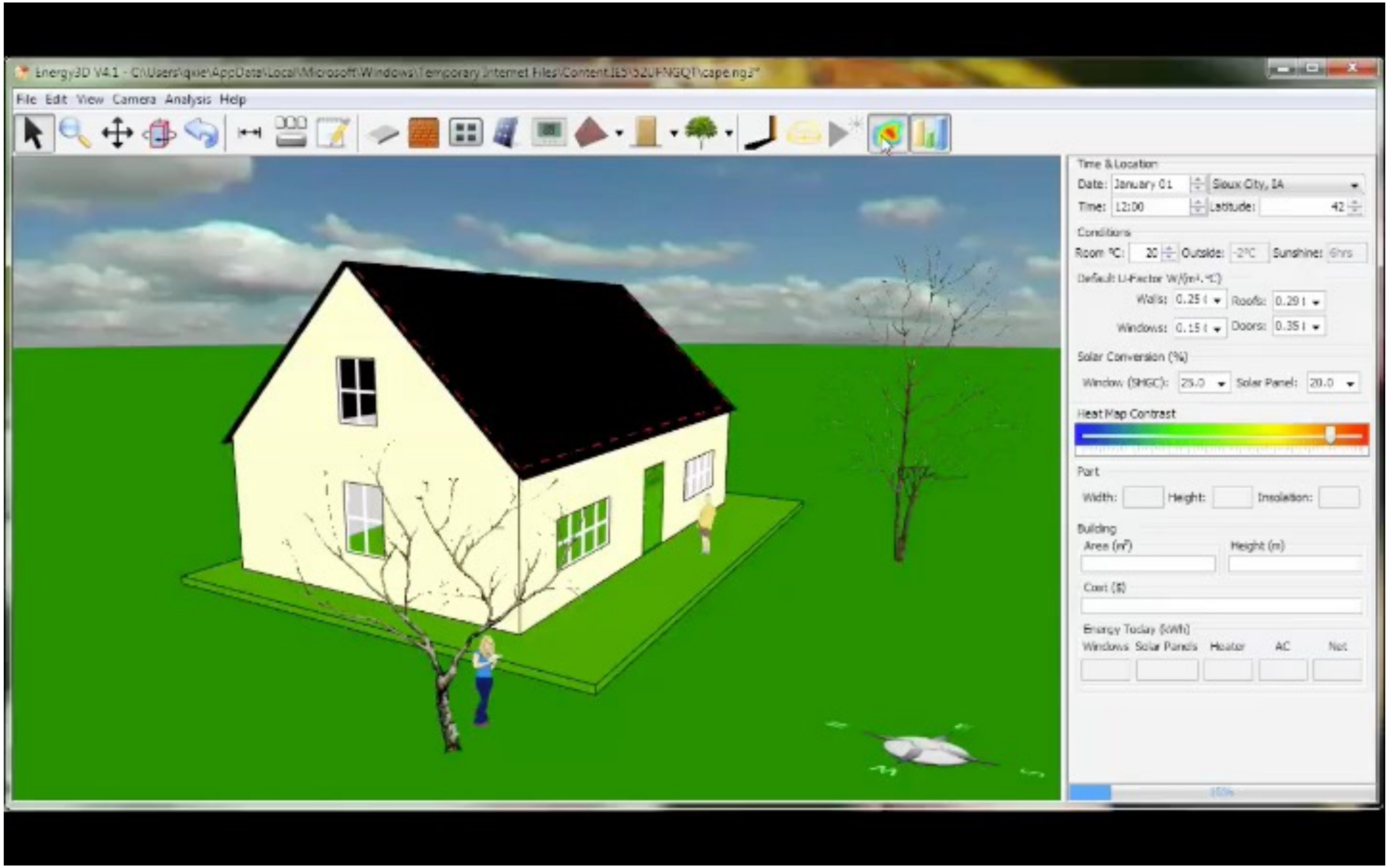
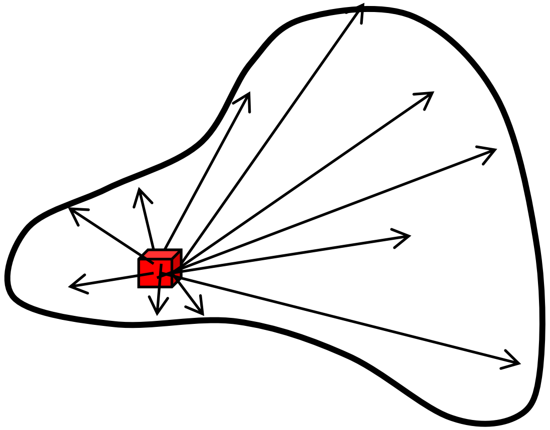
Heat flux

Energy radiating through a surface



CHAPTER 24:
Gauss's Law

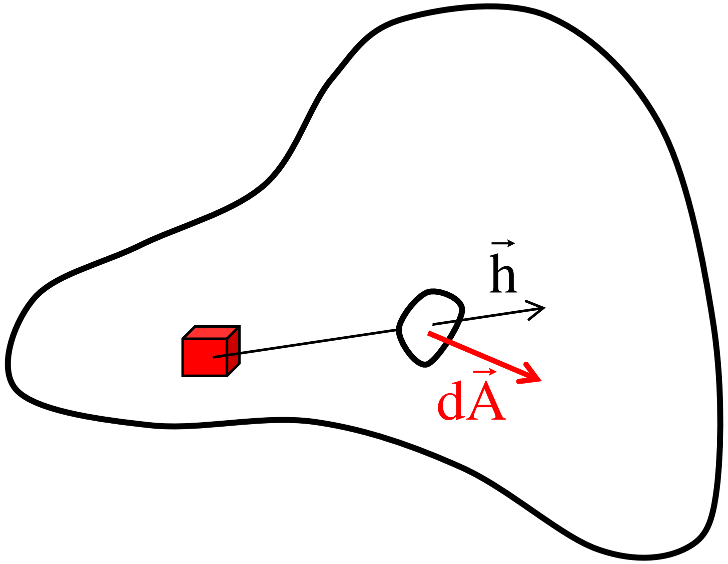
Consider a heat source asymmetrically placed inside an irregularly shaped, closed surface. Assume the amount of *heat per unit area per unit time* passing through the surface is defined by a position function *h* (that is, you give me a coordinate and I give you the goods). How much *heat per unit time* passes through the entire surface?



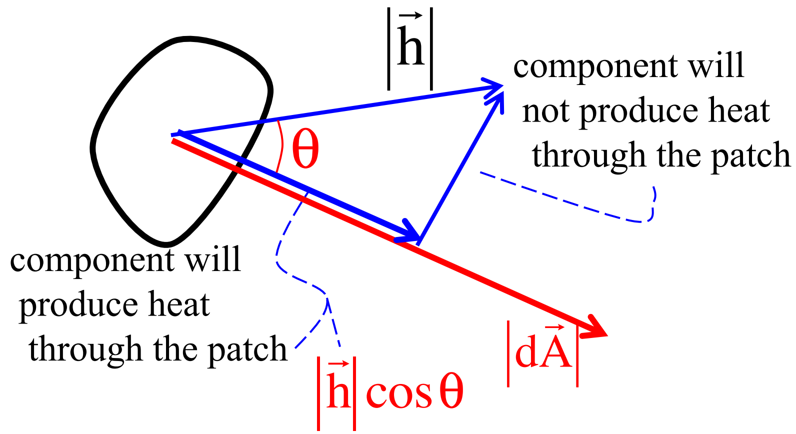
If we could identify a *differential surface area* through which heat was passing, we could multiply the *heat per unit area per unit time* function *h* by that *surface area dA* to produce the *heat per unit time* through that area. Summing (integrating) over all the areas for the entire surface would generate the amount of heat that passes through the *entire surface per unit time*.

So how does the math work on this?

We start by defining on the surface a *differential surface area vector* $d\vec{A}$ whose magnitude *dA* is equal to the surface area of the differentially small patch and whose direction is perpendicularly *out* from the surface. That vector, along with $\vec{h}$, is shown to the right.



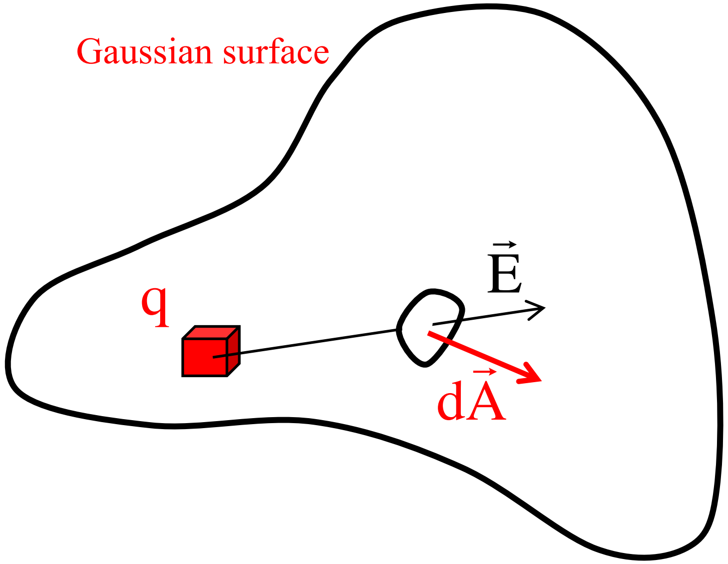
What we are interested in is the component of $\vec{h}$ along the line of $d\vec{A}$ times the magnitude of $d\vec{A}$. In other words, we want the *differential heat flux* through the patch, which will equal:



$$d\Phi_{heat} = \left(|\vec{h}|\cos\theta\right)\,|d\vec{A}|$$

$$= \vec{h} \cdot d\vec{A}$$

This is the mathematical definition of *flux*. Any vector field can have a flux through a surface, whether it be an open surface or a closed surface.



So let's assume our hot object is actually a point charge *q*. What can we do with that situation?

Gauss made the simple but powerful observation that there would be an *electric flux* through the closed surface, called a *Gaussian Surface,* as long as there was *charge enclosed inside the surface*.

What's more, he surmised that the *amount* of flux would have to be proportional to the *amount of charge* enclosed inside the Gaussian surface. In other words, mathematically:

$$\Phi_E = \int \vec{E} \bullet d\vec{A} \text{ proportional to } q_{\text{enclosed}}$$

The proportionality constant that made the relationship into an equality was the inverse of our old friend, the *permittivity of free space* (i.e., $1/\varepsilon_o$), so Gauss's Law is written as:

$$\int \vec{E} \bullet d\vec{A} = \frac{q_{\text{enclosed}}}{\varepsilon_o}$$

(Tiny note: Some books use *dS* as the *surface area vector*, so $\int \vec{E} \bullet d\vec{S} = \frac{q_{\text{enclosed}}}{\varepsilon_o}$.)

Some Observations About Flux:

1.) *Remember back to* the *impulse relationship* $F_x \Delta t = \Delta p_x$ and the instances when you were asked to determine the impulse on an object. You had the choice of determining the namesake of the operation ($F_x \Delta t$) OR, you could determine Δp_x, depending upon what you knew.

You have a similar situation when asked to determine the *electric flux* through a closed surface. You *can* determine $\int \vec{E} \bullet d\vec{A}$ over the surface, OR you could just determine the total net charge enclosed inside the surface and use $q_{enclosed} / \varepsilon_o$ to determine the flux. The two will be the same.

2.) *The units* for *electric flux* are $N \bullet m^2 / C$.

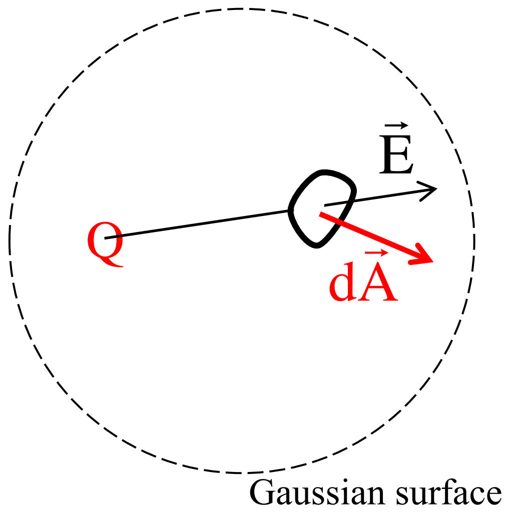
3.) *The area vector dA* used in *electric flux calculations* is ALWAYS defined perpendicularly outward from the Gaussian surface.

4.) *Charge outside* a *Gaussian surface* will generate no net flux through the closed surface. (Think about it, charge outside the surface will produce electric field lines that first pass into the surface, but then will pass out of the surface on the other side generating *no net flux* through the surface).

Back to Gauss's Law

Gauss's Law is ALWAYS TRUE, no matter how the geometry shakes out, but it is pretty useless unless you can exploit symmetry in a problem.

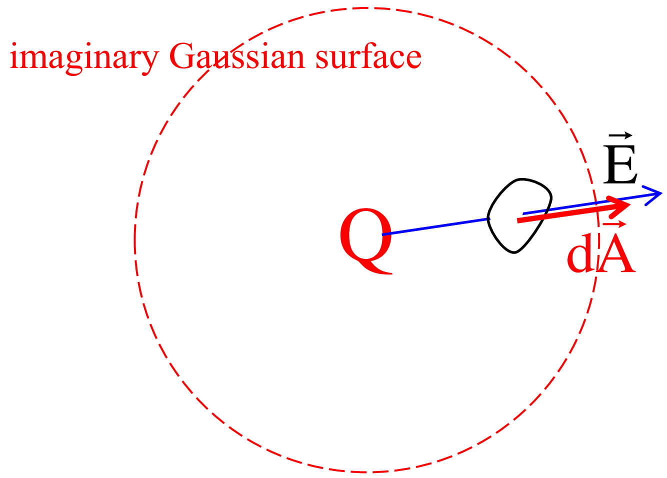
Example 1: Use *Gauss's Law* on the spherical surface of radius *R* and charge *Q* as shown.



