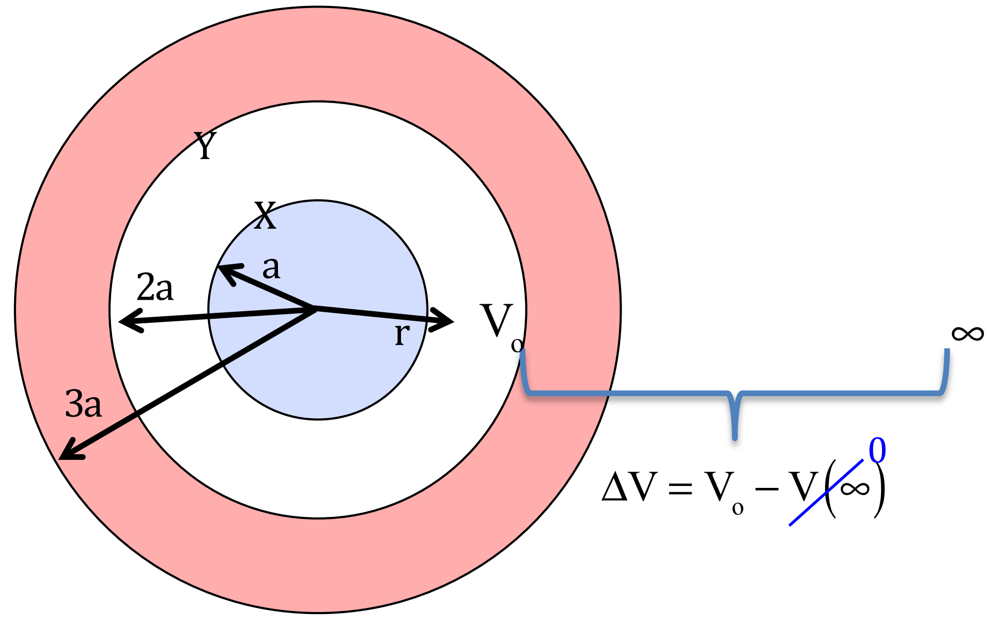
etc.

This horror clearly can't be the expected approach as this is an AP problems, and doing all of this would take an eternity, relatively speaking.

So how to proceed?

With the voltage at infinity zero, let's define V_o as the voltage on the inside surface of the outer shell due to the charge shot through the outside shell. The inside sphere will look like a point charge from the outside, so the net voltage at "r," where "r" is between "a" and "2a," will be the superposition of the voltage due to a point charge and V_o. That is

