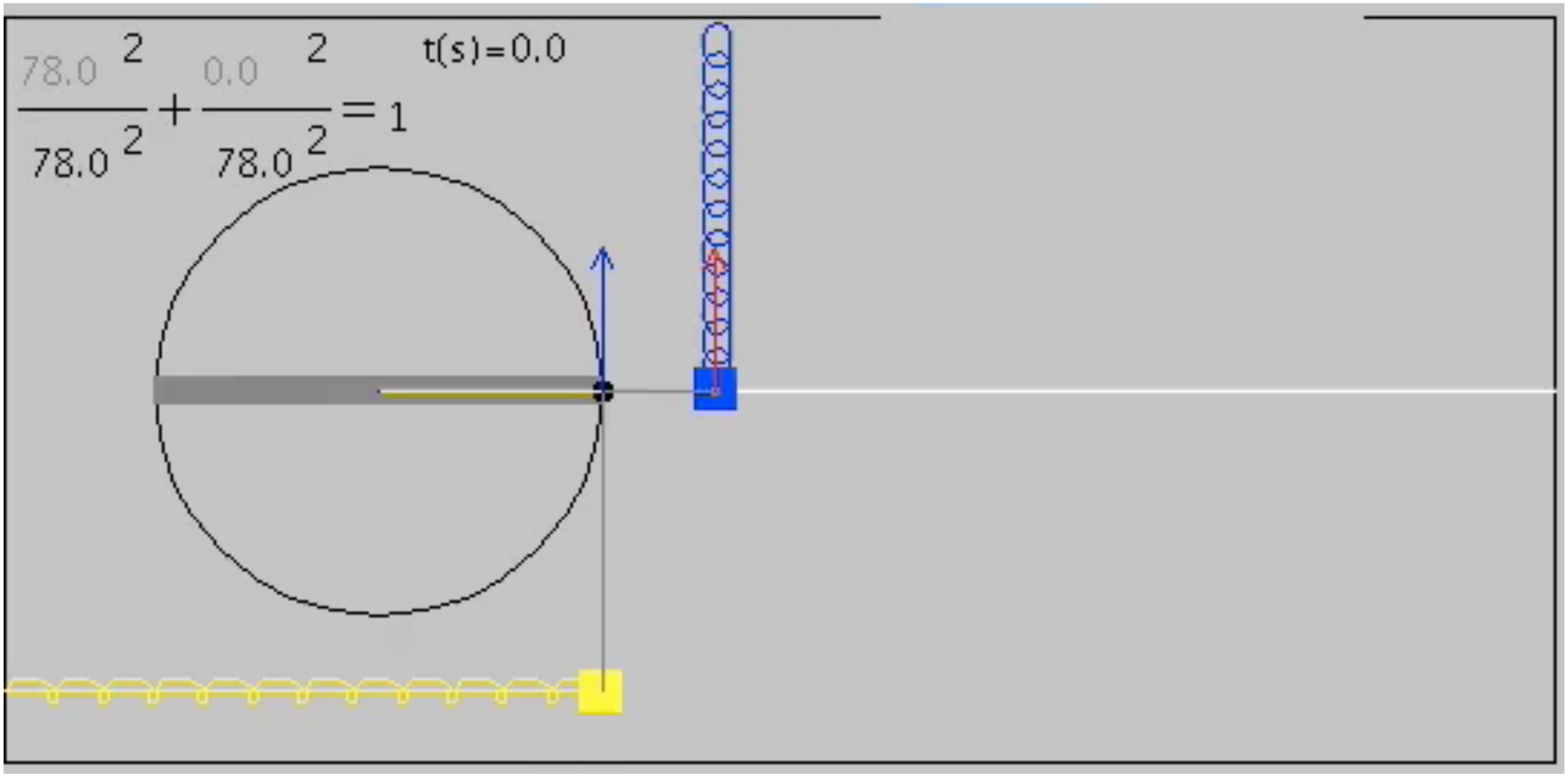
CHAPTER 15:
Vibratory Motion



courtesy of Richard White

The Island Series:

You have been kidnapped by a crazed physics nerd and left on an island with twenty-four hours to solve the following problem. Solve the problem and you get to leave. Don't solve the problem and you don't.

The problem: Your demented host has a precocious child who wants to dig a hole through the center of the earth and jump into it. Thinking the kid is a genius who can do anything he puts his mind to, the parent assumes the kid will succeed, periodically re-emerging at the mouth of the hole after the jump (he'll accelerate down through the earth to the other side, then come back). To get a picture of the kid every time he shows himself at the original hole site, the parent wants to hire a satellite that will orbit at just the right altitude so that every time the kid's head pops out of the ground, the satellite will be overhead to take a picture. To get off the island, you must determine the appropriate orbital altitude the satellite needs to maintain to be able to do this?

Solution to Island Problem

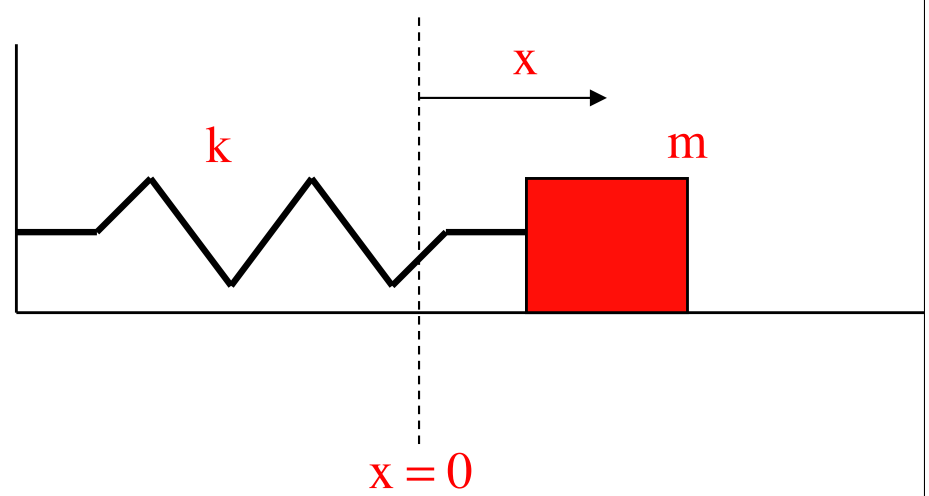
You'll see!