What the integral $\int_s \vec{E} \bullet d\vec{A}$ is apparently asking us to do is to define an arbitrary, differential surface area *dA* (remember, *dA* has a magnitude equal to the area of the enclosed surface and is directed perpendicularly outward from the surface), evaluate both the *direction* and *magnitude* of the electric field *at that surface*, dot the $\vec{E}$ and $d\vec{A}$ into one another, then do that for all the differential surfaces over the entire structure and sum them by integrating.

Gauss was right. That flux WOULD equal Q/ε_o. But because no two points on the surface are the same distance from the charge Q, and no two dot products are going to be the same (angles different), doing that integral would be a NIGHTMARE. In short, this is an impossible problem to do!!!!!!!

Example 2: This is *Example 1* done more reasonable: Derive an expression for the *electric field function* for a point charge *Q*.

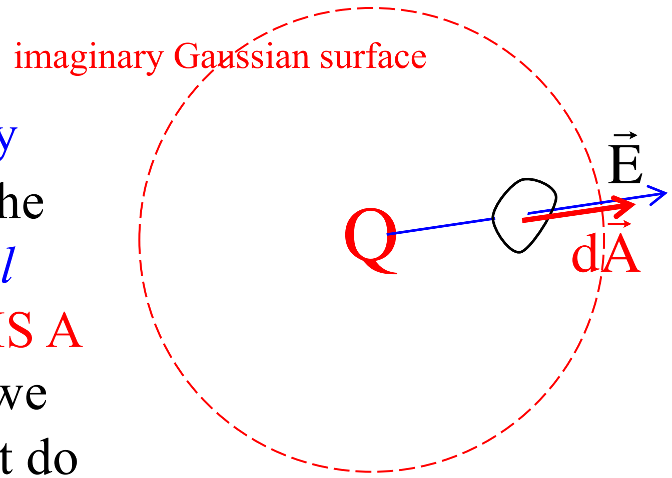


Important observation: There is *no given Gaussian surface* to begin with in this problem, just a hanging charge. We need to create an *imaginary Gaussian surface around the charge*, one that *exploits the symmetry* of the charge's electric field. That is, we need to create a surface such that *ever point* on the surface is *equidistant from the charge*.

With the imaginary Gaussian surface centered on the charge, ANY differential area vector *dA* will be radially outward, which is to say, *in the direction of E*, and the angle between the electric field vector and the differential area vector will be zero (so the cosine in the dot product will equal 1). With that, we can draw (see sketch), then write:

$$\int_S \vec{E} \bullet d\vec{A} = \frac{q_{cnclosed}}{\varepsilon_o}$$

$$\Rightarrow \int_S \left|\vec{E}\right|\left|d\vec{A}\right| \overset{1}{\cancel{\cos 0^0}} = \frac{Q}{\varepsilon_o}$$



Herein lies the beauty of the method. Because every point on the surface is equidistant from the charge, the evaluation of the magnitude of *E* at *every differential surface dA* WILL BE THE SAME, which is to say, IS A CONSTANT VALUE, and because it is a constant, we can pull it out of the integral. (Note that we couldn't do that with the original Example 1 because each point was a different distance from Q.) With that, we can write:

$$\int_S |\vec{E}||d\vec{A}| = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow \quad |\vec{E}| \int_S |d\vec{A}| = \frac{Q}{\varepsilon_o}$$

That makes life wonderful, as now the only thing inside the integral is the differential surface area *dA*, and summing that over the surface simply yields the total surface area of the sphere ($4\pi R^2$) . . . So we can further write

$$|\vec{E}| \int_S |d\vec{A}| = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow \quad |\vec{E}|\left(4\pi R^2\right) = \frac{Q}{\varepsilon_o}$$

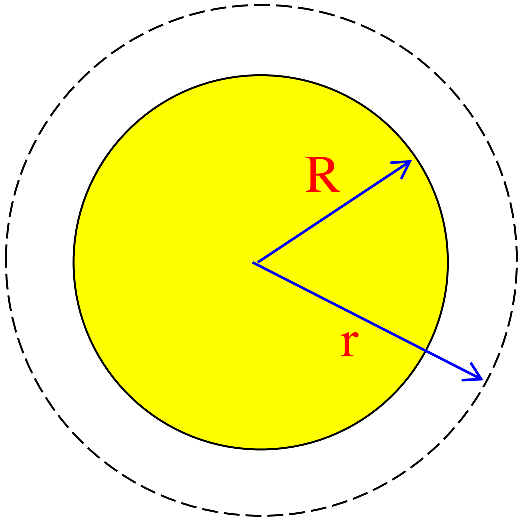
$$\Rightarrow \quad |\vec{E}| = \frac{Q}{4\pi\varepsilon_o R^2}$$

Look familiar? It should. It's the same as the *electric field function* we derived for a *point charge* using Coulomb's Law!

Insulators and Gauss's Law

What is characteristic of *insulators* is that free charge stays put in an insulator, it doesn't migrate around in response to an electric field. With that in mind:

Example 3: Derive the *electric field function*, both inside and outside, for a spherical insulator of radius R, that has charge -Q evenly distributed throughout its volume.



a.) for r>R: Start with an imaginary Gaussian sphere of radius r, where $r > R$, and write out *Gauss's Law* for the situation (there are notes on the process on the next page):

$$\int_S \vec{E} \bullet d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \int_S E\,dA\cos 0^\circ = \frac{-Q}{\varepsilon_o}$$

$$\Rightarrow E\int_S dA = \frac{-Q}{\varepsilon_o}$$

$$\Rightarrow E\left(4\pi r^2\right) = \frac{-Q}{\varepsilon_o}$$

$$\Rightarrow E = -\frac{Q}{4\pi\varepsilon_o r^2}$$

And yes, this looks like a point charge!

Subtleties:

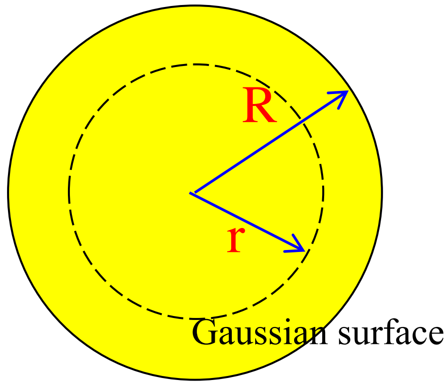
1.) *The LEFT SIDE* of *Gauss's Law* is always the EASY PART of the equation. Assuming you've picked the appropriate geometry, the magnitude of $\vec{E}$, which is what the *dot product* is interested in, should be the same everywhere. That means $|\vec{E}|$ will be a constant and you will ALWAYS pull it out of the integral which, in turn, means you will ALWAYS end up integrating over *dA*. So for *spherical symmetry*, the left side will ALWAYS end up equaling:

$$E(4\pi r^2)$$

2.) *The dot product* deals with magnitudes, so the *E* being determined by Gauss's Law should be the magnitude of the electric field as evaluated at (or on) the Gaussian surface. So what's the deal with the negative sign?

To make the math easy, always assume that $\vec{E}$ is *outward* in the same direction as $d\vec{A}$. That will make the *dot product* positive. If you are clever, though, and include the sign of the charge in $q_{enclosed}$, a negative sign showing up in the final expression for *E* will tell you that you've *assumed the wrong direction for* $\vec{E}$ (i.e., it's inward, not outward). In other words, the negative sign doesn't *technically* means a negative electric field direction, but it sorta does . . . This will be a lot more important when you get to complicated charge configurations.

b.) for r<R: Again, start with an imaginary Gaussian sphere of radius r, this time with $r < R$, and write out *Gauss's Law* for the situation.



Note that we to need to figure out how much charge is inside r. We can do this two way. I'll show both.

Straight ratio (works only for constant density)

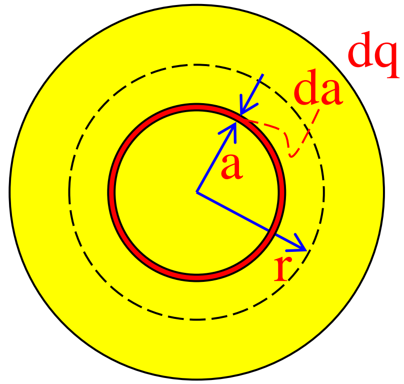
$$q_{enclosed} = (\text{fraction of Q inside r})(-Q)$$

$$= \left[\frac{\left(\frac{4}{3}\cancel{\pi} r^3\right)}{\left(\frac{4}{3}\cancel{\pi} R^3\right)}\right](-Q)$$

$$= -\left(\frac{r^3}{R^3}\right)Q$$

Using density function (works for all situations)

Noting that: $\rho = \frac{\text{chg}}{\text{vol}} = \frac{-Q}{\frac{4}{3}\pi R^3}$ and $dV_{shell} = \underbrace{4\pi a^2}_{\text{surface area}} da$



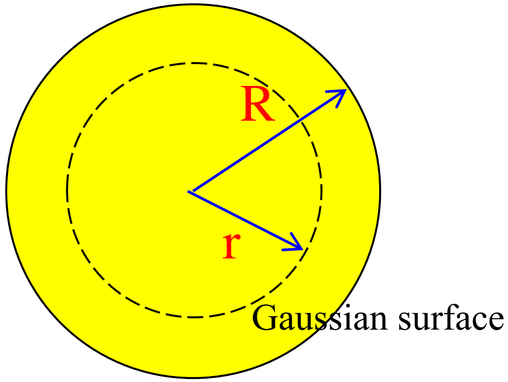
$$q_{enc} = \int dq = \int_{a=0}^{r} \rho \, dV$$

$$= \rho \int_{a=0}^{r} (4\pi a^2) da$$

$$= \frac{-Q}{\cancel{\frac{4}{3}}\cancel{\pi} R^3}(\cancel{4\pi}) \int_{a=0}^{r} a^2 \, da$$

$$= \frac{3(-Q)}{R^3}\left(\frac{a^3}{3}\right)\Big|_{a=0}^{r} = -\left(\frac{r^3}{R^3}\right)Q$$

b.) for r<R: (con't.—doing this with the *density function* . . . though either way would do here)



$$\int_S \vec{E}\bullet d\vec{A} = \frac{\int_{a=0}^{r} \rho\, dV}{\varepsilon_o}$$

$$\Rightarrow \int_S E\, dA \cos 0^o = \frac{\int_{a=0}^{r}\left[\left(\frac{-Q}{\left(\cancel{4}/3\,\cancel{\pi} R^3\right)}\right)\right]\left[\cancel{4}\cancel{\pi} a^2 da\right]}{\varepsilon_o}$$

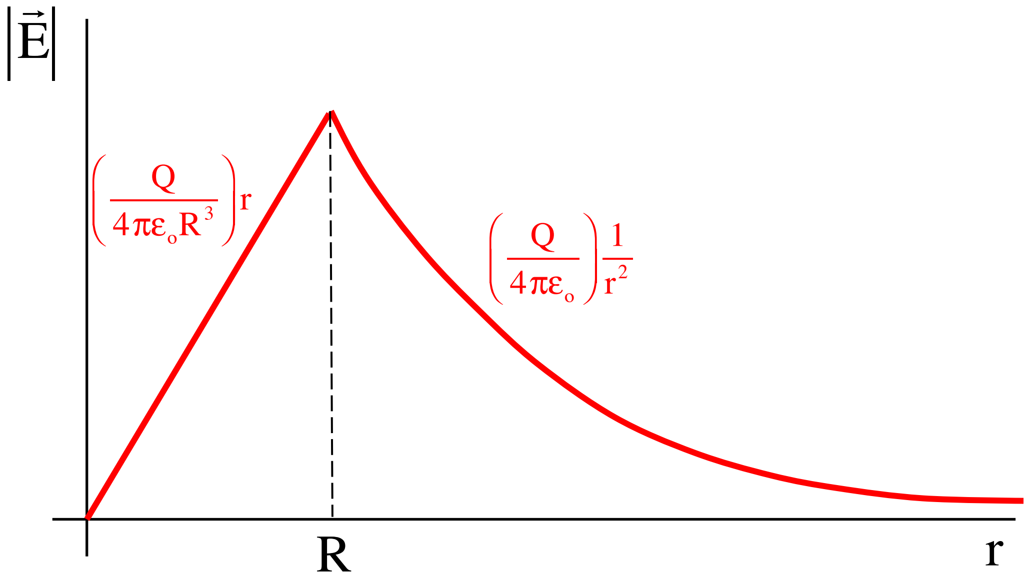
$$\Rightarrow E\int_S dA = \frac{3(-Q)\int_{a=0}^{r} a^2\, da}{\varepsilon_o R^3} = \frac{\cancel{3}(-Q)}{\varepsilon_o R^3}\left(\frac{a^3}{\cancel{3}}\bigg|_{a=0}^{r}\right)$$

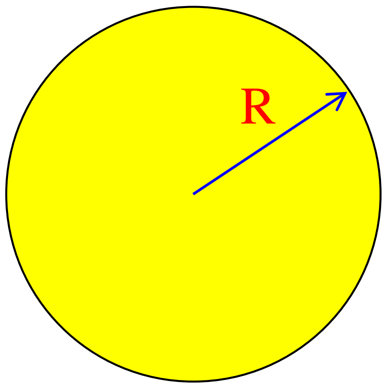
$$\Rightarrow E\left(4\pi r^2\right) = \frac{-Q}{\varepsilon_o R^3} r^3$$

$$\Rightarrow E = -\frac{Q}{4\pi\varepsilon_o R^3} r$$

Note that this is a *linear E-field* inside the sphere!