$$V(r) = V_o + k\frac{Q}{r}$$

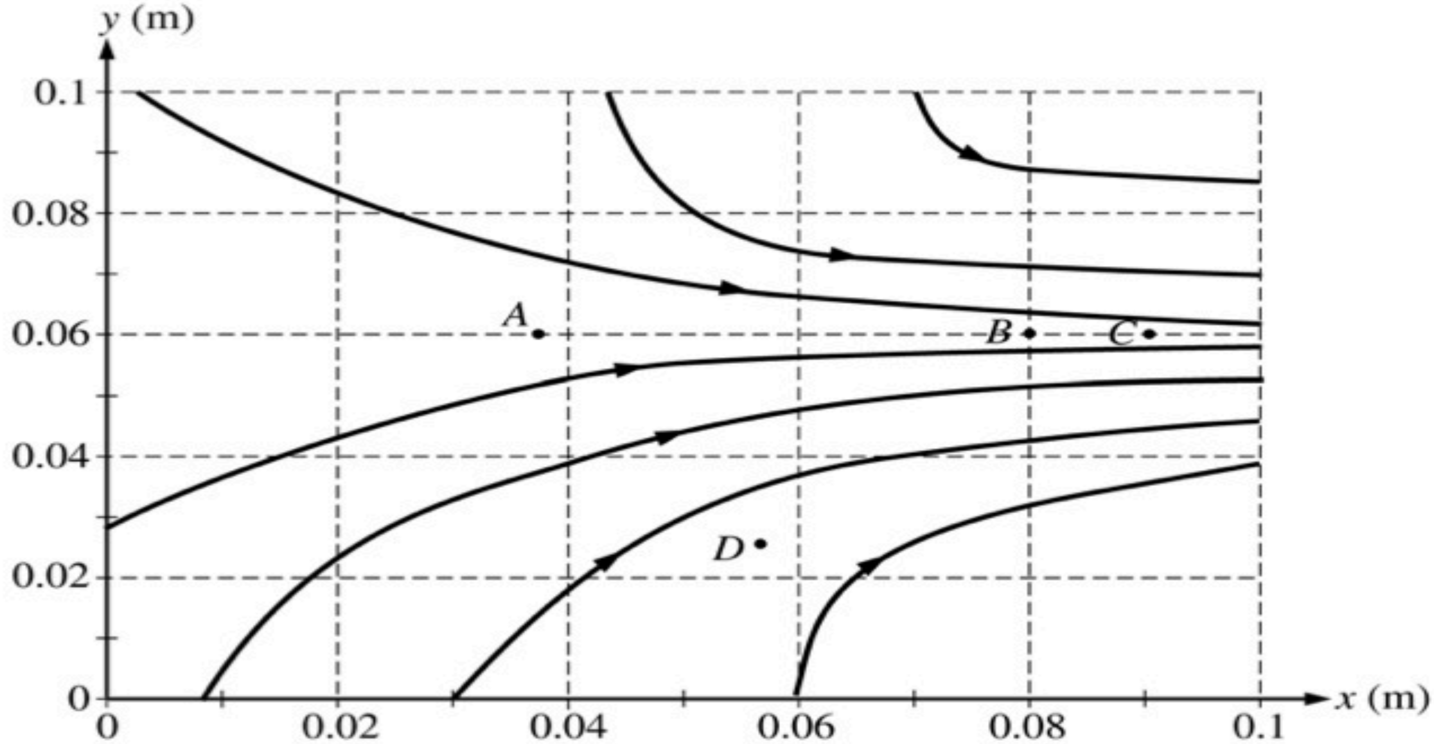


Evaluating this relationship at X (where r = a) and Y (where r = 2a), then subtracting the two to get the voltage difference between the two points, yields:

$$V_{X-Y} = \left(V_o + k\frac{Q}{a}\right) - \left(V_o + k\frac{Q}{2a}\right)$$

$$= k\frac{Q}{a} - k\frac{Q}{2a} = \left(\frac{1}{4\pi\varepsilon_o}\right)\left(\frac{1}{a} - \frac{1}{2a}\right)$$

$$= \left(\frac{1}{4\pi\varepsilon_o}\right)\left(\frac{1}{2a}\right) = \frac{1}{8\pi\varepsilon_o a}$$



E&M. 1.

Consider the electric field diagram above.

(a) Points *A*, *B*, and *C* are all located at $y = 0.06$ m .

i. At which of these three points is the magnitude of the electric field the greatest? Justify your answer.

ii. At which of these three points is the electric potential the greatest? Justify your answer.

(b) An electron is released from rest at point *B*.

i. Qualitatively describe the electron's motion in terms of direction, speed, and acceleration.