Two glaring observations can be made from the graphic on the previous slide:

1.) *The PROJECTION of a point* on a circle moving with constant *angular velocity* ω follows the same path as a mass attached to an ideal *hanging spring* as the spring oscillates up and down. And:

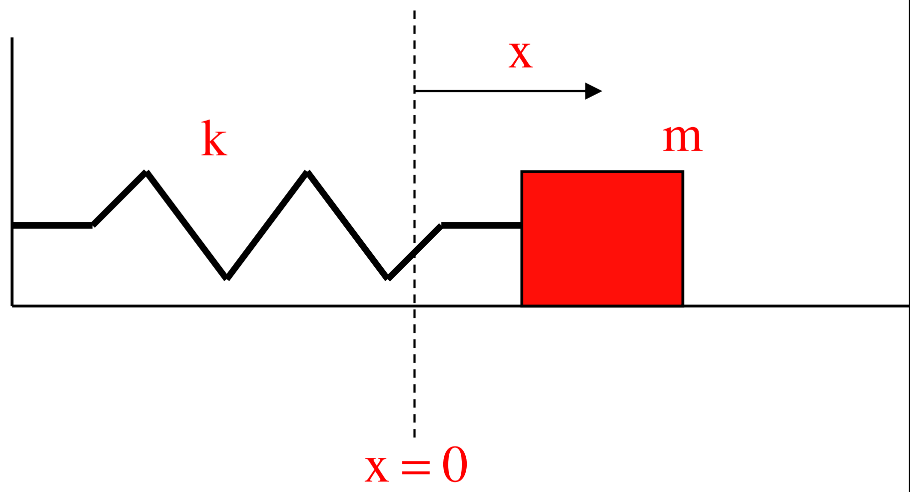
2.) *If you track* the oscillation in time, it traces out a sinusoidal path.

Both of these observations fall out from the math if we start with *Newton's Second Law* applied to a mass *m* attached to an ideal spring oscillating over a frictionless, horizontal surface.



That analysis follows:

Keeping the sign of the acceleration embedded (it will be either positive or negative, depending upon the point in time, so we'll leave it implicit), and noting that the spring force in the x-direction on a mass attached to the spring will be $F_{spring} = -(kx)\hat{i}$ (Hooke's Law), Newton Second Law yields:



$$\sum F_x :$$

$$-kx = ma_x$$

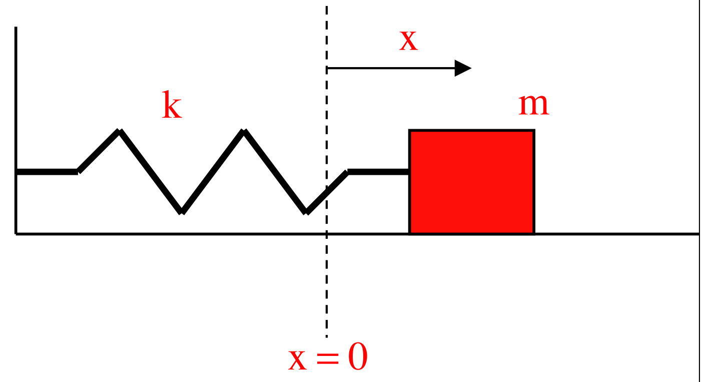
$$= m\left(\frac{d^2x}{dt^2}\right)$$

Pulling the $-kx$ to the right side and dividing by *m* yields the relationship:

$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$$

This is the characteristic equation of *SIMPLE HARMONIC MOTION*.

This relationship:



$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$$

essentially asks us to find a function *x* such that when you take its second derivative and add it to a *constant* times itself, the sum will always add to zero.

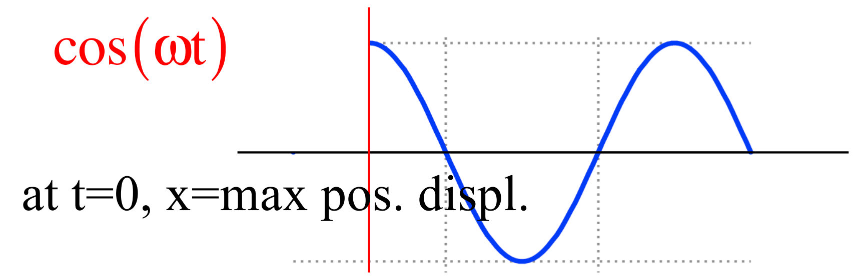
The function that does this is either a cosine or a sine (I usually use a *sine*, but your book for no particularly good reason uses a cosine, so we'll use that).

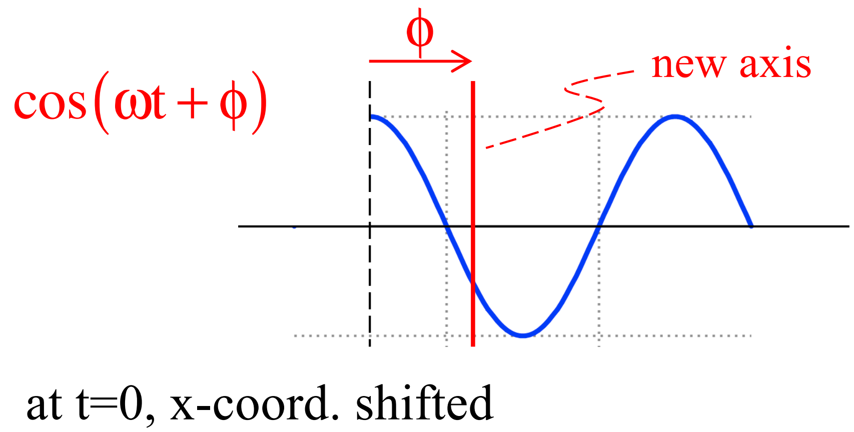
There are restrictions on the *cosine function* we need. In fact, we want the most general form possible. Specifically:

1.) *We need* the cosine's angle to be *time dependent*, so instead of using an angle θ, we will use a *constant* times *t*, where the *constants units* have to be *radians/second*. As we have already run into a variable with those units (ω), we will use that symbol. (Interesting note: If the *angular velocity* of the rotating point-on-the-circle shown in the first slide had been ω, the constant in question for the vibratory motion's cosine function would have been that same number ω.)

2.) *We need* the ability to start the clock when we want. A simple *cosine function* sets the position to be at a positive maximum at t = 0 (see sketch).

We need to be able to shift the axis by some *phase shift* amount ϕ, essentially starting the clock (i.e., setting t = 0) when the body is at *any chosen* x-coordinate.





3.) *Lastly,* we need to be able to accommodate motion whose *maximum displacement* is other than one.