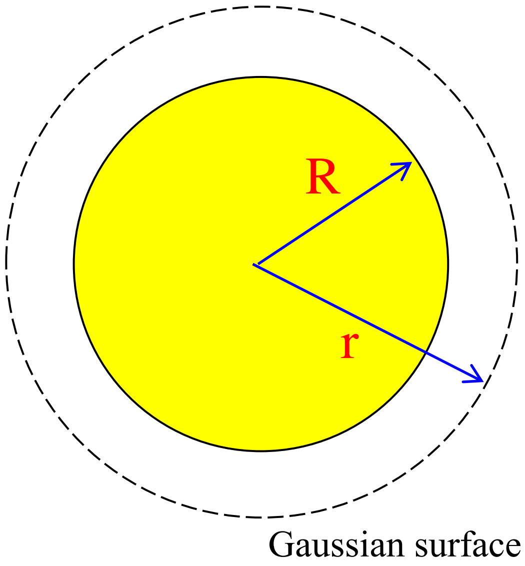
c.) What does the graph look like for *electric field magnitude* versus *position*?





Conductors and Gauss's Law

What is characteristic of conductors is that free charge can move around freely within the conductor, migrating in response to an *electric field*. With that in mind:



Example 4: Derive the *electric field function*, both inside and outside, for a spherical conductor of radius R, that has charge *Q* placed on it.

a.) for r>R: The imaginary Gaussian sphere of radius *r*, where *r* > *R*, looks just like it did for the insulator, except the charge is not shot through the volume but resides on the volume's surface (like-charge attempts to get as far away from like charge as possible). So . . .

$$\int_S \vec{E} \bullet d\vec{A} = \frac{q_{\text{cnclosed}}}{\varepsilon_o}$$

$$\Rightarrow \int_S E\, dA \cos 0^o = \frac{Q}{\varepsilon_o}$$

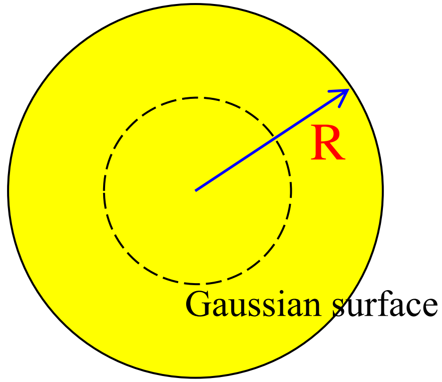
$$\Rightarrow E\int_S dA = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow E\left(4\pi r^2\right) = \frac{Q}{\varepsilon_o}$$

$$\Rightarrow E = \frac{Q}{4\pi\varepsilon_o r^2}$$

Still looks like a point charge!

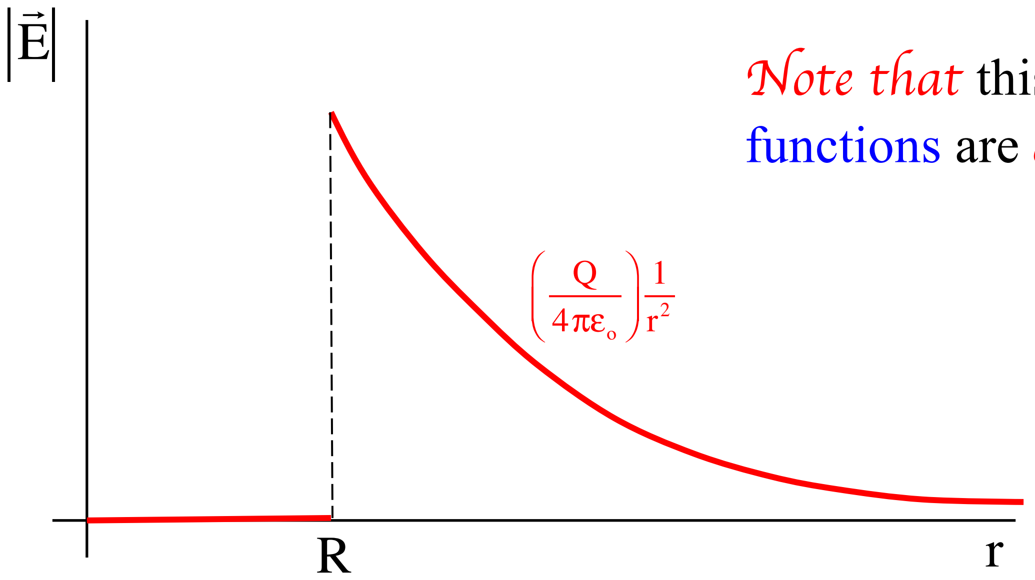
b.) for r<R: This is easy. With all the charge on the surface, the charge enclosed inside the Gaussian surface is zero and:



$$\int_S \vec{E} \bullet d\vec{A} = \frac{0}{\varepsilon_o}$$

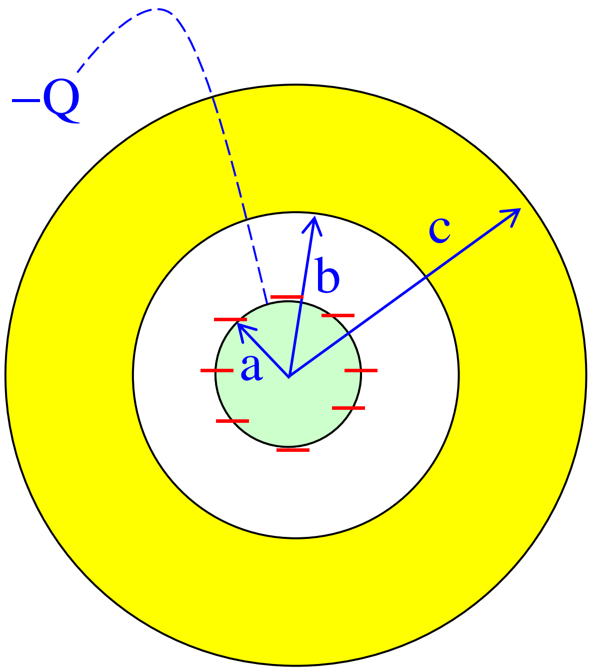
$$\Rightarrow \; E = 0$$

c.) What does the graph look like for *E-field* magnitude versus *position*?



Note that this suggests that *E-fld* functions are *discontinuous* . . .

Now for the Real Fun



Example 5: A thick skinned, conducting, *spherical shell* with *inside radius b* and *outside radius c* has a net charge *3Q* on it. At its center is a *solid conducting sphere* of radius *a* that has a net charge of *–Q* on it.

a.) How will charge distribute itself on these structure?

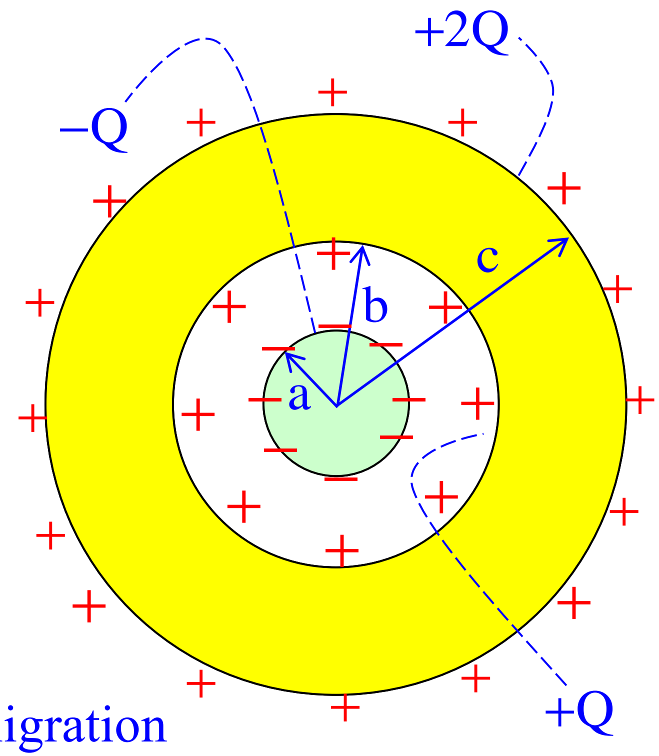
OBSERVATIONS:

1.) The electric field inside a *conductor* in a static situation will always be ZERO. Why? Because if an electric field existed, electrons would respond to it and move until the field was nullified.

2.) For the E-field to be zero inside the *conducting sphere*, the *net charge* inside a Gaussian surface in that region must be zero. How can this be? Free charge in the amount *–Q* (i.e., electrons) must migrate to the *outside surface* of the conductor (hence the total charge inside the Gaussian surface inside the shell will sum to zero).

a.) (con't)

OBSERVATIONS (con't)



3.) *Being a conductor,* the electric field inside the *shell* must also be zero. As there is *–Q*'s worth of charge on the solid sphere at the center, *electrons in the shell* will be repulsed and migrate out from the *inside* to *outside wall* leaving the inside surface with +Q's worth of charge on it and no *charge enclosed* for a Gaussian surface inside the shell.

4.) *IF the shell* had been *electrically neutral*, the migration of electrons in the shell due to the *–Q at the center* would have left the *inside surface* at *+Q* and the *outside surface* at *–Q*.

5.) *IF the charge at the center didn't exist,* on the other hand, the *+3Q* net charge on the shell (meaning 3Q's worth of electrons had been removed) would reside on the outside surface.

6.) *Combining observations 4 and 5* leaves us with a net charge on the outside surface of *+2Q* and a *net charge on the shell* of *+3Q* (look at sketch).

b.) Derive an expression for the *E-fld* for $r > c$.

Using an imaginary Gaussian sphere of radius *r*, where *r > c*, and including ALL the charge enclosed, we can write:

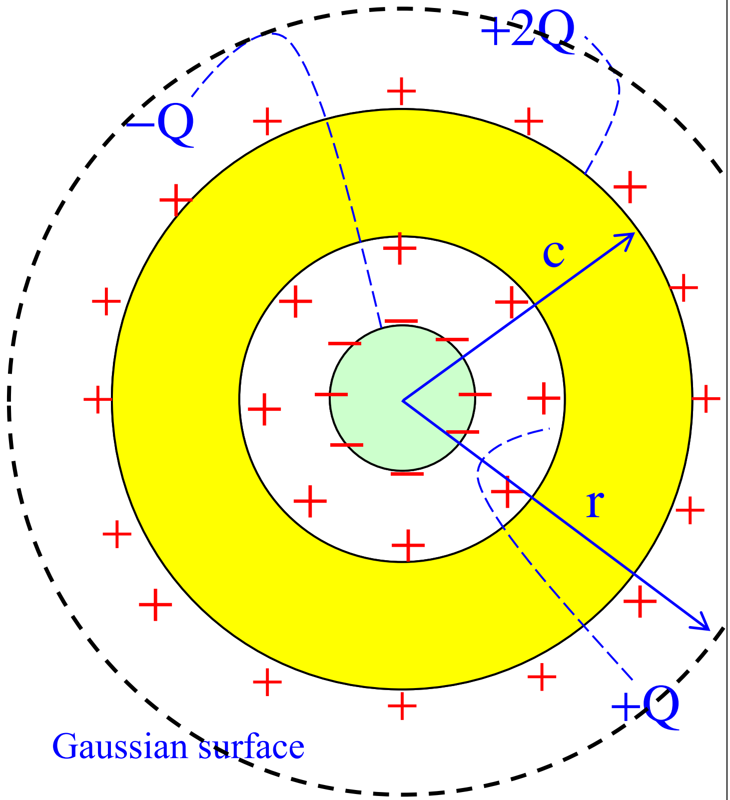
$$\int_S \vec{E} \bullet d\vec{A} = \frac{q_{cnclosed}}{\varepsilon_o}$$

$$\Rightarrow \quad \int_S E\, dA \cos 0^o = \frac{+2Q + Q - Q}{\varepsilon_o}$$

$$\Rightarrow \quad E \int_S dA = \frac{2Q}{\varepsilon_o}$$

$$\Rightarrow \quad E\left(4\pi r^2\right) = \frac{2Q}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{2Q}{4\pi\varepsilon_o r^2}$$



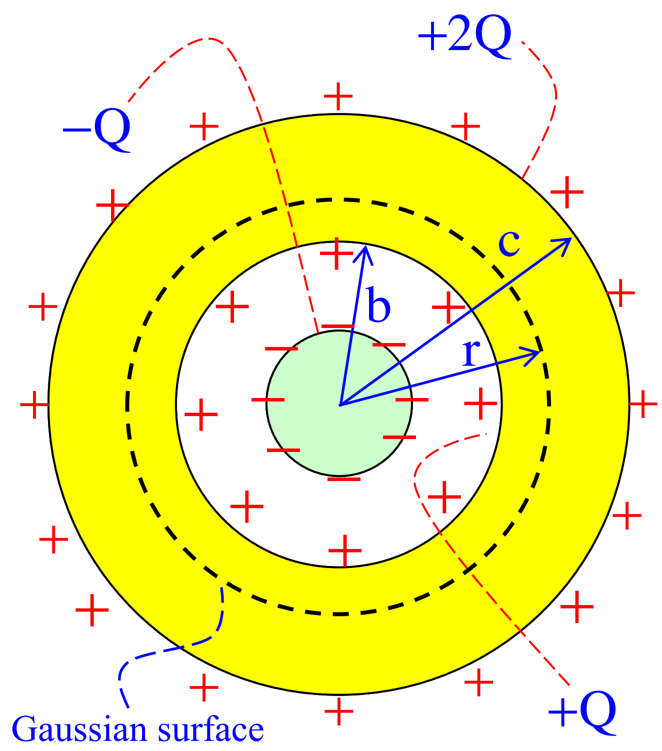
c.) Derive an expression for the *E-fld* for b<r<c .