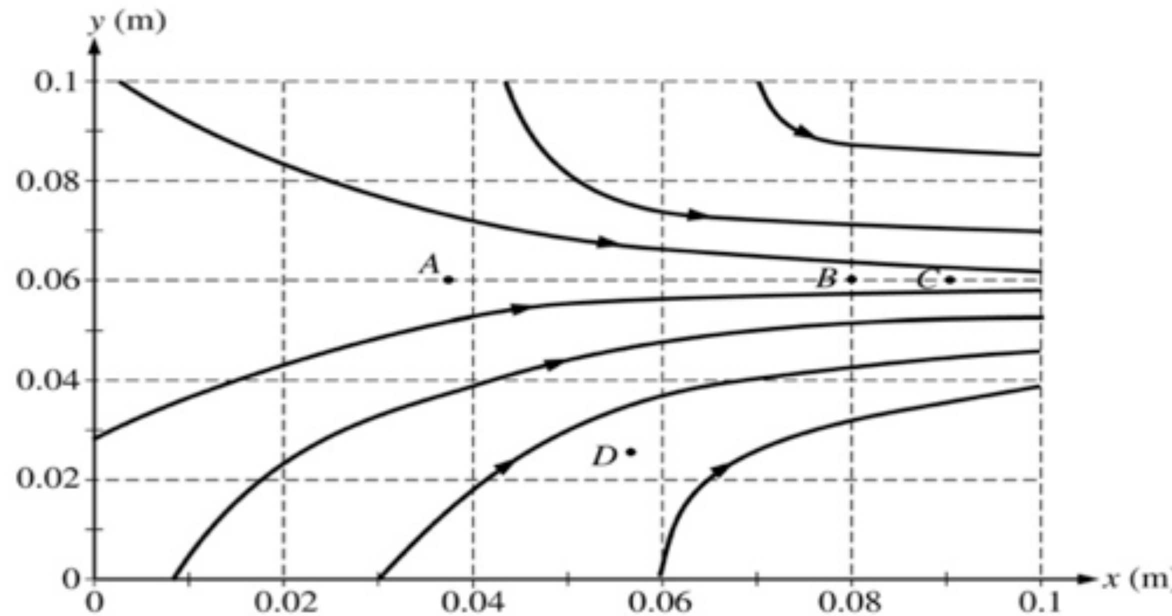
ii. Calculate the electron's speed after it has moved through a potential difference of 10 V.

(c) Points *B* and *C* are separated by a potential difference of 20 V. Estimate the magnitude of the electric field midway between them and state any assumptions that you make.

(d) On the diagram, draw an equipotential line that passes through point *D* and intersects at least three electric field lines.

60.)

E&M 1: Consider the E-fld lines:



a-i.) Between A, B and C, where is the E-fld the greatest?

The E-fld is greatest at Point C as the field lines are the closest together at that point.

a-ii.) Between A, B and C, where is the electric potential the greatest?

The electric potential is greatest at Point A as electric potentials are always greatest upstream in E-flds (positive charges move along E-fld lines, and E-fld lines move from higher to lower voltages)

b-i.) Describe the motion of an electron released at Point B.

Electrons accelerate opposite the direction of E-flds, so an electron will accelerate toward the left. The acceleration will be a function of the magnitude of the field. From observation, the field lines get farther apart as one moves toward the left, so the E-fld gets smaller in that direction. That means the electric force and, hence, acceleration, will diminish as the electron proceeds, though the velocity will continually increase (just not at a steady pace).

b-ii.) The electron's speed after accelerating through a 10 volt potential difference?

Assuming the electron accelerates, for the sake of argument, from 2 volts to 12 volts, conservation of energy yields:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

$$0 \quad + \left(-eV_2\right) + \quad 0 \quad = \frac{1}{2}mv^2 + \left(-eV_{12}\right)$$

$$\Rightarrow \quad v = \left(\frac{2e}{m}\left(V_{12} - V_2\right)\right)^{1/2}$$

$$= \left(\frac{2\left(1.6\text{x}10^{-19}\text{C}\right)}{\left(9.1\text{x}10^{-31}\text{kg}\right)}\left(12\text{V} - 2\text{V}\right)\right)^{1/2}$$