The function that does all of this for us is:

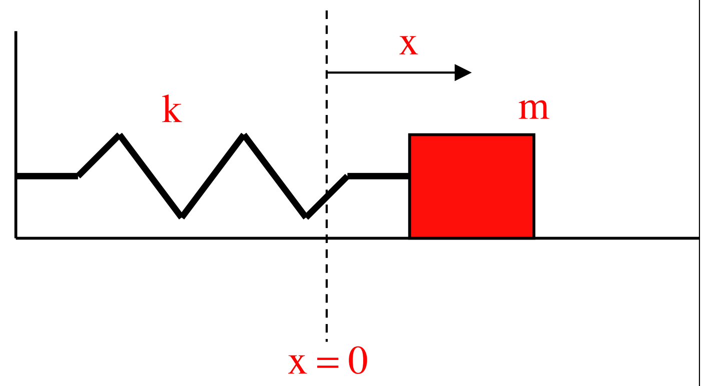
$$x = A\cos(\omega t + \phi)$$

So back to the problem at hand. Does

$$x = A\cos(\omega t + \phi)$$

satisfy

$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0 ?$$



The only way to tell is to try it out:

$$\frac{dx}{dt} = \frac{d\left(A\cos(\omega t + \phi)\right)}{dt}$$

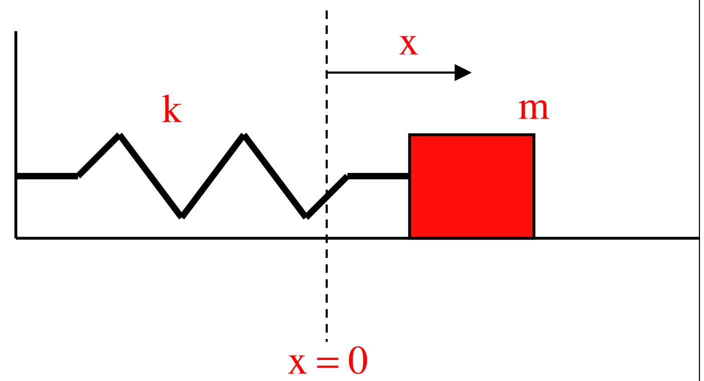
$$= -\omega A\sin(\omega t + \phi)$$

This, by the way, is the *velocity function*. And as a *sine function* can never be larger than one, this means the magnitude of the maximum velocity for this oscillatory motion will be:

$$v_{max} = \omega A$$

This will happen when the force is completely spent, or at equilibrium.

Continuing:



$$\frac{d^2x}{dt^2} = \frac{d\left(-\omega A \sin(\omega t + \phi)\right)}{dt}$$

$$= -\omega^2 A \cos(\omega t + \phi)$$

Another side point: This means the magnitude of the *maximum acceleration*, which happens at the extremes where the spring force is maximum, will be:

$$a_{max} = \omega^2 A$$

Plugging everything in:

$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right) x = 0$$

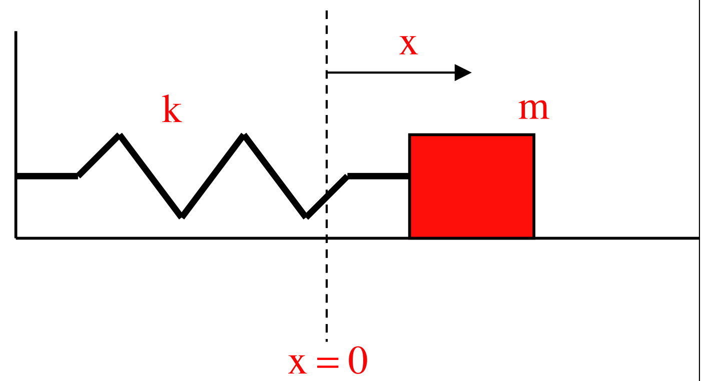
$$\left[-\omega^2 \cancel{A\cos(\omega t + \phi)}\right] + \left(\frac{k}{m}\right)\left[\cancel{A\cos(\omega t + \phi)}\right] = 0$$

$$\Rightarrow \quad -\omega^2 + \left(\frac{k}{m}\right) = 0$$

$$\Rightarrow \quad \omega = \left(\frac{k}{m}\right)^{1/2}$$

In other words, the *differential equation*

$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$$



is satisfied by the position function

$$x = A\cos(\omega t + \phi)$$

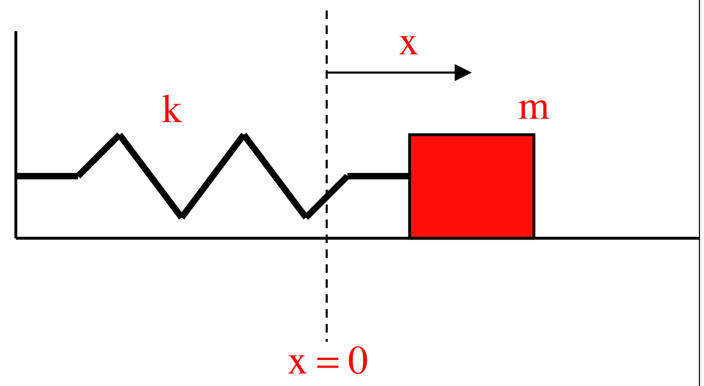
as long as the *angular frequency* ω satisfies: $\omega = \left(\frac{k}{m}\right)^{1/2}$

Big Note: Notice that in concluding that $\omega = \left(\frac{k}{m}\right)^{1/2}$, we are saying that the *angular frequency* of our oscillating system is equal to the *square root of the constant* that sits in front of the *position* term in the *Newton's Second Law* equation! Put a little differently, if you can get a N.S.L. evaluation into the form:

$$\text{acceleration} + (\text{constant})(\text{position}) = 0$$

you will know the oscillation is *simple harmonic* in nature AND you will know that the *angular frequency* of the system is $\omega = (\text{constant})^{1/2}$.

Minor point: So what is the angular frequency ω really doing for us? It is simply another way to identify how quickly the system is oscillating back and forth. But instead of telling us the frequency ν in *cycles per second*, it is telling us how many *radians* being swept through per cycle. Noting that there are 2π *radians per cycle*, the relationship between *frequency* and *angular frequency* is:



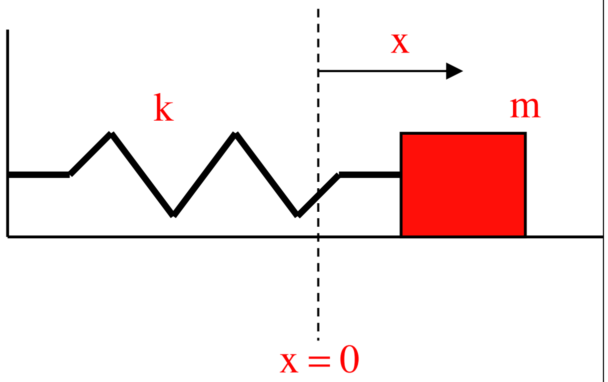
$$\omega = 2\pi\nu$$

And as the frequency ν in *cycles per second* is the inverse of the number of seconds required to traverse one cycle (or the period *T* in *seconds per cycle*), we can also write:

$$T = \frac{1}{\nu}$$

In short, if we can derive an expression for ω for a system, we also know ν and *T*.