Using an imaginary Gaussian sphere of radius *r*, where *r is inside the shell*, we can write:

$$\int_S \vec{E} \bullet d\vec{A} = \frac{q_{cnclosed}}{\varepsilon_o}$$

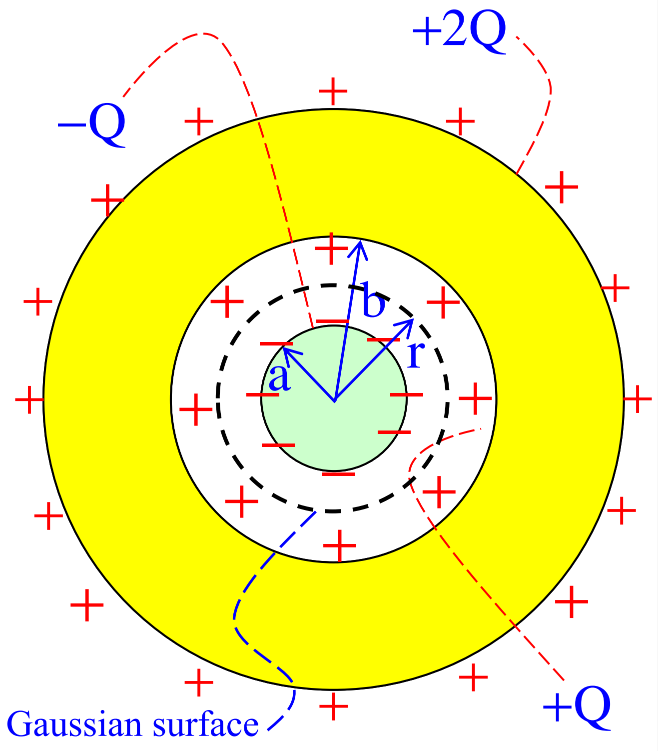
$$\Rightarrow \int_S E\, dA \cos 0^o = \frac{+Q - Q}{\varepsilon_o}$$

$$\Rightarrow \quad E = 0$$



d.) Derive an expression for the *E-fld* for a<r<b.

Using an imaginary Gaussian sphere of radius *r*, where *r is between the shell and the sphere*, we can write:



$$\int_S \vec{E} \bullet d\vec{A} = \frac{q_{\text{cnclosed}}}{\varepsilon_o}$$

$$\Rightarrow \quad \int_S E\,dA\cos 0^o = \frac{-Q}{\varepsilon_o}$$

$$\Rightarrow \quad E\int_S dA = \frac{-Q}{\varepsilon_o}$$

$$\Rightarrow \quad E\left(4\pi r^2\right) = \frac{-Q}{\varepsilon_o}$$

$$\Rightarrow \quad E = -\frac{Q}{4\pi\varepsilon_o r^2}$$

where the negative sign tells us the *E-fld* is not outward as assumed but, instead, *opposite the assumed direction* and inward (you'd expect this with net negative charge on the inside of the region and net positive charge on the outside).

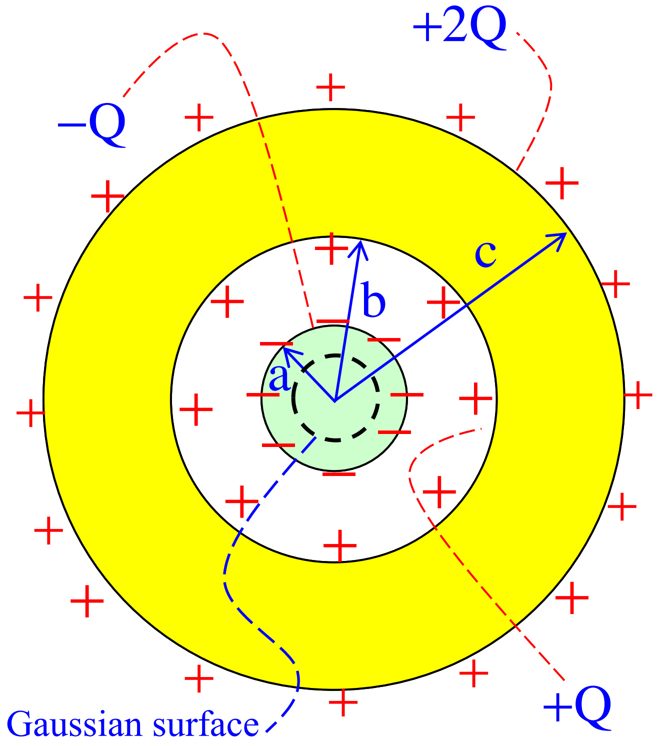
e.) Derive an expression for the *E-fld* for $r < a$.

Using an imaginary Gaussian sphere of radius *r*, where *r* is inside the sphere:

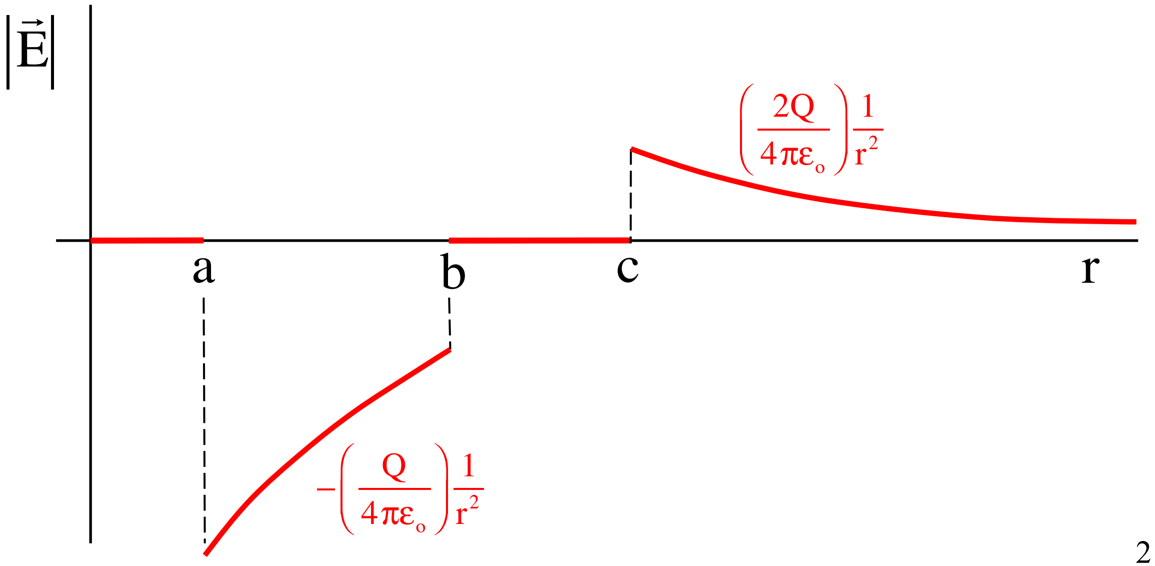
$$\int_S \vec{E} \bullet d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \int_S E\, dA \cos 0^\circ = \frac{0}{\varepsilon_o}$$

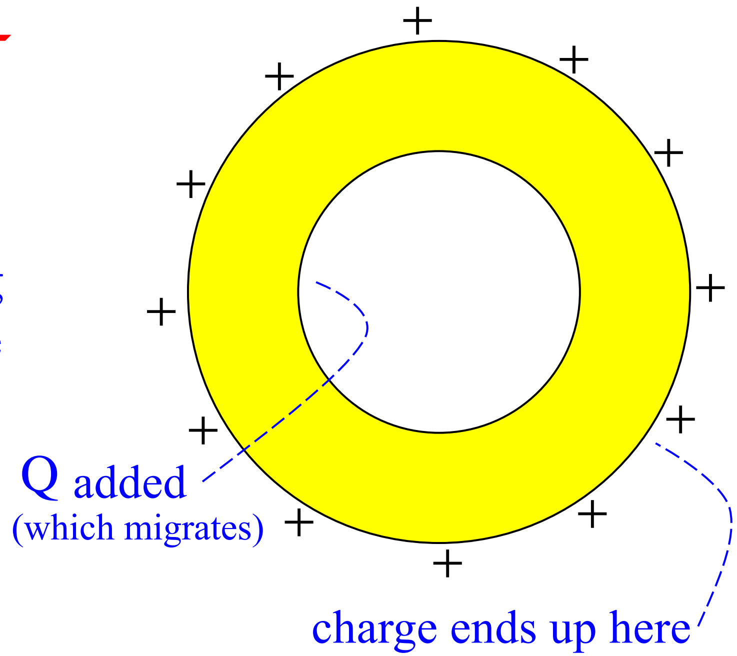
$$\Rightarrow \quad E = 0$$



f.) What does the graph look like for *E-field* as a vector versus *position*?



Real Fun With a Twist



Example 6: Let's assume that someone touches an insulator to the inside of the conducting shell, and in doing so transfers *Q's* worth of charge to the shell. What is going to happen in that case?

1.) *An important point,* which I haven't mentioned yet, is that physicist have no qualms about looking at situations like this from the perspective of the motion of POSITIVE CHARGE. That is not what is really happening, but you can view theoretical situations as though it *was* happening, and come out with reasonable conclusion.

Doing that in this case suggests the following will happen: The bits of positive charge making up the transferred *Q* will migrate to the outside edge of the shell leaving it with a net charge of *Q* on that surface.

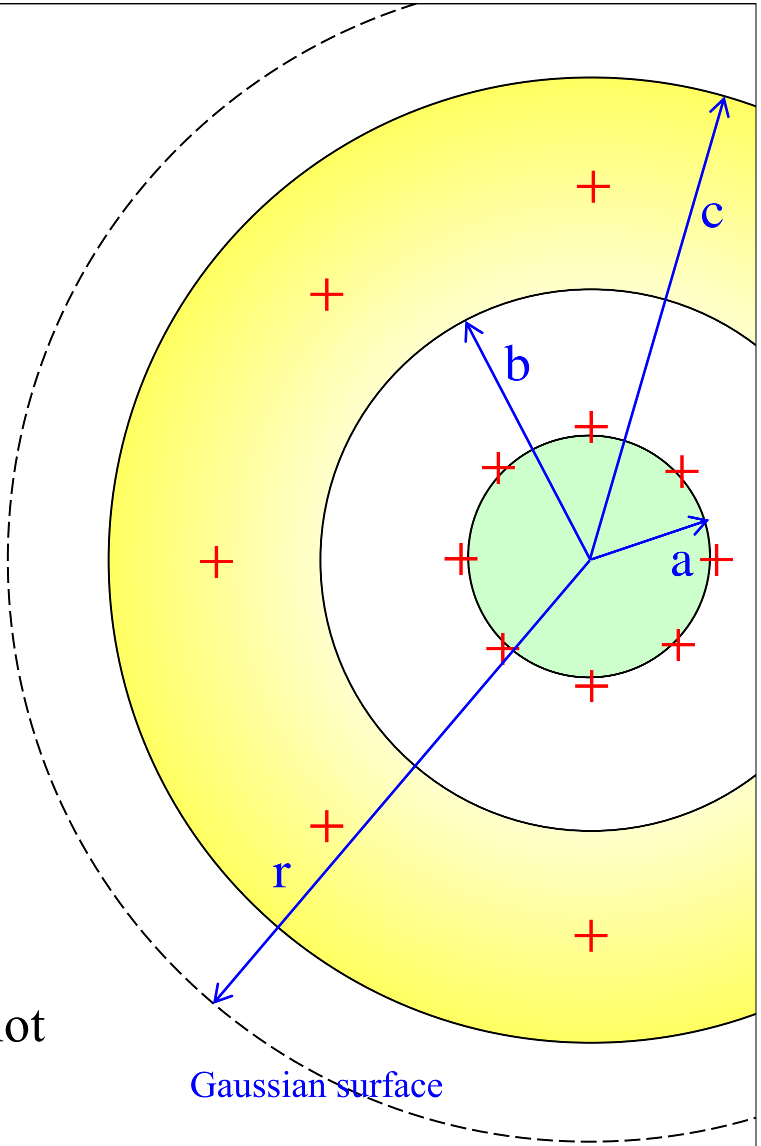
2.) *In reality, electrons* are drawn from the shell's outside edge, leaving it with a net charge of *Q,* migrating through the shell to the inside edge where they jump to the insulator (that's how the shell becomes more positive).