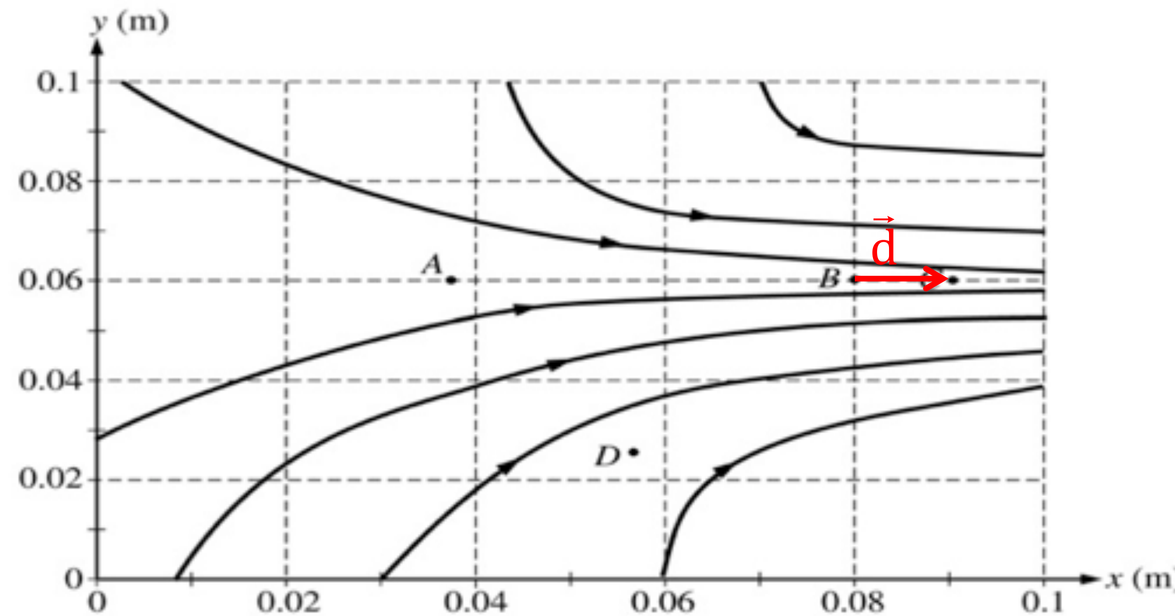
$$= 1.88\text{x}10^{6}\,\text{m/s}$$

c.) If the voltage between B and C is 20 volts, estimate the magnitude of the E-fld halfway between them. State assumptions made:

Assuming the electric field is constant, or nearly constant in that region, assume (for the sake of the math) that $V_B = 30V$ and $V_C = 10V$ (thereby generating the 20 volt potential difference between the two), and noticing