Energy in a Spring System



At the extremes where the velocity is *zero* and acceleration a maximum:

$F_{max} = -kA$, where *A* is the *maximum displacement* (the amplitude), so:

$$E_{total} = \frac{1}{2}m\cancelto{0}{v^2} + \frac{1}{2}kA^2$$
$$= \frac{1}{2}kA^2$$

At the equilibrium where the acceleration is *zero* and velocity a maximum:

$$E_{total} = \frac{1}{2}m{v_{max}}^2 + \frac{1}{2}k\cancelto{0}{x^2}$$
$$= \frac{1}{2}m(\omega A)^2$$

Summary of Relationships

Relationships always true:

$\frac{d^2x}{dt^2} + (\kappa)x = 0$ or $\alpha + (\kappa)\theta = 0$	characteristic equation for simple harmonic motion
$\omega = (\kappa)^{1/2}$	angular frequency from characteristic equation
$\omega = 2\pi\nu$	angular frequency and frequency related
$T = \frac{1}{\nu}$	period inversely related to frequency
$x = A\cos(\omega t + \phi)$	position function for s.h.m.
$v = dx/dt$ and $a = dv/dt = d^2x/dt^2$	velocity and acceleration functions
$v_{max} = \omega A$	maximum velocity (happens at equilibrium)
$a_{max} = \omega^2 A$	maximum acceleration (happens at extremes)

Summary of Relationships

For a spring:

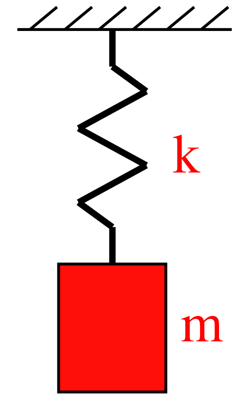
$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0$$ characteristic equation for simple harmonic motion

$$\omega = \left(\frac{k}{m}\right)^{1/2}$$ angular frequency from characteristic equation

$$E_{tot} = \frac{1}{2}kA^2$$ total mechanical energy in system

The period of oscillation for a spring is *constant* no matter what the spring's amplitude. How so? A larger displacement will require more distance traveled to execute a single cycle, but because force is a function of displacement, it will also generate a larger maximum force, so the *period* will stay the same no matter what!

REALLY SIMPLE Example 1: A spring with spring constant *k* is hung from the ceiling. A mass *m = .4 kg* is attached and allowed to gently elongate the spring until it comes to rest at a point .6 meters below its free-hanging position. The mass is then pulled an additional .2 meters down and released.



a.) What is the spring's spring constant?

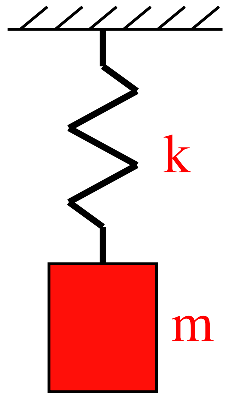
The spring constant identifies how much *force* is required to displacement the spring one meter (units: N/m). It measures, essentially, stiffness. We've been told that a mass *m*, whose weight is *mg*, displaced the spring a distance *d = .6 meters,* so we know:

$$k = \frac{F}{x} = \frac{mg}{d}$$

$$= \frac{(.4\text{ kg})(9.8\text{ m/s}^2)}{(.6\text{ m})}$$

$$= 6.53\text{ N/m}$$

b.) What is the amplitude of the resulting oscillation?



Being elongated an additional .2 meters from equilibrium means *m* will oscillate about the equilibrium point .2 meters above and below, which means the motion's amplitude is .2 meters.

c.) What is the motion's angular frequency?

Knowing m and *k,* we can write:

$$\omega = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{(6.53 \text{ N/m})}{(.4 \text{ kg})}}$$

$$= 4.04 \text{ rad/sec}$$

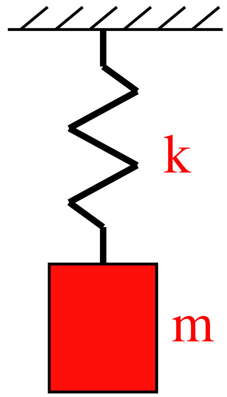
d.) What is the motion's frequency?

$$\omega = 2\pi\nu$$

$$\Rightarrow \nu = \frac{\omega}{2\pi} = \frac{4.04 \text{ rad/sec}}{2\pi}$$

$$= .643 \text{ cycles/sec} \quad \text{(or Hertz)}$$

e.) What is the *period* of the motion?



$$T = \frac{1}{\nu} = \frac{1}{.643\ \text{cycles/sec}}$$

$$= 1.56\ \text{sec/cycle}$$

f.) What is the *maximum velocity*?

$$v_{max} = \omega A$$

$$= (4.04\ \text{rad/sec})(.2\ \text{m})$$

$$= .81\ \text{m/s}$$

g.) Where is the *velocity* a *maximum*? at equilibrium

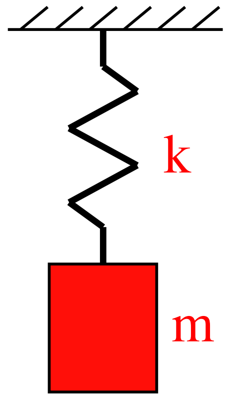
f.) What is the *maximum acceleration*?

$$a_{max} = \omega^2 A$$

$$= (4.04\ \text{rad/sec})^2 (.2\ \text{m})$$

$$= 3.26\ \text{m/s}^2$$

g.) Where is the acceleration a maximum? at the extremes *h.)* How much *mechanical energy* is in the system?

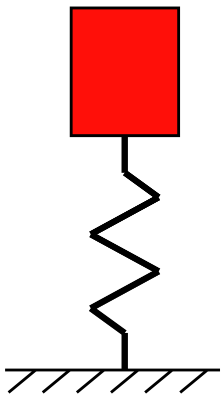


$$E = \frac{1}{2}kA^2$$

$$= \frac{1}{2}(6.53 \text{ N/m})(.2 \text{ m})^2$$

$$= .13 \text{ joules}$$

i.) How would this problem have changed if the spring had been inverted?



it wouldn't

Physical Characteristics of Springs

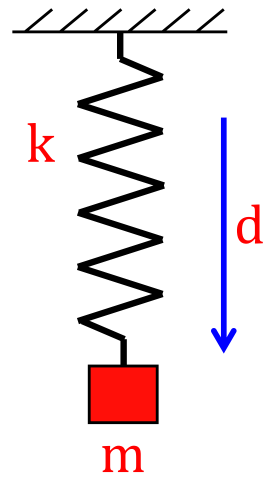
Consider a spring of spring constant k that has a mass m attached to it. What happens to the spring constant if:

a.) You cut the spring in half?

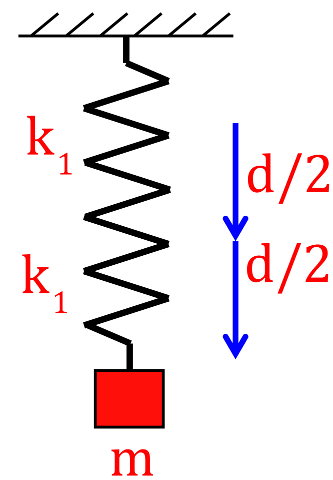
It doubles for each new spring, but how so?

Assume the mass m elongates the spring a distance d down to its equilibrium position. In that case,

$$mg = kd$$



If we put a mark halfway down the spring and think of it as two springs, one on top of the other, with each having its own new spring constant k_1, and if we notice that the force on the upper spring has to be the same as the force on the lower spring (note that this is called a *series combination* and is associated with situations in which the *stress* is the same for all parts involved) we can write:

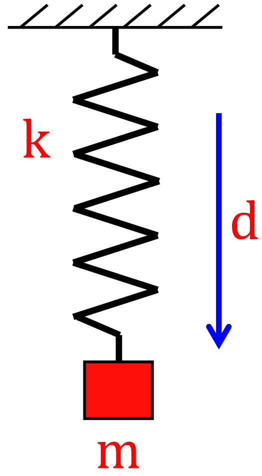


$$\begin{aligned} mg &= k_1 y_1 \\ &= k_1\left(\frac{d}{2}\right) = kd \quad \Rightarrow \quad k_1 = 2k \end{aligned}$$

a.) What happens if you cut the spring in half? (con't.)

Another way to look at it is through energy. The spring potential energy for an elongated spring stretched from its equilibrium position is:

$$U = \frac{1}{2}kd^2$$

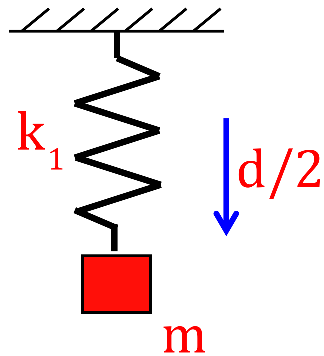


If we think of the spring as two springs attached one on top of the other, the energy content won't change and we can write:

$$U = \frac{1}{2}kd^2 = \frac{1}{2}k_1\left(\frac{d}{2}\right)^2 + \frac{1}{2}k_1\left(\frac{d}{2}\right)^2$$

$$\Rightarrow \quad kd^2 = 2k_1\left(\frac{d^2}{4}\right)$$

$$\Rightarrow \quad k_1 = 2k$$

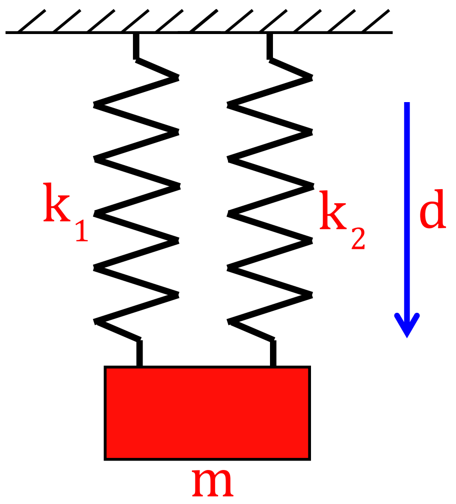


Still another way to look at it: Elongating the original spring to the 1.0 meter mark is relatively easy because there is a lot of spring to unfurl (so k, the force/unit length required to hold the spring at a particular length, is relatively small). But cut the spring in half and you are physically stretching a lot less spring to that 1.0 meter point, which means more force/length will be required to hold at that point (so bigger k).

b.) What happens if you double the spring's length?

Playing off the reasoning from above, the spring constant would halve.

c.) What happens if you put two springs side by side?



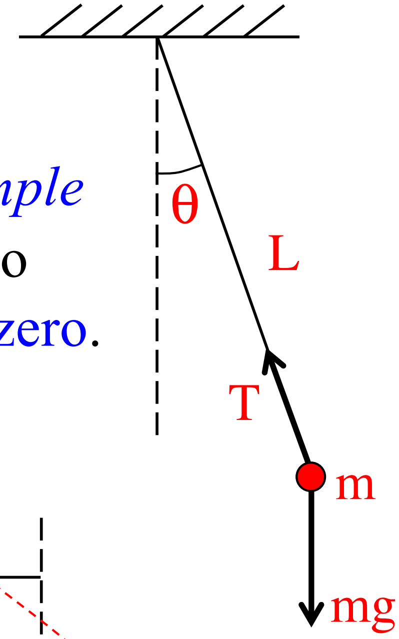
This is a parallel combination where the stress is distributed between the members, and in it the effective spring constant is the sum of the individual spring constants involved. How so?

$$F_{net} = F_1 + F_2 = k_1 y + k_2 y$$

$$\Rightarrow \quad mg = (k_1 + k_2)d$$

$$\Rightarrow \quad k_{new} = k_1 + k_2$$

Example 2: A simple pendulum of length L is observed as shown to the right. What is its period of motion?



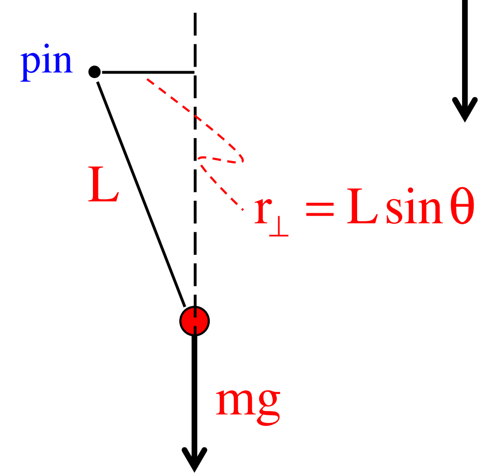
If we can show that this system's N.S.L. expression conforms to *simple harmonic motion*, we have it. As the motion is rotational, we need to sum torques about the pivot point. The torque due to the *tension* is zero. Noting that *r-perpendicular* for gravity is $L\sin\theta$, we can write:

$$\sum \tau_{pin}:$$

$$-(\cancel{m}g)(\cancel{L}\sin\theta) = I_{piin}\alpha$$

$$= (\cancel{m}L^{\cancel{2}})\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \left(\frac{g}{L}\right)\sin\theta = 0$$



This isn't quite the right form, but if we take a *small angle approximation*, we find that for $\theta <<,\ \sin\theta \to \theta$ and we can write:

$$\frac{d^2\theta}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$$

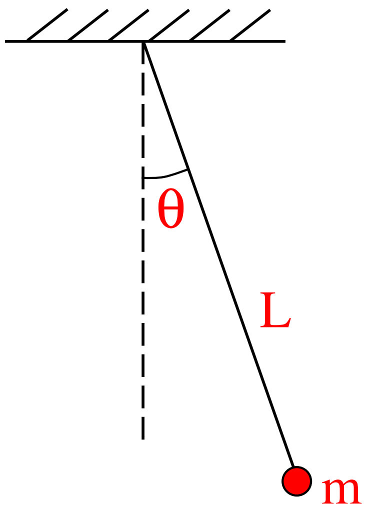
Minor Note: How do we know that the $\sin\theta = \theta$ for $\theta <<$?

The Taylor expansion for a sine function is:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$$

which suggests that for small angles, $\sin\theta \Rightarrow \theta$.

With $\frac{d^2\theta}{dt^2} + \left(\frac{g}{L}\right)\theta = 0$



We can see we are dealing with *simple harmonic motion*, which means:

$$\omega = \left(\frac{g}{L}\right)^{1/2}$$

and

$$\omega = 2\pi\nu$$

$$\Rightarrow \quad \nu = \frac{\omega}{2\pi}$$

$$\Rightarrow \quad \nu = \frac{1}{2\pi}\left(\frac{g}{L}\right)^{1/2}$$

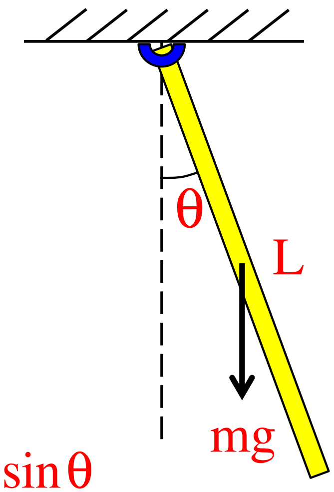
and

$$T = \frac{1}{\nu}$$

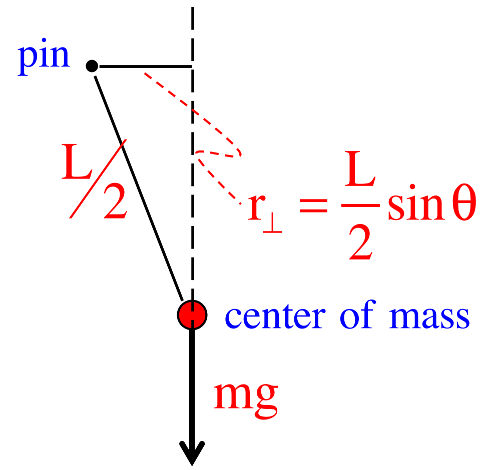
$$\Rightarrow \quad T = 2\pi\left(\frac{L}{g}\right)^{1/2}$$

And wasn't that fun . . .

Example 3: A physical pendulum of length *L* takes the form of a beam pinned at one end as shown to the right. What is its period of motion?



Again, starting with N.S.L.:



$$\sum \tau_{pin}:$$

$$-(\cancel{m}g)\left(\frac{\cancel{L}}{2}\sin\theta\right) = I_{piin}\alpha$$

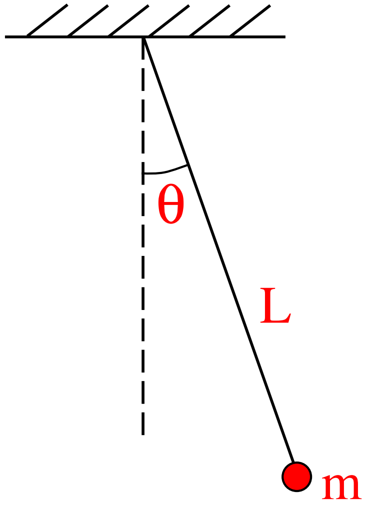
$$= \left(\frac{1}{3}\cancel{m}L^{\cancel{2}}\right)\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \left(\frac{3g}{2L}\right)\sin\theta = 0$$

Again, this isn't quite the right form, but if we take a *small angle approximation*, we find that for $\theta <<,\ \sin\theta \rightarrow \theta$ and we can write:

$$\frac{d^2\theta}{dt^2} + \left(\frac{3g}{2L}\right)\theta = 0$$

With $\frac{d^2\theta}{dt^2} + \left(\frac{3g}{2L}\right)\theta = 0$



We can see we are dealing with *simple harmonic motion*, which means:

$$\omega = \left(\frac{3g}{2L}\right)^{1/2}$$

and

$$\omega = 2\pi\nu$$

$$\Rightarrow \quad \nu = \frac{\omega}{2\pi}$$

$$\Rightarrow \quad \nu = \frac{1}{2\pi}\left(\frac{3g}{2L}\right)^{1/2}$$

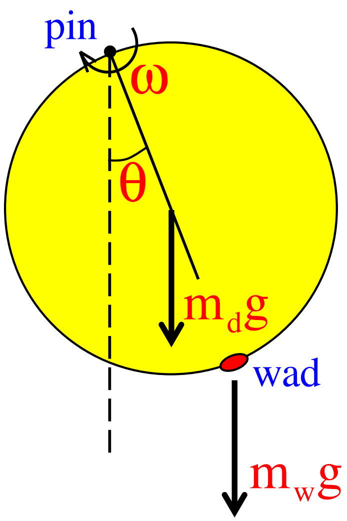
and

$$T = \frac{1}{\nu}$$

$$\Rightarrow \quad T = 2\pi\left(\frac{2L}{3g}\right)^{1/2}$$

And wasn't THAT fun . . .

Example 4: An oddball physical pendulum consists of a disk of mass *m* and radius *R* with a wad of mass *.2m* attached at its edge as shown. What is its period of motion?



Using the parallel axis theorem on the disk and the definition of moment of inertia for a point mass on the wad, we can determine the net *moment of inertia* about the pin as:

$$I_{pin} = \left(I_{cm/disk} + M_{disk}R^2\right) + m_{wad}\left(2R\right)^2$$

$$= \left(\frac{1}{2}mR^2 + mR^2\right) + \left(.2m\right)\left(2R\right)^2$$

$$= 2.3mR^2$$

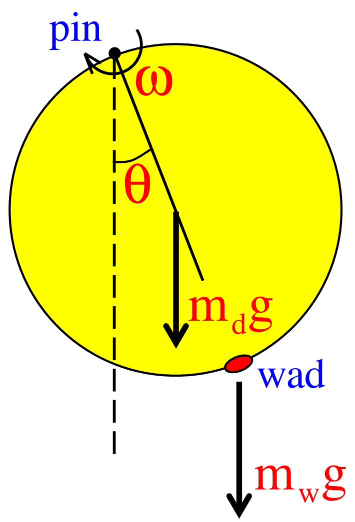
Taking the torque about the pin, yields:

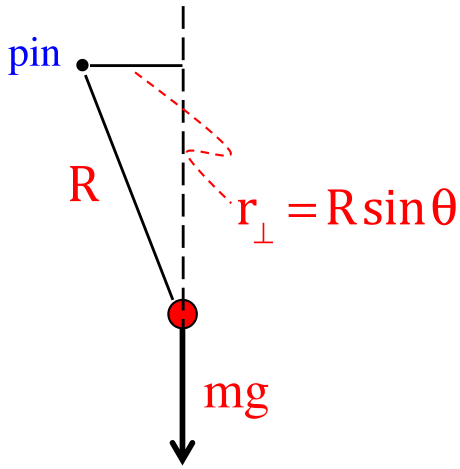
$$\sum \tau_{pin}:$$

$$-(mg)(R\sin\theta)-(.2mg)(2R\sin\theta)=I_{pin}\alpha$$

$$\Rightarrow \quad -1.4mgR\sin\theta=(2.3mR^2)\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2}+\left(\frac{.61g}{R}\right)\sin\theta=0$$





With $\theta <<, \ \sin\theta \rightarrow \theta$ and $\frac{d^2\theta}{dt^2}+\left(\frac{.61g}{R}\right)\theta=0$

$$\Rightarrow \quad \omega=\left(\frac{.61g}{R}\right)^{1/2} \quad \Rightarrow \quad \nu=\frac{\omega}{2\pi}=\frac{1}{2\pi}\left(\frac{.61g}{R}\right)^{1/2}$$

$$\Rightarrow \quad T=\frac{1}{\nu}=2\pi\left(\frac{R}{.61g}\right)^{1/2}$$

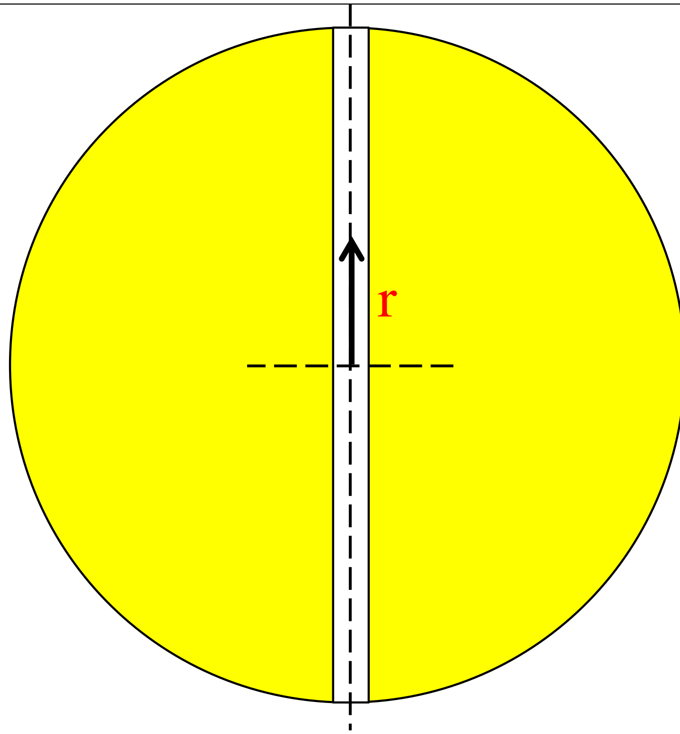
Example 5--an old AP question: A spring whose force magnitude is equal to $|\vec{F}| = -kx^3$ is *displaced* a distance *A* and released whereupon its *period* is determined to be *T*. If it is then *displaced* a distance 2A, will its *period* go up, go down or stay the same?

This is a very cool problem because it makes you think about what you know, then extrapolate to this new situation.

What you know is that for a "normal," Hooke's Law spring where the force magnitude is $|\vec{F}| = -kx$, the *period will be constant*. That is, if the *displacement is small*, the force will be small and it will take some amount of time to cover one cycle. And if the displacement is large, the proportionally larger force will allow the additional distance to be covered in the same time . . . hence a constant period.

In this scenario, increasing the displacement means that the force will be larger, but not proportionally larger—it will be GREATER than proportionally larger. With the *larger than expected force*, the mass should cover the ground in *less time* than "usual," meaning the period should *diminish*. Bizarre, but that the case.

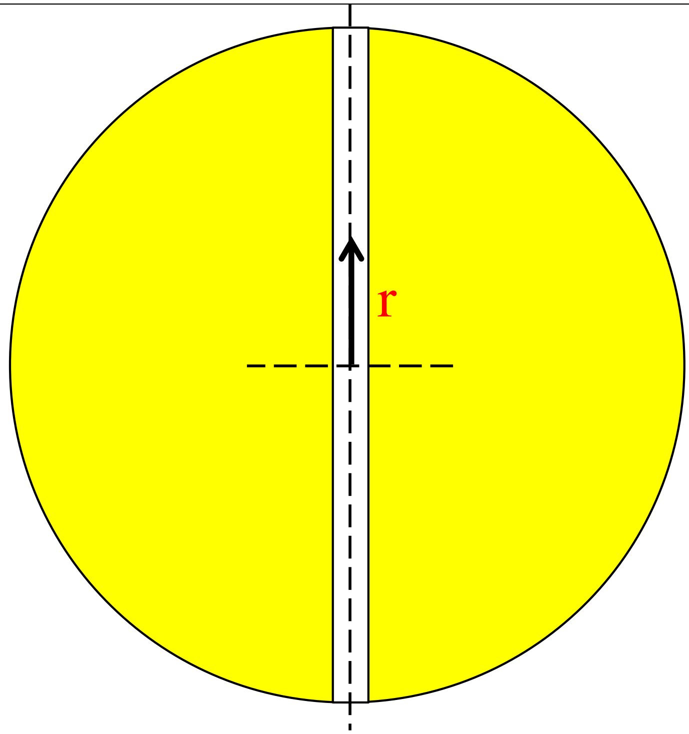
Example 6: A great problem! A kid jumps into a tunnel drilled through the earth from pole to pole. He accelerates toward the center of the earth, reaches that point whereupon he begins to negatively accelerate as he travels back toward the surface on the other side of the planet. Once at the surface on the other side, he stops, then start back toward the center of the earth again. In other words, the kid oscillates back and forth. His father, wanting to see his kid occasionally, commi-sions a satellite to orbit in just the right circular path so as to be overhead every other time the kid's head emerges from the hole. Remembering that the force due to gravity inside a solid sphere is $\left|\vec{F}_g\right| = Gm\left(M/R^3\right)r$ (this was derived in the last chapter), what altitude above the earth's surface must the satellite be to effect this feat?



The key to this problem is in finding the frequency (or period) of motion of the kid's oscillation between the poles. This will be the same as the period of the satellite's motion. So how do you determine that period?

If we could show that the kid was executing *simple harmonic motion*, complete with its characteristic equation, we'd be set. So let's try that!

Writing out Newton's Second Law on the kid yields:



$$\sum F_r :$$

$$-G\not{m}_k\left(\frac{m_e}{R^3}\right)r = \not{m}_k \frac{d^2r}{dt^2}$$

$$\Rightarrow \quad \frac{d^2r}{dt^2} + \left(G\frac{m_e}{R^3}\right)r = 0$$

This is the characteristic equation of S.H.M. (an acceleration plus a constant times a position equals zero). That means the *angular frequency* of the oscillation is:

$$\omega = \left(G\frac{m_e}{R^3}\right)^{1/2}$$

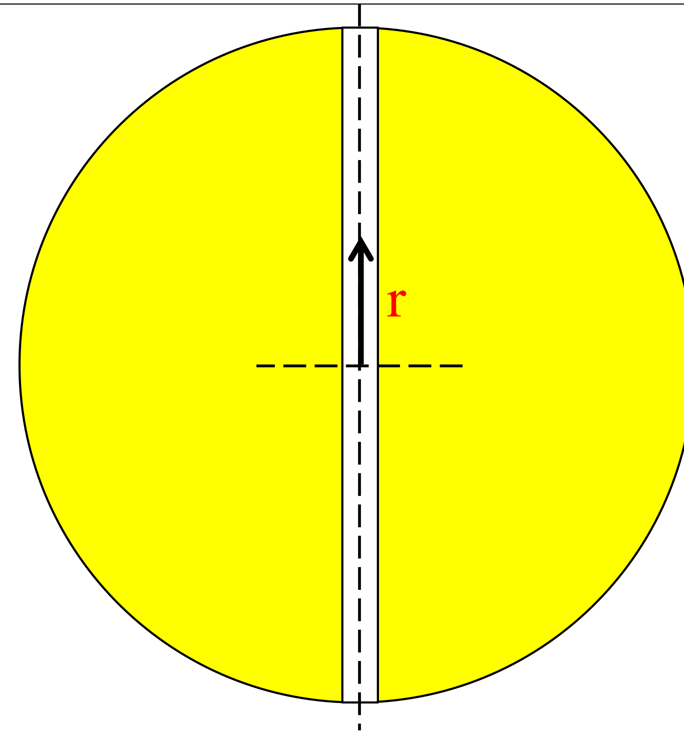
But:

$$\omega = 2\pi\nu = \frac{2\pi}{T}$$

$$\Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(G\frac{M}{R^3}\right)^{1/2}}$$

Knowing that:

$$T = \frac{2\pi}{\left(G\, {}^{m_e}\!/\!_{R^3}\right)^{1/2}}$$

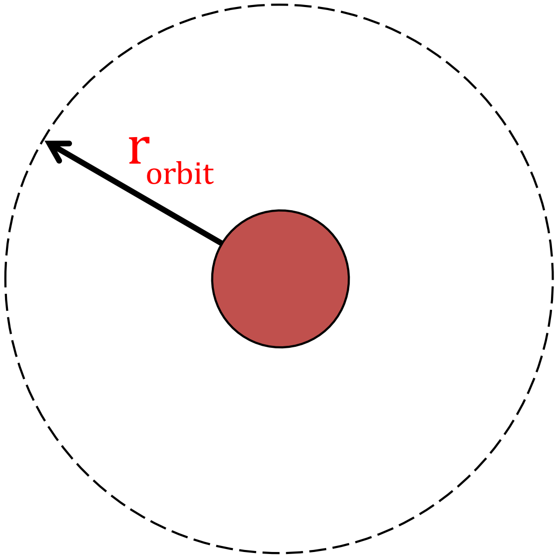


the satellite's velocity will be:

$$v = \frac{2\pi\left(R + r_{orbit}\right)}{T} = \frac{\cancel{2\pi}\left(R + r_{orbit}\right)}{\left[\cancel{2\pi}\Big/\left(G\, {}^{m_e}\!/\!_{R^3}\right)^{1/2}\right]}$$

$$= \left(R + r_{orbit}\right)\left(G\, {}^{m_e}\!/\!_{R^3}\right)^{1/2}$$

We can use Newton's Second Law on the satellite:



$$\sum F_r:$$

$$-G\frac{m_e m_s}{\left(R+r_{orbit}\right)^2} = -m_s\frac{v^2}{\left(R+r_{orbit}\right)}$$

$$\Rightarrow \quad -G\frac{m_e m_s}{\left(R+r_{orbit}\right)^2} = -m_s\frac{\left(\left(G\frac{m_e}{R^3}\right)^{1/2}\right)^2\left(R+r_{orbit}\right)^2}{\left(R+r_{orbit}\right)}$$

$$\Rightarrow \quad -G\frac{m_e m_s}{\left(R+r_{orbit}\right)^2} = -m_s\frac{\left(R+r_{orbit}\right)^2\left(G\frac{m_e}{R^3}\right)}{\left(R+r_{orbit}\right)}$$

$$\Rightarrow \quad R^3 = \left(R+r_{orbit}\right)^3$$

$$\Rightarrow \quad R = R + r_{orbit}$$

$$\Rightarrow \quad r_{orbit} = 0$$

which is to say, the sattellite has to skim the earth's surface to do the deed