And Even More Fun

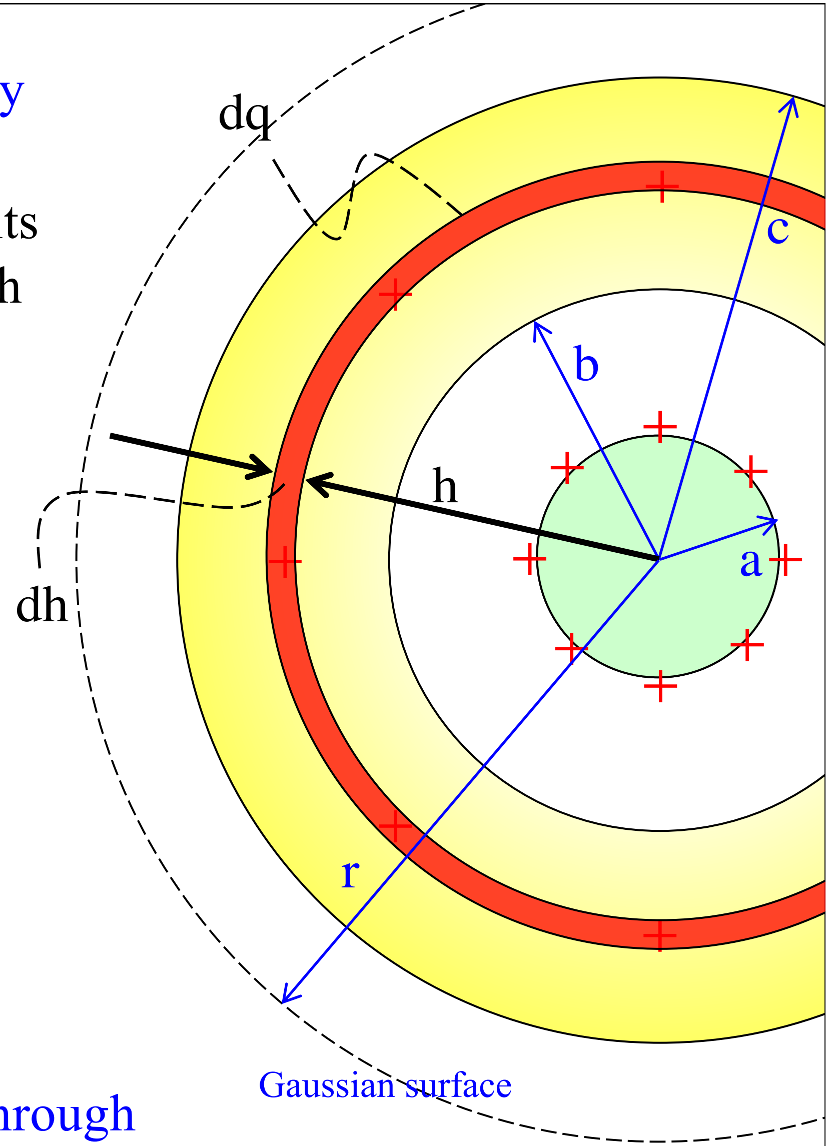
Example 7: A thick skinned, insulating *spherical shell* with *inside radius b* and *outside radius c* has a *volume charge density* $\rho = kr$, where the constant $k = 1\ C/m^4$. At its center is a *solid conducting sphere* of radius *a* that has a net charge of *Q* on it.

a.) Derive an expression for the magnitude of the electric field for $r > c$.

We need the total charge inside a *Gaussian surface* that engulfs the entire charge configuration. The *Q* on the center sphere is easy, but the charge shot through the shell is not so easy as it is slight at the inside and gets heavier as one moves outward (that is, it isn't uniform). To do this, we need to be clever.



Define a differential volume *dV* as a differentially *thin shell* of radius *h* and thickness *dh*. Its magnitude will be its surface area $(4\pi h^2)$ times its differential thickness *dh*, or $dV = (4\pi h^2 dh)$. With a given volume charge density of $\rho = kh$, where ρ has been evaluated at *h,* we can write:



$$\rho = \frac{dq}{dV}$$

$$\begin{aligned} \Rightarrow \quad dq &= \rho dV \\ &= \rho(4\pi h^2 dh) \\ &= (kh)(4\pi h^2 dh) \\ &= 4\pi k h^3 dh \end{aligned}$$

With this, we can determine the charge shot through the entire shell, or through just part of the shell.

With $dq = \rho\left(4\pi h^2 dh\right)$,
and figuring out the charge between *b* and *c* inside the Gaussian radius *r*, we can write:

$$\int_S \vec{E} \bullet d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

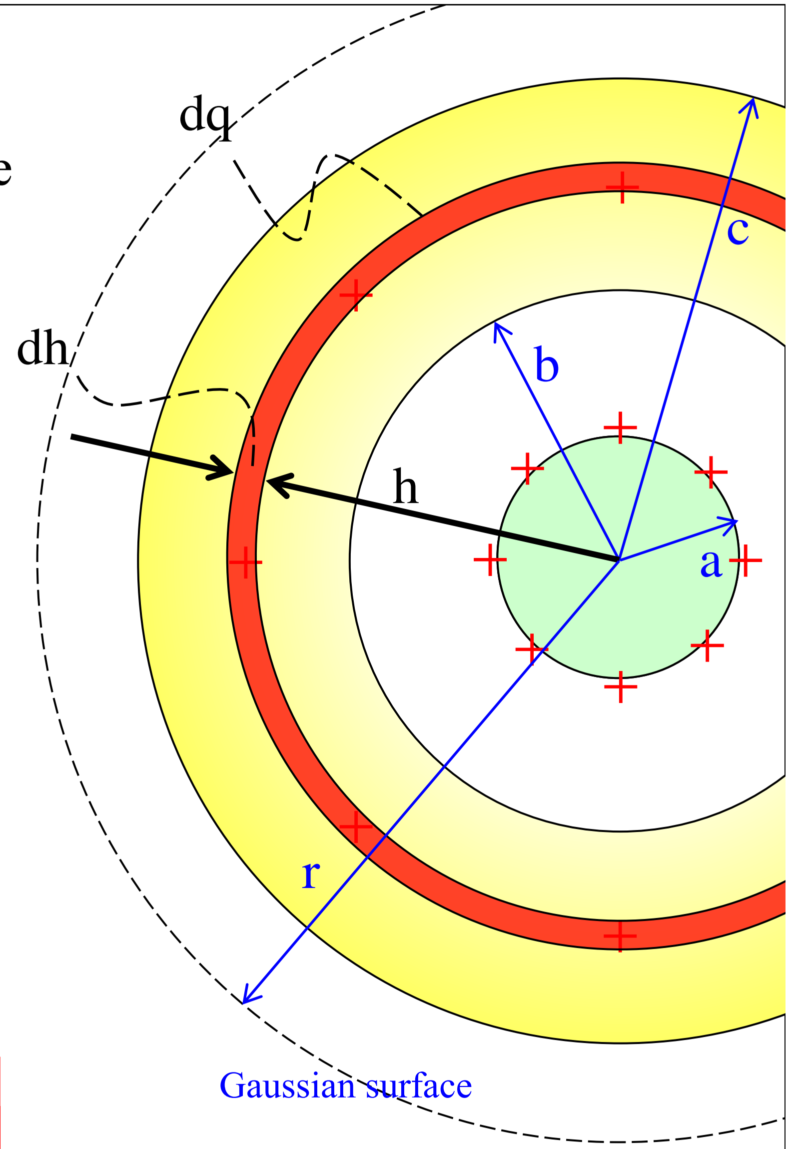
$$\Rightarrow \quad E\left(4\pi r^2\right) = \frac{Q + \int_{h=b}^{c} \rho \, dV}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{Q + \int_{h=b}^{c} (kh)\left(4\pi h^2 dh\right)}{4\pi r^2 \varepsilon_o}$$

$$= \frac{1}{4\pi\varepsilon_o r^2}\left[Q + (4\pi k)\int_{h=b}^{c} h^3 \, dh\right]$$

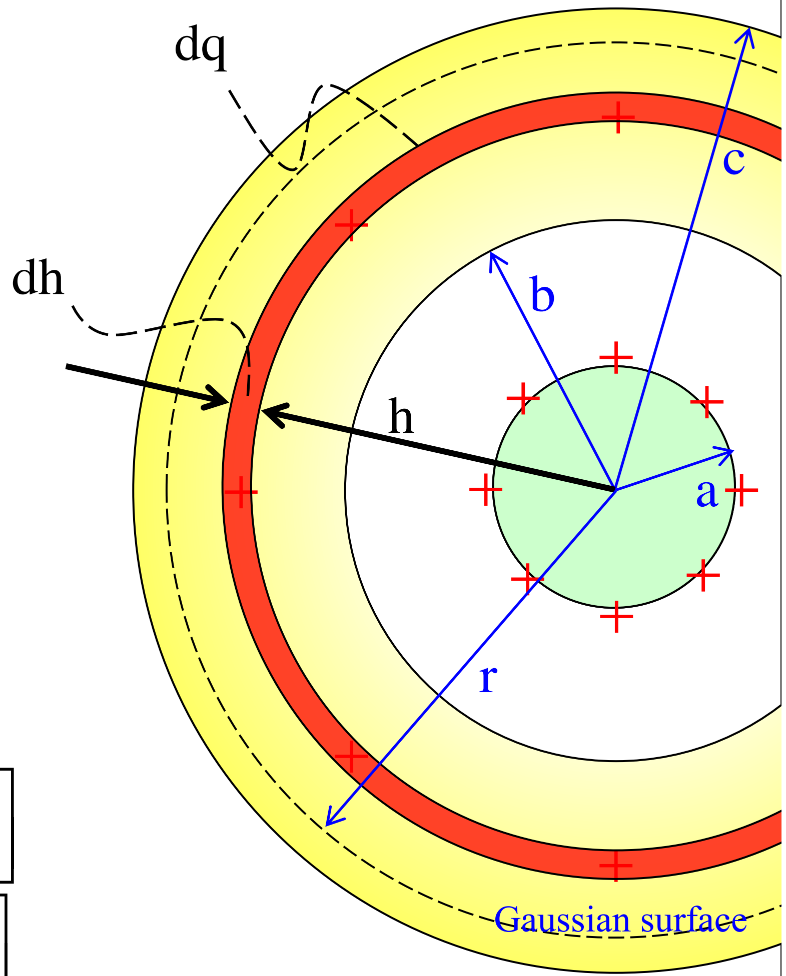
$$= \frac{1}{4\pi\varepsilon_o r^2}\left[Q + (4\pi k)\left(\frac{h^4}{4}\right)\Bigg|_{r=b}^{c}\right]$$

$$= \frac{1}{4\pi\varepsilon_o r^2}\left[Q + (4\pi k)\left(\frac{c^4}{4} - \frac{b^4}{4}\right)\right]$$



b.) Derive an expression for the magnitude of the electric field for $c > r > b$.

This is easy as it's the same as *Part a* except the shell's contribution to the charge is only the charge *out to the Gaussian radius r*. That is:



$$\int_S \vec{E}\bullet d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \quad E\left(4\pi r^2\right) = \frac{Q + \int_{a=b}^{r}(kh)\left(4\pi h^2 dh\right)}{\varepsilon_o}$$

$$= \frac{1}{4\pi\varepsilon_o r^2}\left[Q + (4\pi k)\int_{h=b}^{r} h^3\, dh\right]$$

$$= \frac{1}{4\pi\varepsilon_o r^2}\left[Q + (4\pi k)\left(\frac{h^4}{4}\right)\Bigg|_{r=b}^{r}\right]$$

$$= \frac{1}{4\pi\varepsilon_o r^2}\left[Q + (4\pi k)\left(\frac{r^4}{4} - \frac{b^4}{4}\right)\right]$$

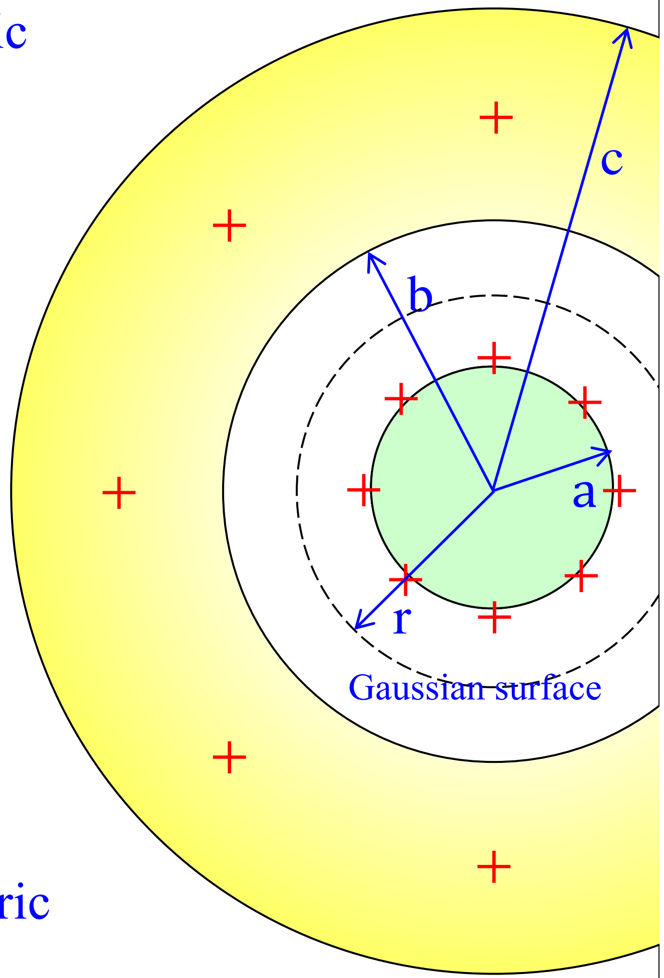
c.) Derive an expression for the magnitude of the electric field for $b > r > R$.

The only charge inside the Gaussian surface for this section is that on the inner conductor, which is Q, so:

$$\int_S \vec{E} \bullet d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \quad E\left(4\pi r^2\right) = \frac{Q}{\varepsilon_o}$$

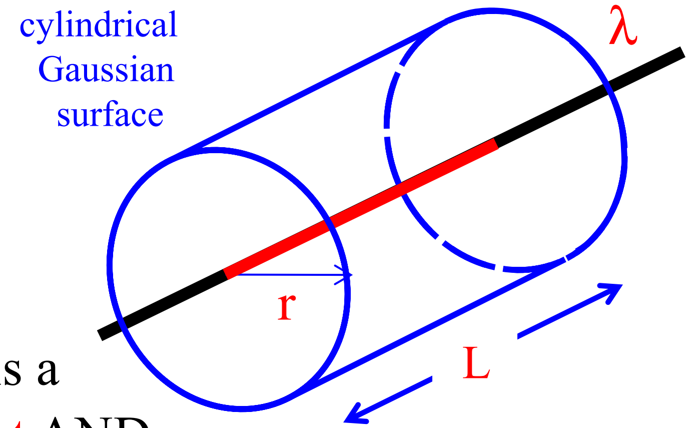
$$\Rightarrow \quad E = \frac{Q}{4\pi\varepsilon_o r^2}$$



d.) Derive an expression for the magnitude of the electric field for $r < R$.