from the graph that the distance between the two points is approximately .01 m, we can write:

$$\vec{E} \bullet \vec{d} = -\Delta V$$

$$\Rightarrow \quad Ed\cos\theta = -\left(V_{final} - V_{initial}\right)$$



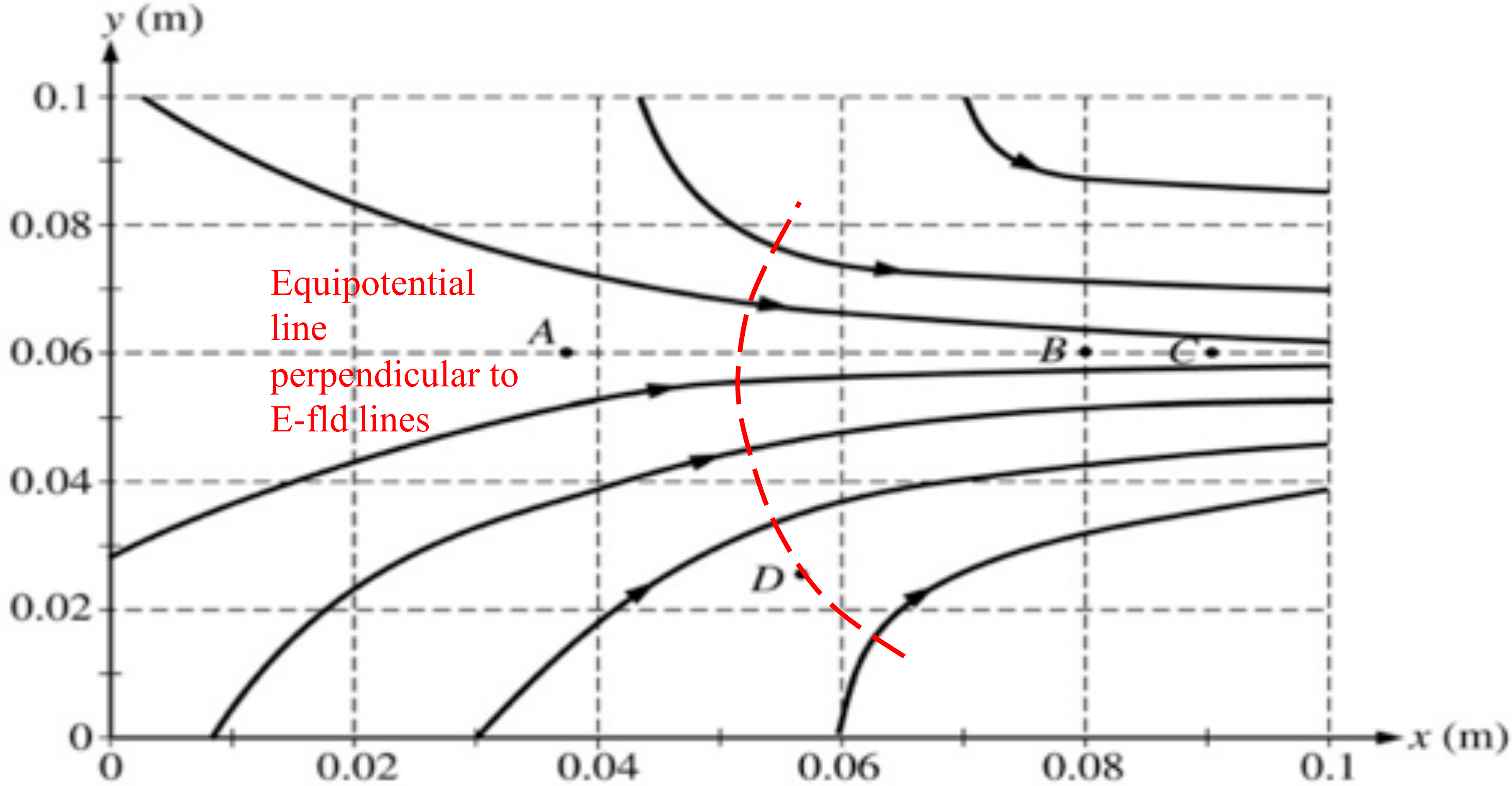
If we take $\vec{d}$ to be a vector between Points B and C in the direction of the E-fld, then the angle between $\vec{E}$ and $\vec{d}$ will be zero and that relationship becomes:

$$E \quad d \quad \cos 0^o = -\left(V_C \; - \; V_B \right)$$

$$E(.01m)\,(1) \quad = -\left(10V - 30V\right)$$

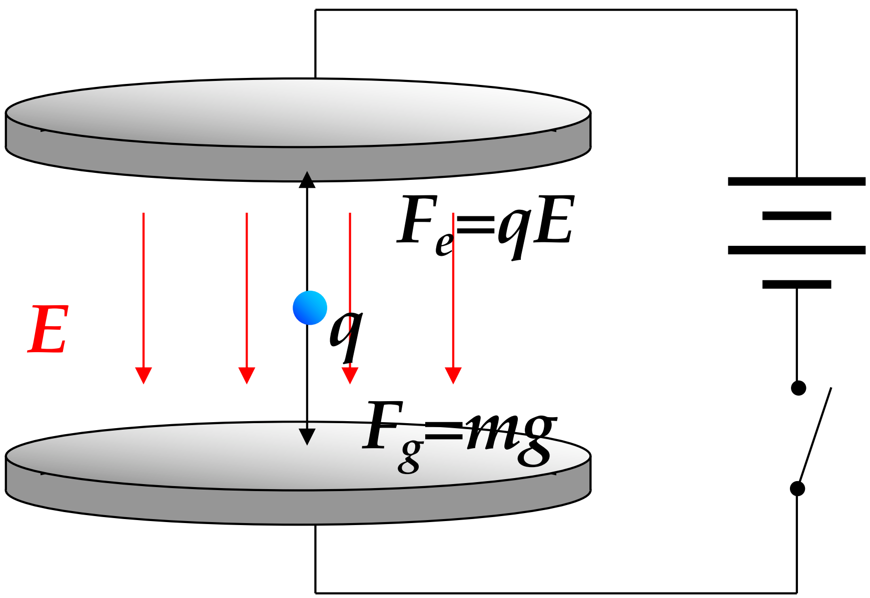
$$\Rightarrow \quad E = 2000\ V/m$$

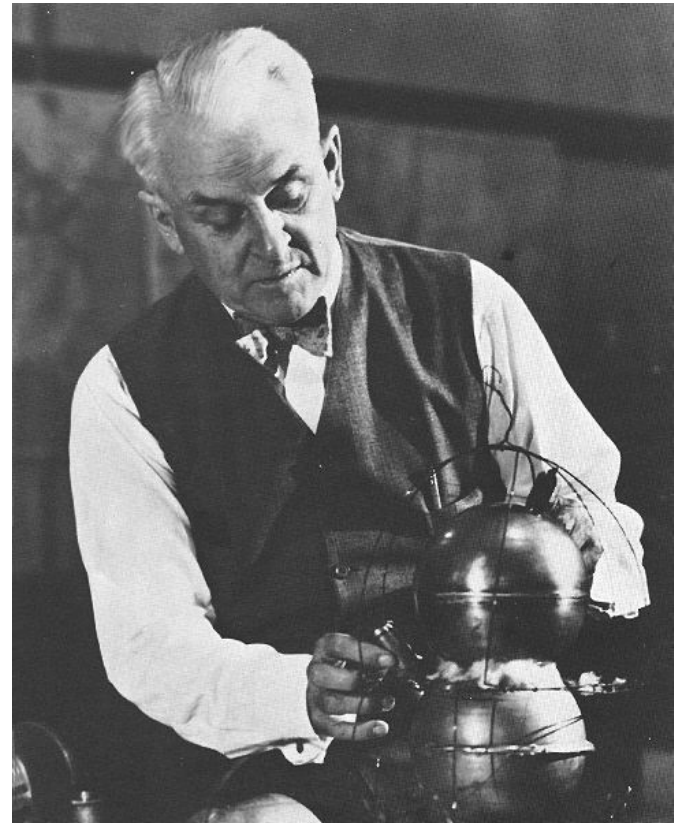
d.) Equipotential line through Point D:



Millikan's Oil Drop Experiments (courtesy of Mr. White)

- Conducted from 1909-1913
- Determined the magnitude of electron charge
- Earned Millikan the Nobel Prize in 1923

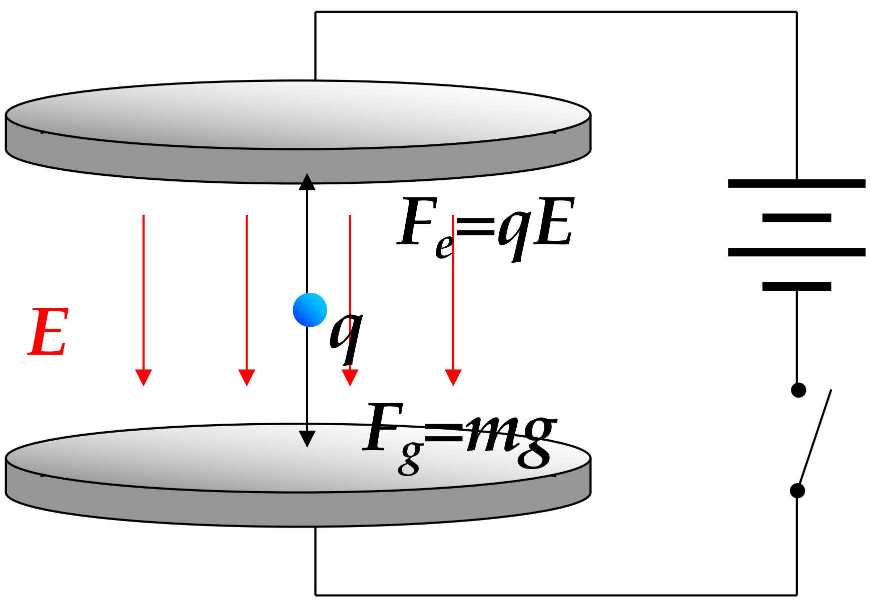




Millikan's Oil Drop Experiments

(courtesy of Mr. White)

Assume charged sphere with mass m, charge q. It should be possible to set up a specific electric field E, based on a specific potential V, that will cause mass to levitate.

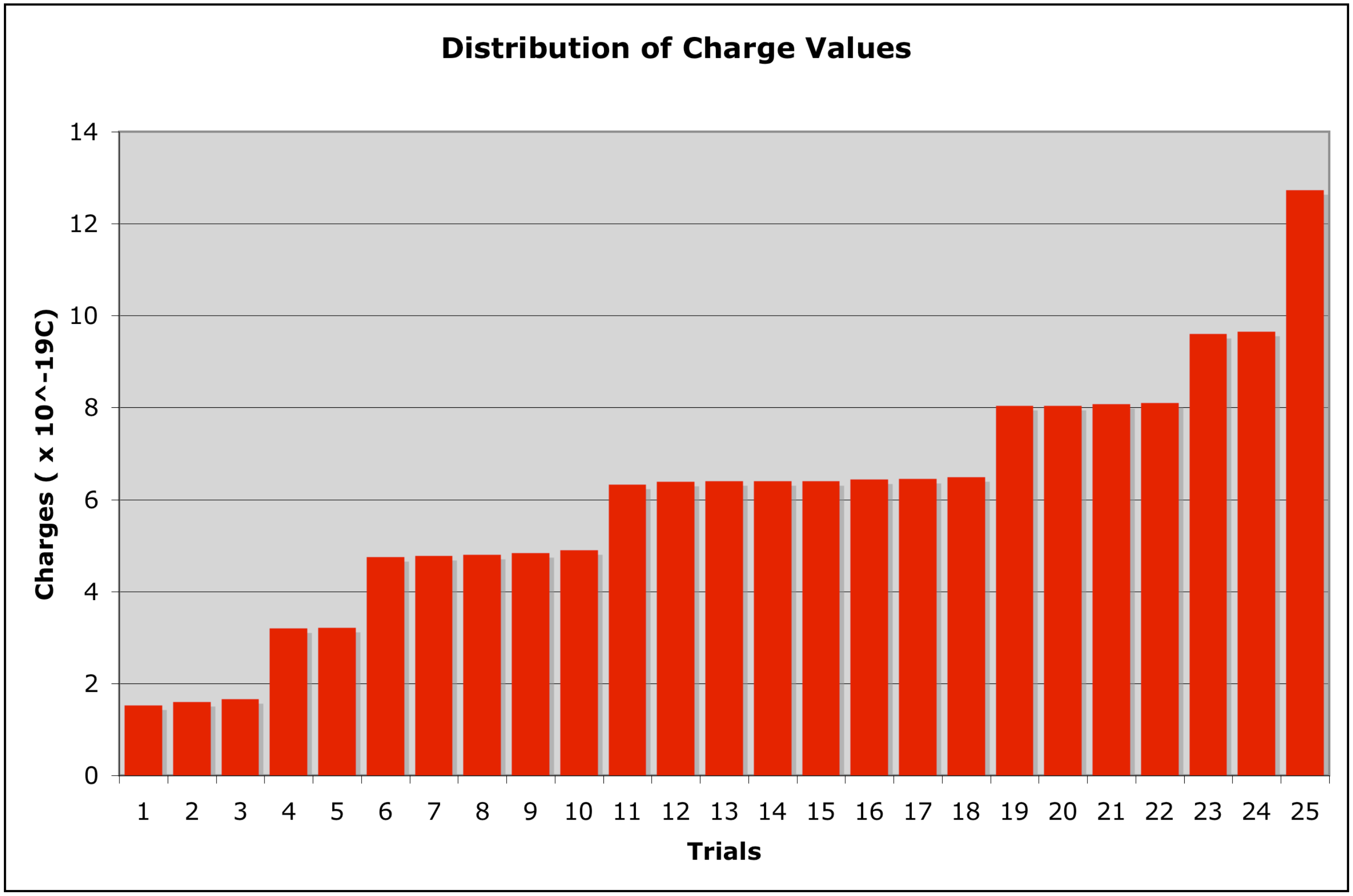


charge (x10^-19 C)
8.04204
4.90212
6.408
6.3279
1.602
12.7359
9.612
6.408
4.806
6.45606
6.408
3.204
1.5219
6.4881
8.04204
6.44004
4.83804
3.22002
6.39198
8.07408
9.66006
8.10612
4.75794
4.77396
1.66608

Millikan's Oil Drop Experiments

(courtesy of Mr. White)

By statistically grouping the results, it's possible to determine the fundamental unit of charge.



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