No charge enclosed inside the Gaussian surface, so $E = 0$.

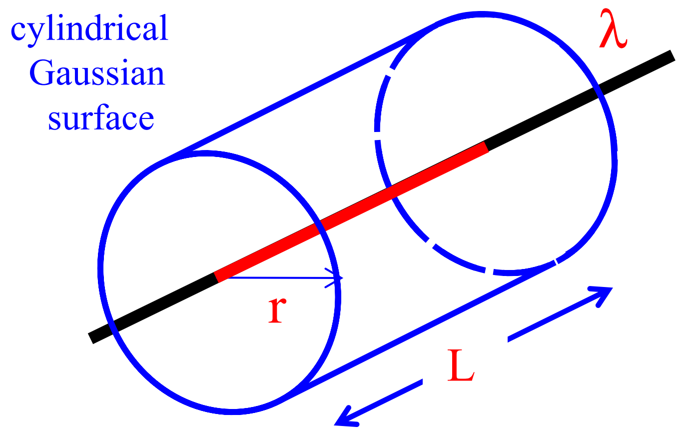
Cylindrical Symmetry and Gauss's Law



Example 8: Derive an expression for the *electric field function* for an "infinitely long" insulating rod whose linear charge density is a constant λ.

What you are looking for in a Gaussian surface is a surface upon which the magnitude of E is constant AND the dot product is either the same everywhere, or zero. With a linear charge distribution, a sphere clearly won't do. What will work, if you are clever, is a cylinder. How so? If the charge configuration is infinitely long (or in a pinch, very, very long), the electric field will be radially outward (or inward, depending upon the charge). A Gaussian *cylinder* will have ends whose *dA's* will be perpendicular to the electric field (hence producing no electric flux and a zero dot product) and a curved surface whose *dA's* are along the line of *E*. In short, a cylindrical Gaussian surface will do the job.

Noting that the amount of charge inside an imaginary, cylindrical, Gaussian surface will be the linear charge density times the length of the surface, and the *surface area* of a cylinder is the circumference $2\pi r$ times the length L, we can write:



$$\int_{\text{curve}} \vec{E} \bullet d\vec{A}_{\text{curve}} + 2\int_{\text{end}} \vec{E} \bullet d\vec{A}_{\text{end}} = \frac{q_{\text{cnclosed}}}{\varepsilon_o}$$

$$\Rightarrow \int_{\text{curve}} E\, dA_{\text{curve}} \cos 0^o + 2\int_{\text{curve}} E\, dA_{\text{end}} \cancelto{0}{\cos 90^o} = \frac{\lambda L}{\varepsilon_o}$$

$$\Rightarrow E\int_{\text{curve}} dA_{\text{curve}} = \frac{\lambda L}{\varepsilon_o}$$

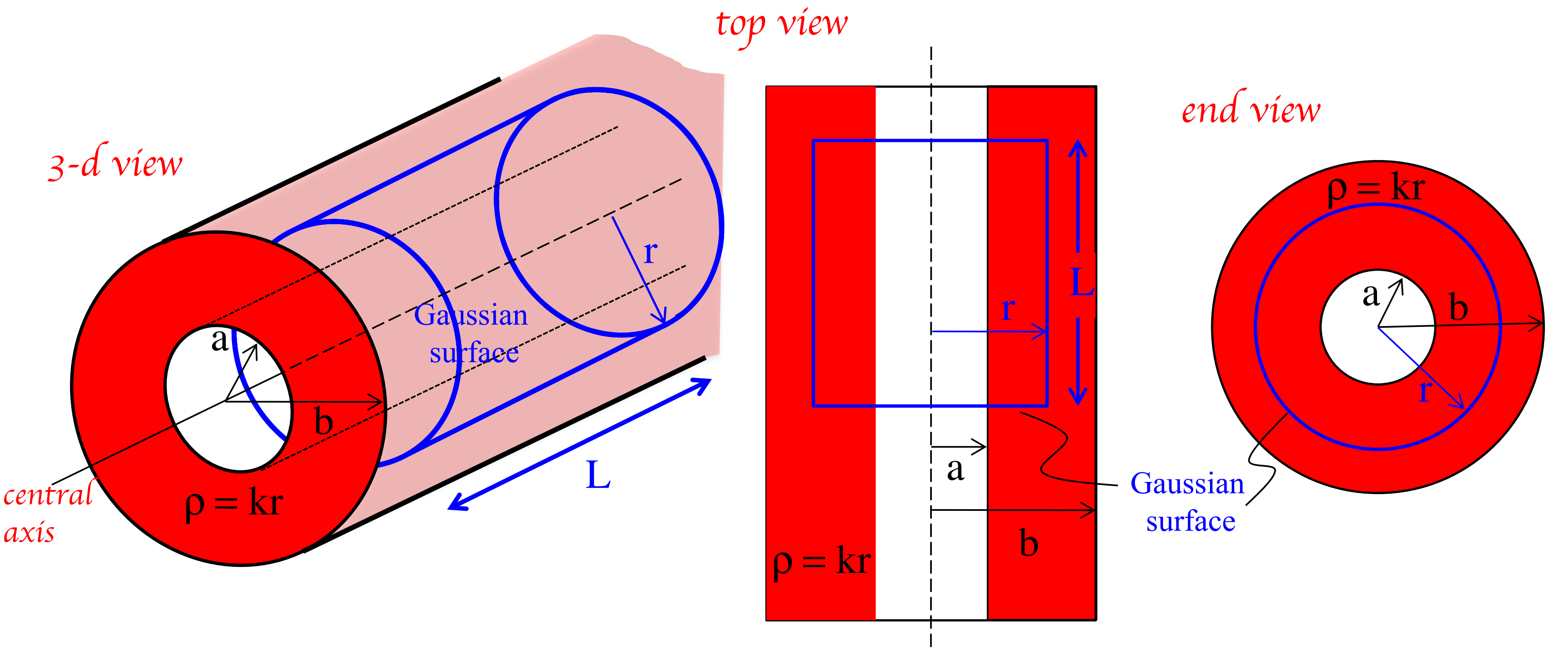
$$\Rightarrow E(2\pi r \cancel{L}) = \frac{\lambda \cancel{L}}{\varepsilon_o}$$

$$\Rightarrow E = \frac{\lambda}{2\pi\varepsilon_o r}$$

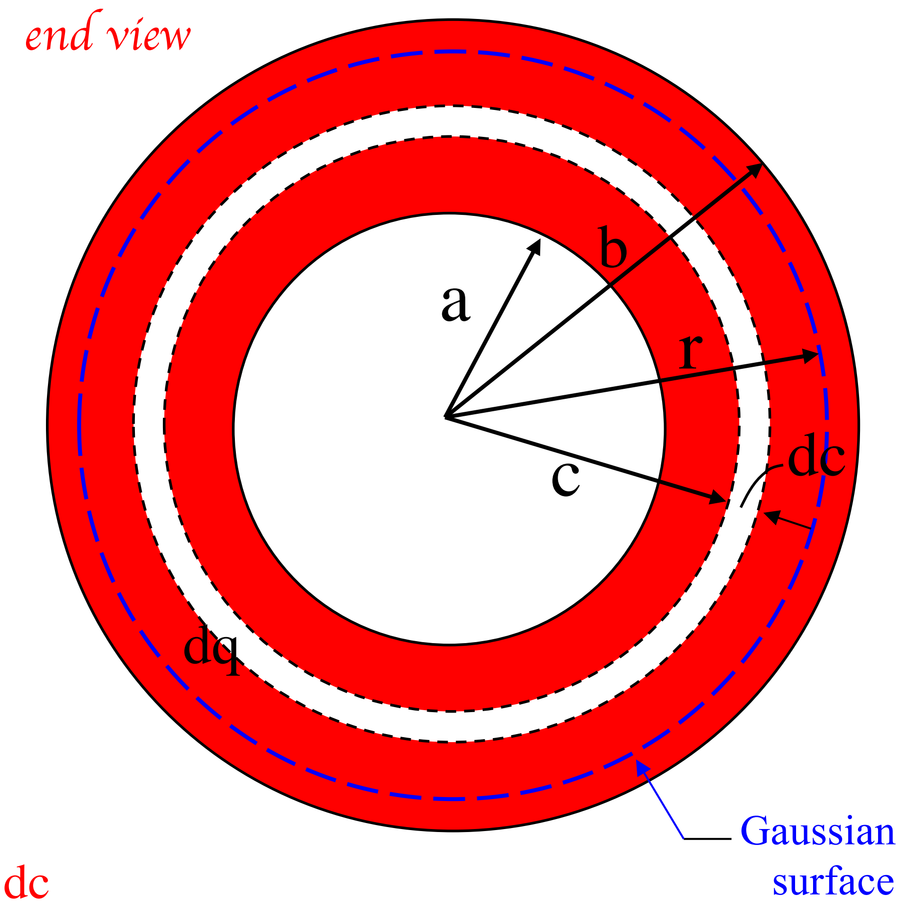
Example 9: A cylinder of inside radius a and outside radius b has a volume charge density that varies as r (i.e., $\rho = kr$, where k is a constant).

a.) Derive the *electric field function* for $a < r < b$:

The set-up is shown below:



We need the total charge inside a Gaussian cylinder, but as the density varies, we need to determine the charge in a *differentially thin cylindrical shell* inside the Gaussian surface, then integrate over all the shells.



The differential volume of a *differentially thin cylindrical shell* of radius *c* is the circumference of the shell ($2\pi c$) times the differential thickness of the shell *dc* times the length of the Gaussian cylinder *L*. That is: $dV = (2\pi cL)dc$

Knowing that $dq = \rho dV$, we can write *dq* evaluated at *c* as:

$$\begin{aligned} dq &= \rho dV \\ &= (kc)[(2\pi cL)dc] \end{aligned}$$

With that, Gauss's Law becomes:

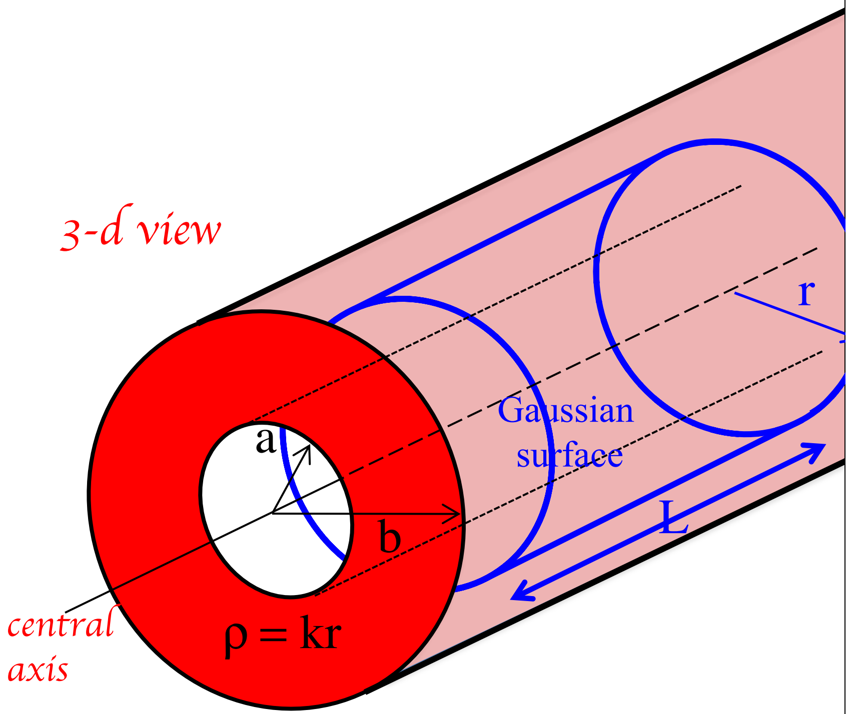
$$\int_S \vec{E}\bullet d\vec{S} = \frac{\int_{c=a}^{r}\rho\, dV}{\varepsilon_o}$$

$$\Rightarrow \quad E(2\pi rL) = \frac{\int_{c=a}^{r}(kc)\left[(2\pi cL)dc\right]}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{\cancel{2}\cancel{\pi}k\cancel{L}}{\cancel{2}\cancel{\pi}\varepsilon_o r\cancel{L}}\int_{c=a}^{r} c^2\, dc$$

$$= \frac{k}{\varepsilon_o r}\left(\frac{c^3}{3}\right)\Bigg|_{c=a}^{r}$$

$$= \frac{k}{3\varepsilon_o r}\left(r^3 - a^3\right)$$



b.) Derive an *electric field* for $r < a$: (It's zero as no charge inside Gaussian surface.)

c.) Derive an *electric field* for $r > b$:

Same problem as *Part a* exception of the limits of the integration are different (you are now adding up ALL the charge inside the cylinder, so the limits go from *c = a* to *c = b* instead of *c = a* to *c = the Gaussian radius r*.)

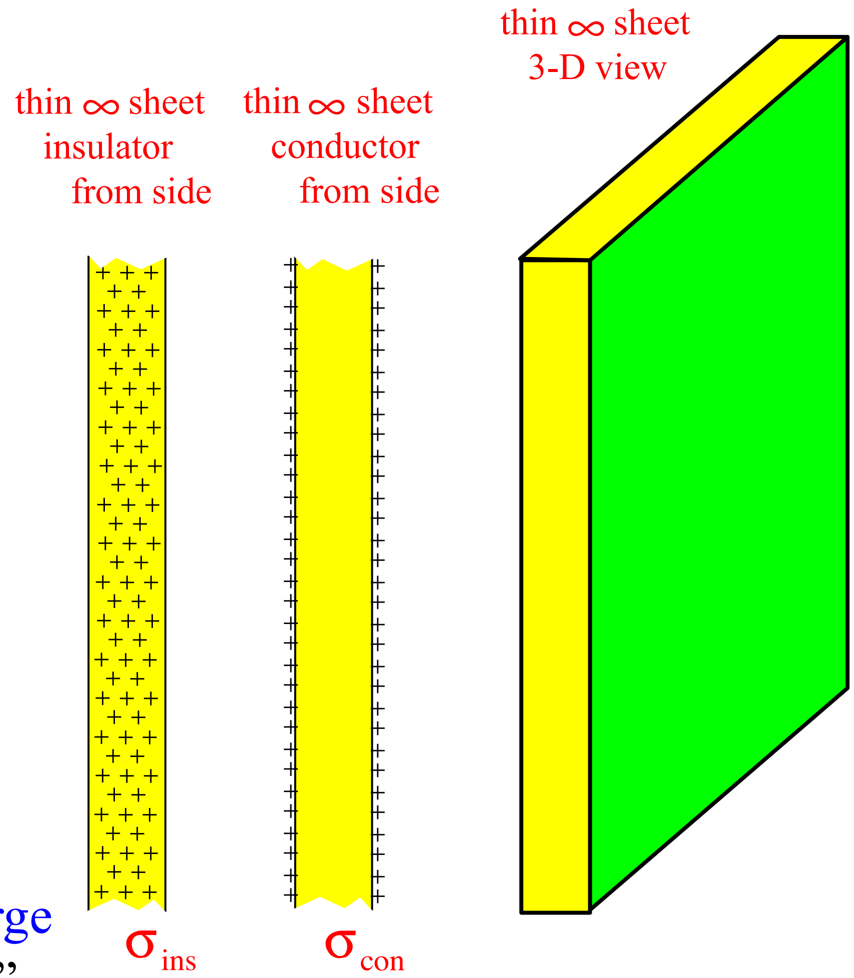
Flat Sheets of Charge and Gauss's Law

There is a subtlety we need to look at before we can make sense out of sheets of charge.

--*An insulator* typically has free charge infused throughout its volume. For *insulators*, an *area charge density function* σ_{ins} say, "Multiply me by a surface area and you get how much charge is *behind that surface* shot through the structure."

--*A conductor* has charge on its surface with NO free charge in its interior. For *conductors*, an *area charge density function* σ_{con} say, "Multiply me by a surface area and you get how much charge is *on the surface* of the area in question."

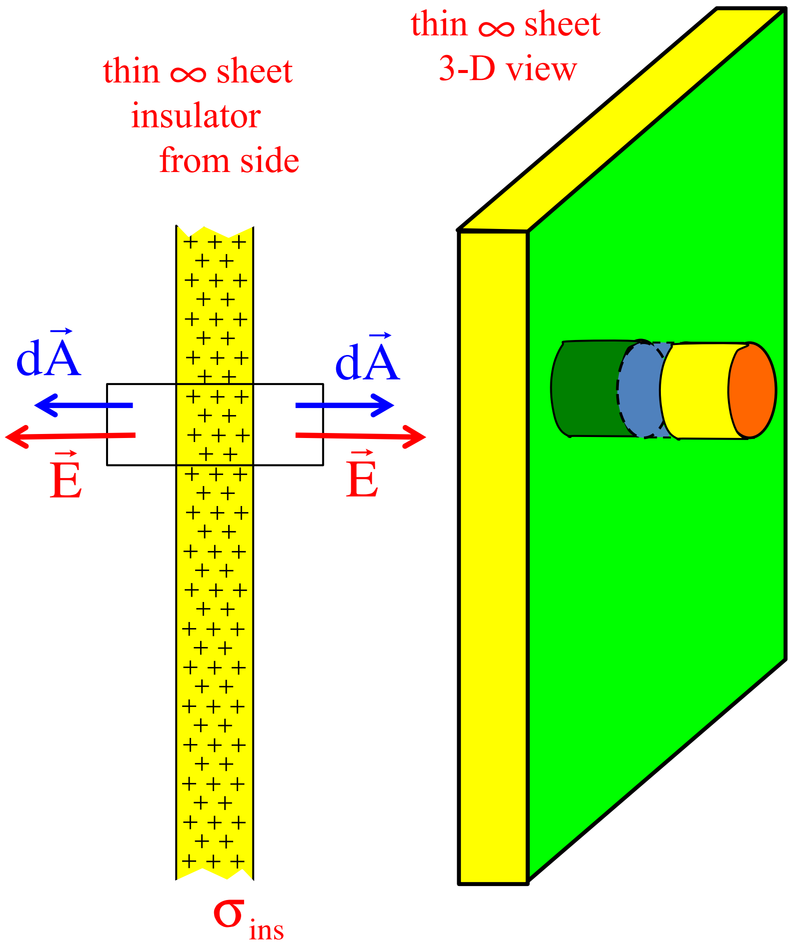
These are two very different critters!



With that in mind:

Example 10: Derive an expression for the *electric field function* for an "infinite" sheet of insulating material whose area charge density is a constant σ_{ins}.

We need a Gaussian surface whose faces either yield *zero flux* or a *flux* whose *electric field evaluation* is a constant. It turns out that a plug whose end-faces are symmetrically placed on either side of the slab will do the job. The flux through the curved section will be zero as E and dA will be at right-angles with one another, and with dA defined as outward on both outside faces, E will both have the same evaluation and will provide the same *dot product*. That is:



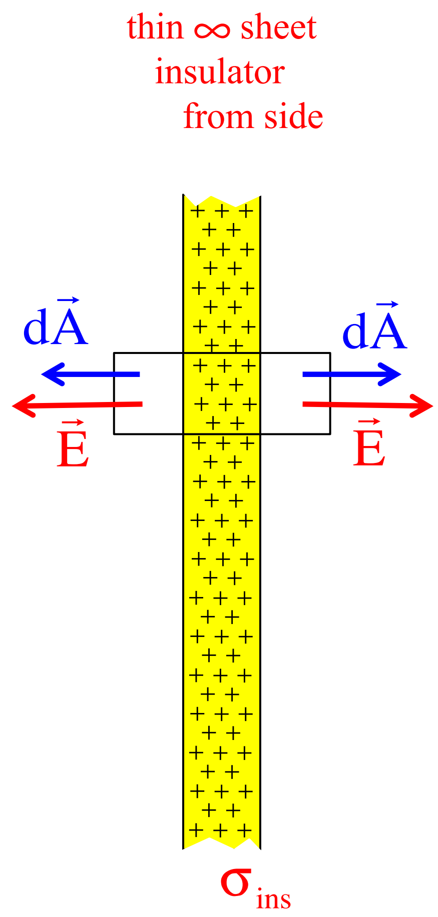
$$\int_{curve} \vec{E} \bullet d\vec{A}_{curve} + 2\int_{end} \vec{E} \bullet d\vec{A}_{end} = \frac{q_{cnclosed}}{\varepsilon_o}$$

$$\Rightarrow \int_{curve} (E)\,dA_{curve} \cancelto{0}{\cos 90^o} + 2\int_{end} (E)\,dA_{end} \cos 0^o = \frac{\sigma_{ins} A_{end}}{\varepsilon_o}$$

$$\Rightarrow 2E\int_{end} dA_{end} = \frac{\sigma_{ins} A_{end}}{\varepsilon_o}$$

$$\Rightarrow 2E\cancel{A_{end}} = \frac{\sigma_{ins} \cancel{A_{end}}}{\varepsilon_o}$$

$$\Rightarrow E = \frac{\sigma_{ins}}{2\varepsilon_o}$$



Example 11: Derive an expression for the *electric field function* for an "infinite" sheet of conducting material whose area charge density is a constant σ_{con}.

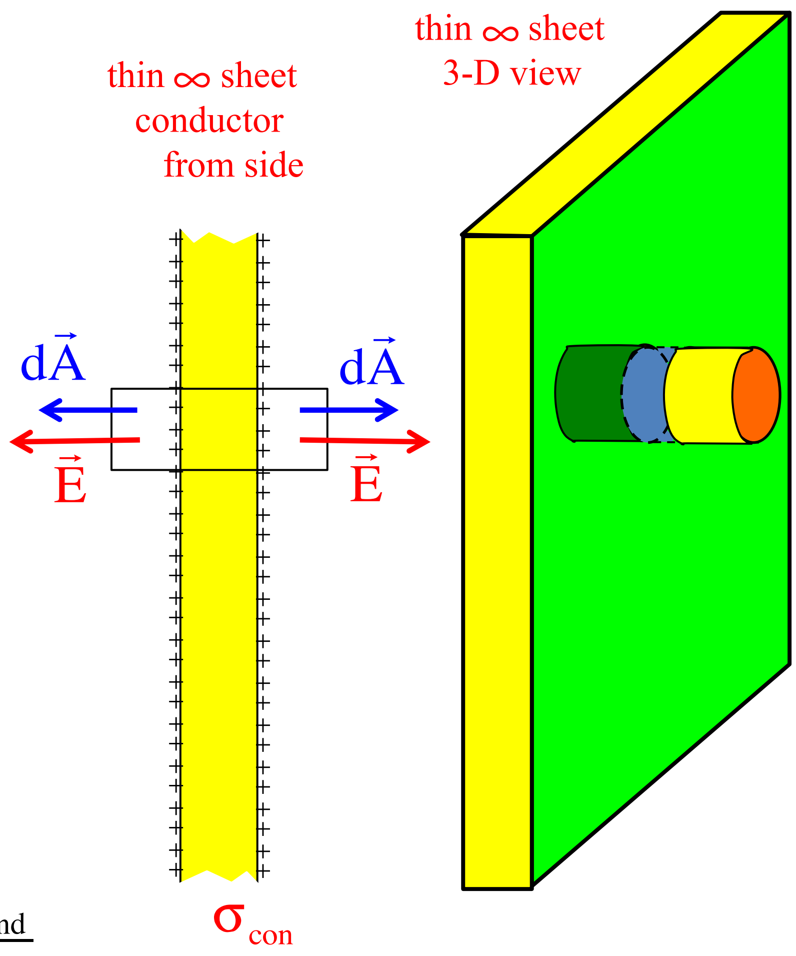
Here is where the difference in the charge configurations comes into play. We could use the same plug we used with the insulator, but there would be two surfaces upon which there was charge placed, each of which would have a charge density of σ_{con}. That means:

$$\int_{curve} \cancelto{0}{\vec{E}\bullet d\vec{A}_{curve}} + 2\int_{end} \vec{E}\bullet d\vec{A}_{end} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \quad 2E\int_{end} dA_{end} = \frac{\sigma_{con}A_{end} + \sigma_{con}A_{end}}{\varepsilon_o}$$

$$\Rightarrow \quad 2E\cancel{A}_{end} = \frac{2\sigma_{con}\cancel{A}_{end}}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{\sigma_{con}}{\varepsilon_o}$$



As a side point, another possibility would be to utilize the fact that *inside* the conductor, the electric field is zero. That means the Gaussian surface *could* have looked like:

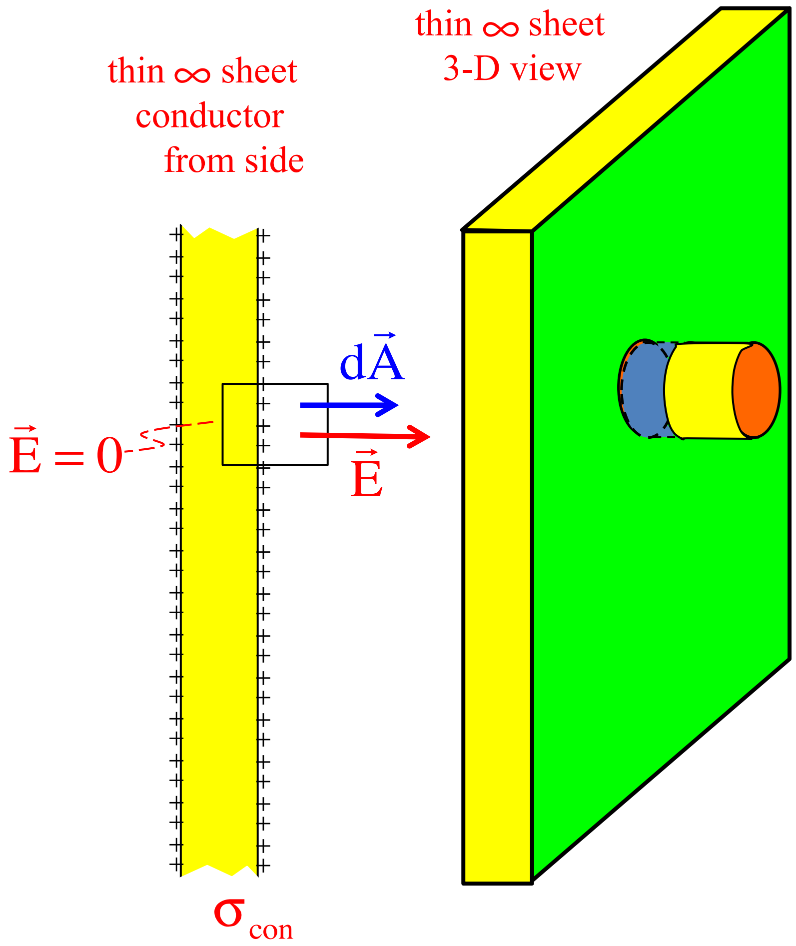
and:

$$\int_{curve} \cancelto{0}{\vec{E} \bullet d\vec{A}_{curve}} + \int_{end} \vec{E} \bullet d\vec{A}_{end} = \frac{q_{cnclosed}}{\varepsilon_o}$$

$$\Rightarrow \quad E\int_{end} dA_{end} = \frac{\sigma_{con} A_{end}}{\varepsilon_o}$$

$$\Rightarrow \quad E\cancel{A}_{end} = \frac{\sigma_{con} \cancel{A}_{end}}{\varepsilon_o}$$

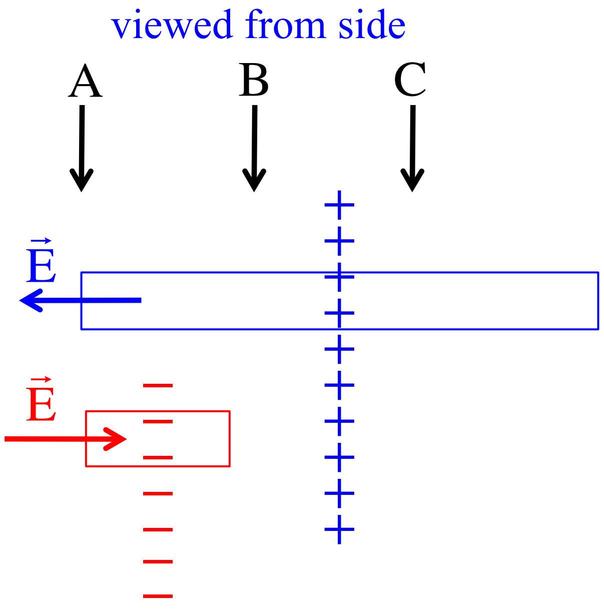
$$\Rightarrow \quad E = \frac{\sigma_{con}}{\varepsilon_o}$$



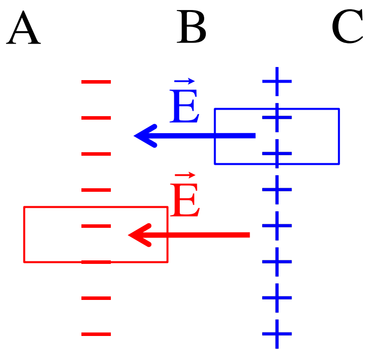
Same result either way!

Parallel Sheets of Charge

Example 12: Two infinite, *parallel sheets of opposite charge* sit side by side and have a charge density of σ on each. What is the electric field intensity on either side, and in-between the configurations?



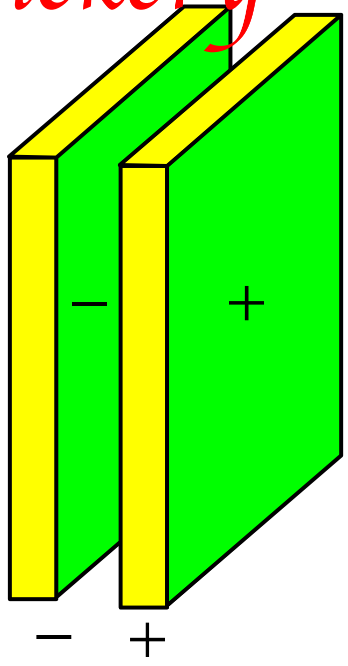
The Gaussian surfaces for both the positive and negative charge configuration is shown for region A. Each is identical to the configuration associated with an insulator. As such, each *electric field magnitude* will be equal to $\sigma/2\varepsilon_o$. As the fields will be in opposite directions, though, they will add to zero. The same will be true in region C.



Between the sheets in region B, the fields are in the same direction with the same magnitude, so we have a net field of

$$2\left(\sigma/2\varepsilon_o\right) = \sigma/\varepsilon_o$$

Parallel Plates and a Bit of Trickery



Example 13: Two infinite *conducting plates* with equal but opposite *charge densities* σ on them sit side by side. What is the electric field intensity between the plates?

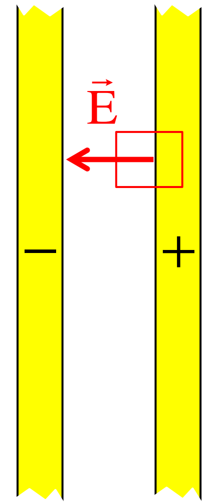
Assuming the plates are very close together (in comparison to their plate area), we can approximate them as infinite conducting sheets. If we do that, Gauss's Law yields:

$$E\int_{end} dA_{end} = \frac{\sigma A_{end}}{\varepsilon_o}$$

$$\Rightarrow \quad EA_{end} = \frac{\sigma A_{end}}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{\sigma}{\varepsilon_o}$$

from side



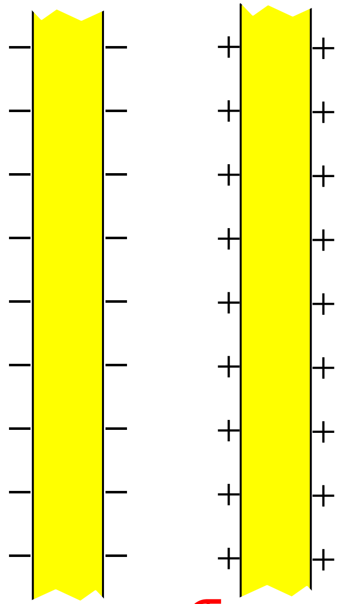
But an intrepid observer will undoubtedly notice that THIS IS THE SAME FUNCTION we got for a SINGLE infinite sheet . . . so how can *that* be ('cause there's two sheets here!)?

The trickiness is, again, in the definition of σ .

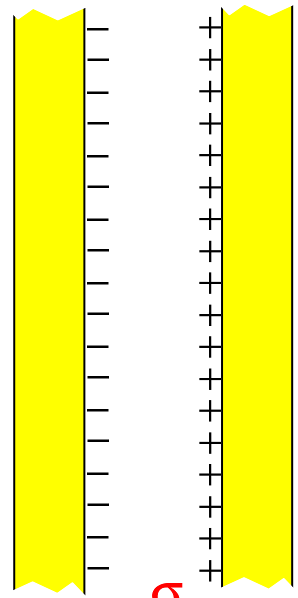
If you start with a sheet that has charge density σ_1 on it, then bring a similar but oppositely charged plate in parallel to it, what's going to happen to the original plate?

Electrons in the original plate will migrate away from the negative charge on the encroaching plate, leaving the original plate's inside surface MORE ELECTRICALLY POSITIVE than it had been (and as that happens, more negative charge will be drawn to the inside surface of the encroaching plate keeping the charge distributions equal and opposite). In other words, it's *plate charge density* will have changed to a new value σ_2.

In other words, Gauss's Law will still worked, but the charge density function will be altered from the original situation.



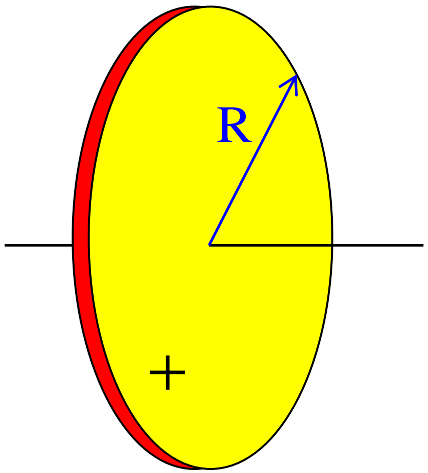
σ_1



σ_2

A Disk

Example 14: Can you use Gauss's Law to determine the *electric field function* for a flat, finite, positively charged disk?



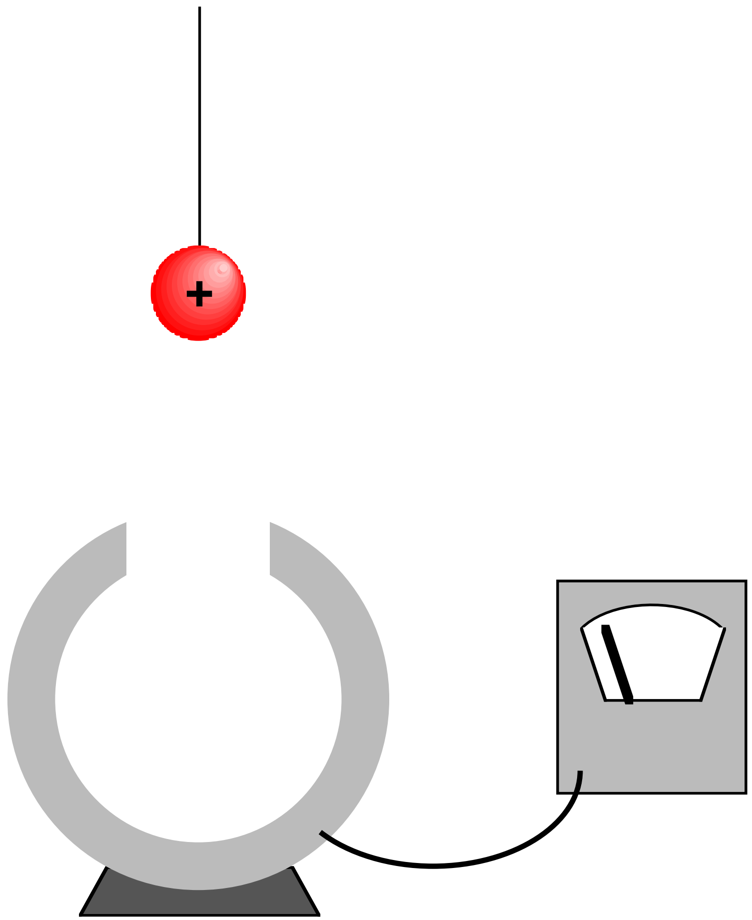
Technically, there is no Gaussian surface that will accommodate the symmetry (or lack of symmetry) associated with a disk. However, you could use a plug and the technique used with the infinite sheet of charge as long as your point of interest was *close to the central axis* and r was small in comparison to the radius R of the disk.

Example 15

Physlet E.24.3

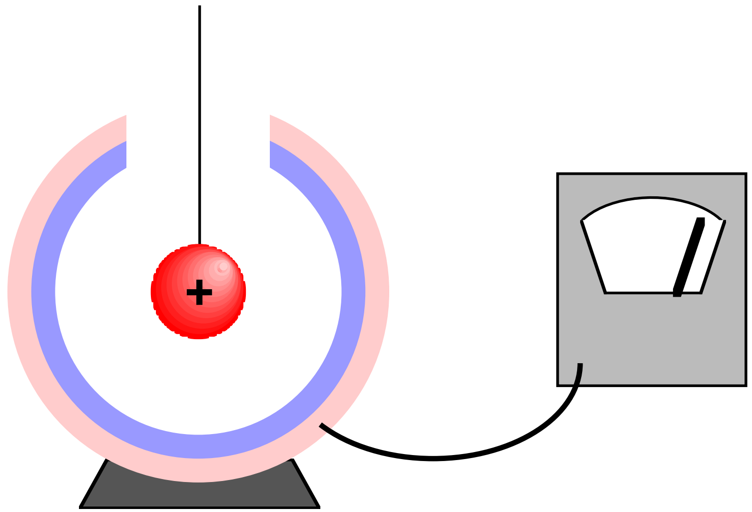
Example 16 - Faraday’s Ice Pail Experiment

Courtesy of Mr. White:



Example 16 - Faraday' s Ice Pail Experiment

Courtesy of Mr. White:



Example 16 - Faraday's Ice Pail Experiment

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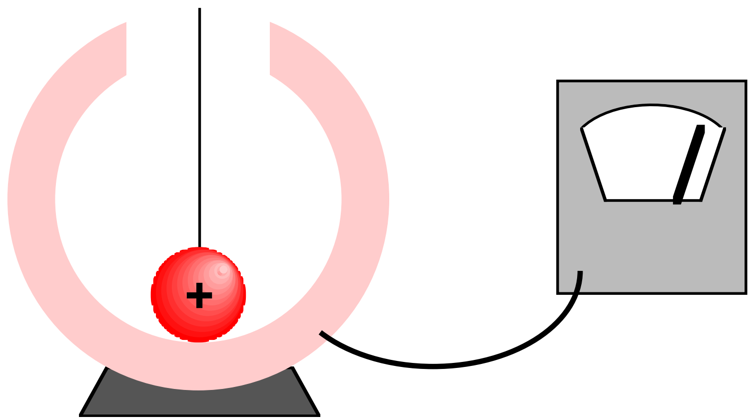

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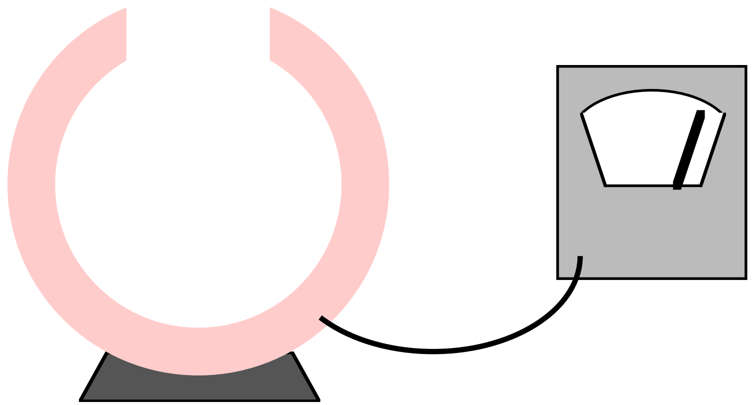

Example 16 - Faraday’s Ice Pail Experiment

Courtesy of Mr. White: