Ch 13 Universal Gravitation

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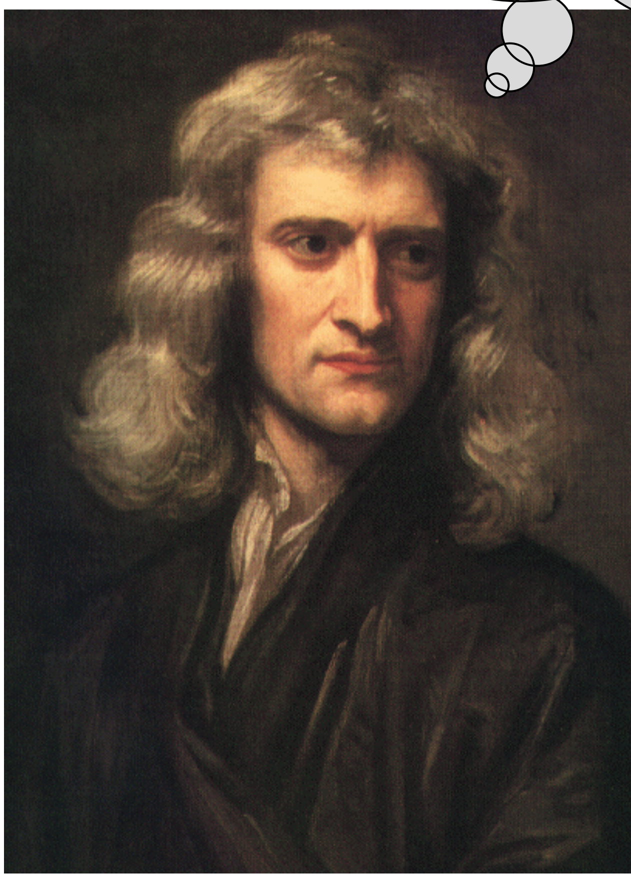
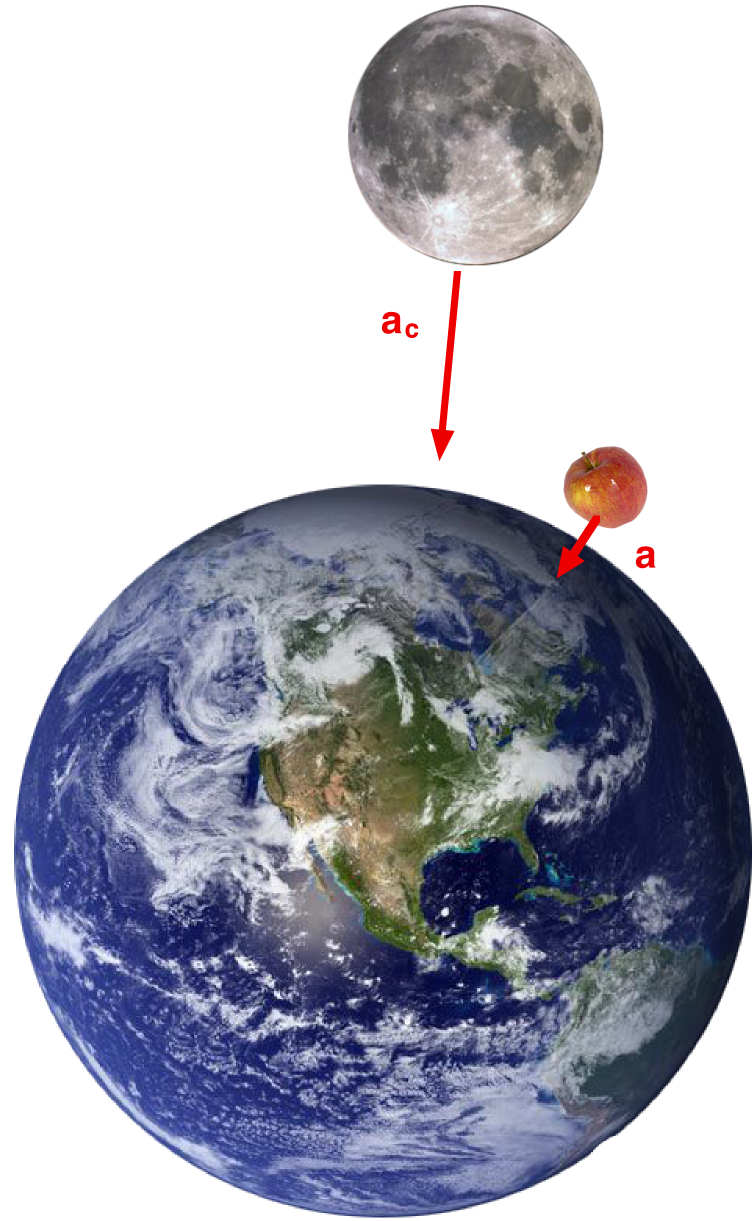
"Why do celestial objects move the way they do?"



- **Kepler (1561-1630)** Tycho Brahe's assistant, analyzed celestial motion mathematically
- **Galileo (1564-1642)** Made celestial observations by telescope
- **Newton (1642-1727)** Developed Law of Universal Gravitation

Newton's Question

What is the relationship between the accelerations of the Apple and the Moon? (The radius of the moon's orbit is approximately 3.85e8 meters; the time for the moon to orbit the earth is approximately 27.3 days.)



Newton's Question

What is the relationship between the accelerations of the Apple and the Moon?

$$a_c = \frac{v^2}{r}$$

$$v = \frac{circumference}{time} = \frac{2\pi r}{time}$$

$$v = \frac{2\pi r}{time} = \frac{2\pi(3.85e8m)}{27.3days \times \frac{24hrs}{1day} \times \frac{3600s}{1hr}} = 1.03e3m/s$$

$$a_c(moon) = \frac{(1.03e3m/s)^2}{3.85e8m} = 2.73e-3m/s^2$$

So, the acceleration of the apple at the surface is $9.80 m/s^2$, and the acceleration of the moon far away is $2.73e\text{-}3\ m/s^2$. If we set these up as a ratio:

$$a_{toward\ the\ earth} = \frac{9.8m/s^2}{2.73e-3m/s^2} \approx \frac{3600}{1}$$

Newton hypothesized that the force of earth's gravity was responsible for this acceleration, but how does that force vary with distance?

$r_{moon} = 3.85e8m$, and $r_{apple} = 6.38e6m$

Ratio of the two distances is $\frac{3.85e8m}{6.38e6m}$

$$\text{Ratio} \approx \frac{60}{1}$$

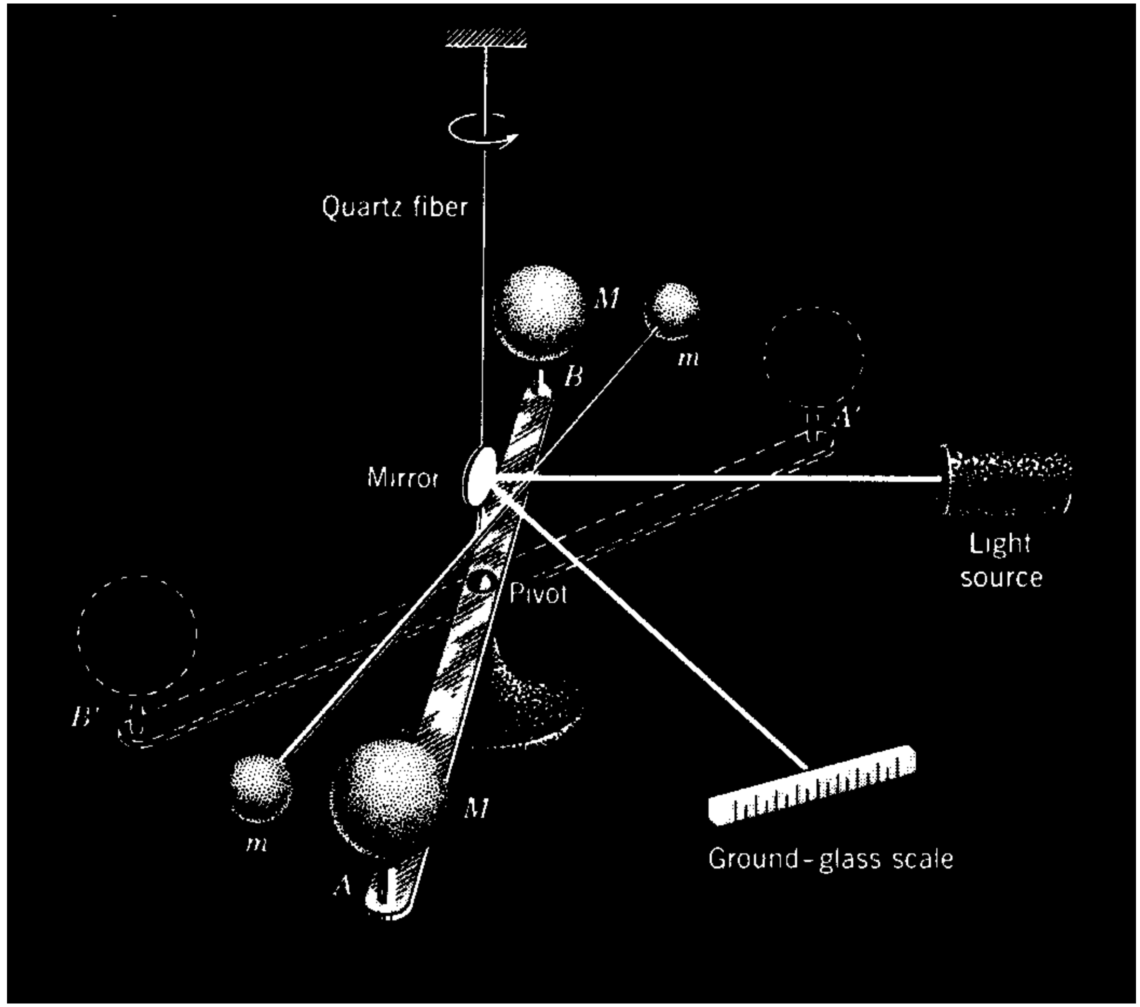
Conclusion: Does force possibly vary inversely with distance?

Law of Universal Gravitation

"Every particle in the universe attracts every other particle with a force that is proportional to the product of their masses, and inversely proportional to the square of the distance between them. This force acts along a line joining the two particles."

$$F_{gravity} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$$

"Weighing the Earth"



$$G = 6.672 \times 10^{-11} Nm^2 / kg^2$$

Example 1

Calculate the mass of the Earth, given that it has a radius of 6.38e6m.

$$F_g(of\ \ any\ \ object) = G\frac{m_{earth}m_{object}}{r^2}$$

$$m_{object}\mathbf{g} = G\frac{m_{earth}m_{object}}{r^2}$$

$$m_{earth} = \frac{\mathbf{g}r^2}{G} = 5.98e24kg$$

Example 2

A 2000 kg space shuttle is orbiting the earth at a distance of 12760 km above the earth's surface (r_{earth}=6.38e6m).

a) What is the acceleration due to earth's gravity acting on the shuttle?

b) What is the acceleration due to earth's gravity acting on an astronaut here?

c) What is the weight of the 60 kg astronaut here?

d) What orbital velocity must the shuttle have to maintain this distance?

e) What distance above the earth is required to maintain a *geosynchronous* orbit?

Example 2

A 2000 kg space shuttle is orbiting the earth at a distance of 12760 km above the earth's surface (r_{earth}=6.38e6m).

a) What is the acceleration due to earth's gravity acting on the shuttle?

$$F_g = F_{centripetal}$$

$$G\frac{m_{earth}m_{satellite}}{r^2} = m_{satellite}a_c$$

$$a_c = G\frac{m_{earth}}{r^2}$$

$$a_c = 6.672e-11\frac{Nm^2}{kg^2}\frac{5.98e24kg}{(6.38e6m + 1.276e7m)^2}$$

$$a_c = 1.09m/s^2$$

b) What is the acceleration due to earth's gravity acting on an astronaut here?

The same

Example 2

A 2000 kg space shuttle is orbiting the earth at a distance of 12760 km above the earth's surface (r_{earth}=6.38e6m).

c) What is the weight of the 60 kg astronaut here?

$$F_g = mg = ma_g$$

$$F_g = (60kg)(1.09m/s^2)$$

$$F_g = 65.4N \text{ (about 15 pounds)}$$

d) What orbital velocity must the shuttle have to maintain this distance?

$$a_c = \frac{v^2}{r}, \; so \;\; v = \sqrt{ra_c}$$

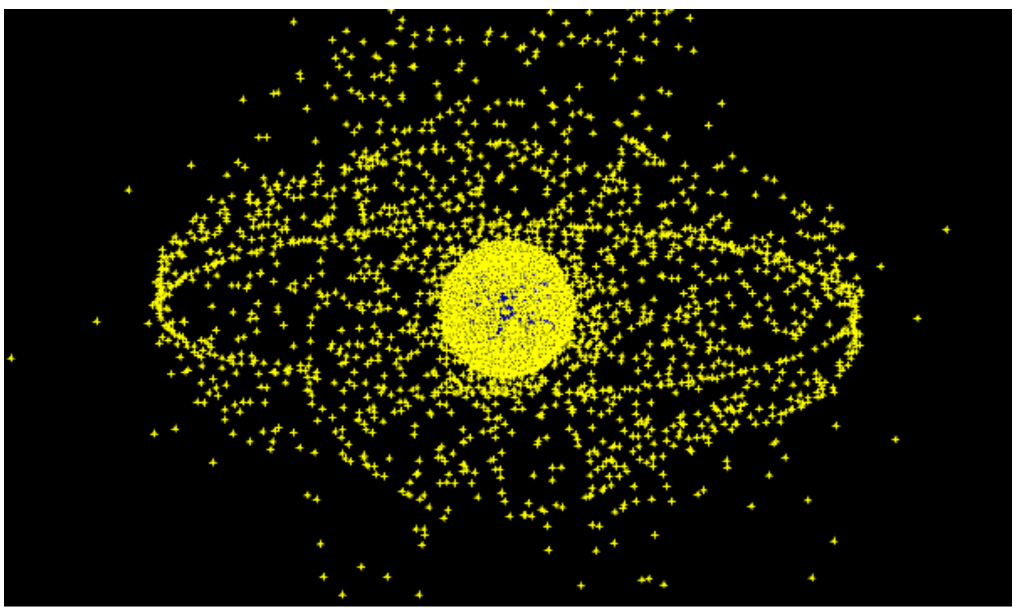
$$v = \sqrt{ra_c}$$

$$v = \sqrt{(6.38e6 + 1.276e7)(1.09m/s^2)}$$

$$v = 4.57e3m/s$$

Example 2

A 2000 kg space shuttle is orbiting the earth at a distance of 12760 km above the earth's surface (r_{earth}=6.38e6m).



e) What distance above the earth is required to maintain a *geosynchronous* orbit?

$$\text{Need } F_g = F_{centripetal}$$

$$\text{Need } \omega_{satellite} = \omega_{earth}$$

$$\omega_{earth} = \frac{1\text{rev}}{24\text{hrs}} = \frac{2\pi}{86400\text{s}}$$

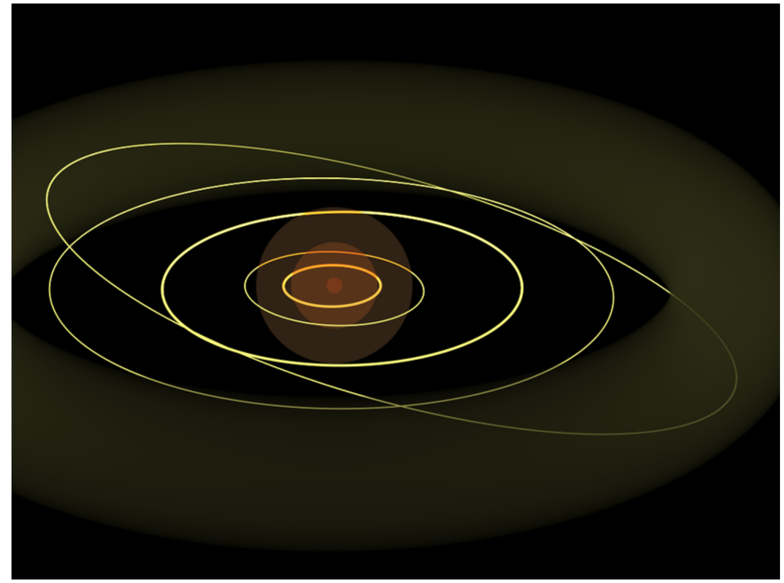
$$v = r\omega_{satellite} = r\left(\frac{2\pi}{86400\text{s}}\right)$$

$$G\frac{m_{earth}\cancel{m_{satellite}}}{r^{\cancel{2}}} = \cancel{m_{satellite}}\frac{v^2}{\cancel{r}}$$

$$G\frac{m_{earth}}{r} = \left(\frac{2\pi r}{86400\text{s}}\right)^2$$

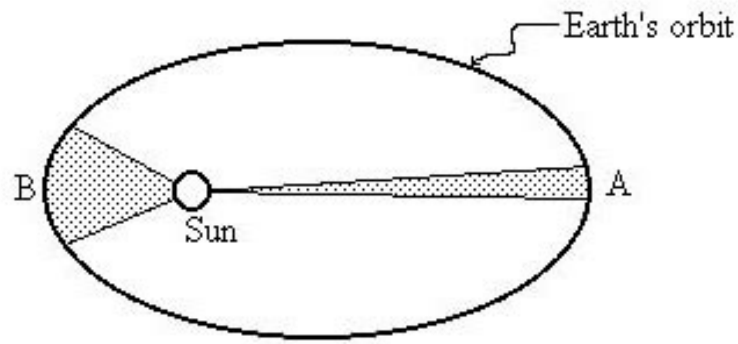
$$r = 4.23\text{e}7\text{m}$$

Kepler's 3 Laws



1. All planets move in elliptical orbits, with the Sun at one of the focal points.

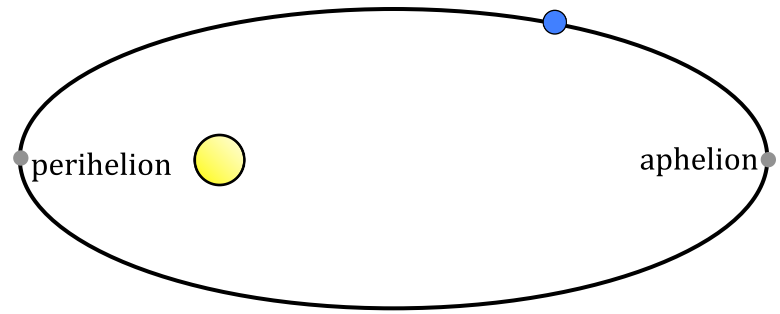
2. The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.



3. $$\frac{T_1^2}{r_1^3} = \frac{T_2^2}{r_2^3}$$ where r is the semi-major axis.

Example 3

A satellite moves in an elliptical orbit about a large body. At aphelion, a distance r_a, the satellite has a speed of v_a. What is the satellite's speed at perihelion, where it has a distance r_p from the body?



$$L_i = L_f$$

$$r_a m v_a = r_p m v_p$$

$$v_p = \frac{r_a}{r_p} v_a$$

"Force?" or "Field?"

Particle near a mass experiences a gravitational force due to that mass.

Particle *in a gravity field* experiences a gravitational force due to that gravity field.

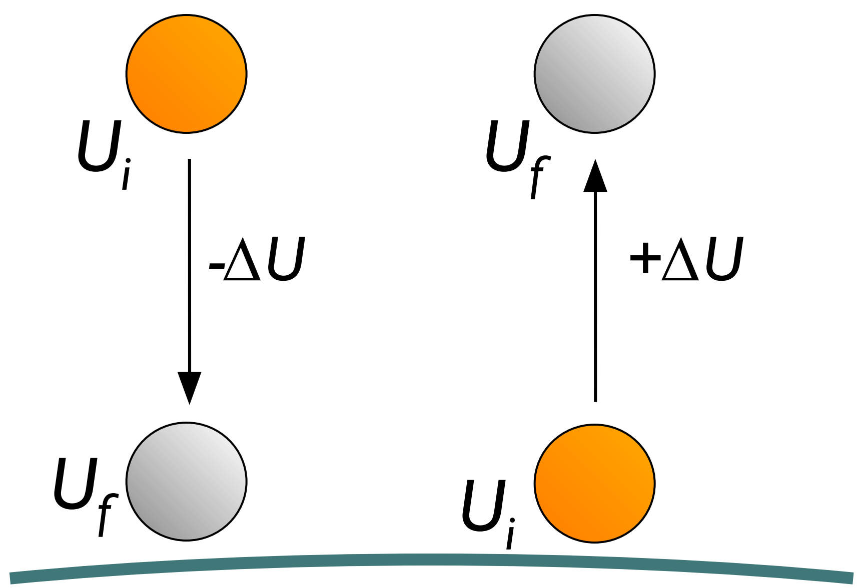
$\mathbf{F_g} = m\mathbf{g}$

Gravity field $\mathbf{g} = \mathbf{F}_{\mathrm{g}} / m$

U_g when g ≠ 9.80?

$$W_{gravity} = U_i - U_f = -\Delta U$$

$$\Delta U = U_f - U_i = -W_{gravity} = -\int_{x_i}^{x_f} F \bullet dx$$



U_g (new def.)

$$U(x) - U(\infty) = -\int_{r=\infty}^{x} \left(-G \frac{M_{earth} m}{r^2} \hat{r} \right) \bullet (dr\ \hat{r})$$

$$U(x) = -\int_{r=\infty}^{x} \left(G \frac{M_{earth} m}{r^2} \right) (dr) \cos 180^{o}$$

$$= GM_{earth} m \int_{r=\infty}^{x} \left(\frac{1}{r^2} \right) dr$$

$$= GM_{earth} m \left(-\frac{1}{r} \right) \Bigg\|_{\infty}^{x}$$

$$= -GM_{earth} m \left(\frac{1}{x} - \frac{1}{\infty} \right)$$

$$= -G \frac{M_{earth} m}{x}$$

U_g (new def.)

$$U_f - U_i = -\int_{x_i}^{x_f} F \bullet dx$$

$$U_f - U_i = -\int_{r_i}^{r_f} -G\frac{M_{Earth}m}{r^2} \bullet dr$$

$$U_f - U_i = -GM_{Earth}m\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

U_g, but where is *r*=0?

$$U_f - U_i = -GM_{Earth}m\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

As always, we need to choose a position where the potential energy U will be 0. Our custom is to let potential energy $U_i=0$ at a position $r_i= \infty$. Then we can write

$$\boxed{U = -\frac{GM_{Earth}m}{r}}$$

This is the energy of the Mm *system*. It's not just the small mass that has the potential energy.

Example 4

An apple is released from a height of 10,000 m above the surface of the earth. How fast is it traveling right before it hits the surface? (Assume no air friction.)

$$U_i + K_i = U_f + K_f$$

$$-\frac{GMm}{r} + 0 = -\frac{GMm}{r} + \frac{1}{2}mv^2$$

$$GMm\left(\frac{-1}{r_i} + \frac{1}{r_f}\right) = \frac{1}{2}mv^2$$

$$v = \sqrt{2GM\left(\frac{1}{r_f} - \frac{1}{r_i}\right)}$$

$$v = \sqrt{2(6.672e-11)(5.98e24)\left(\frac{1}{6.38e6} - \frac{1}{6.38e6+10{,}000}\right)}$$

$$v = 442m/s$$

U_g for many mass(es)

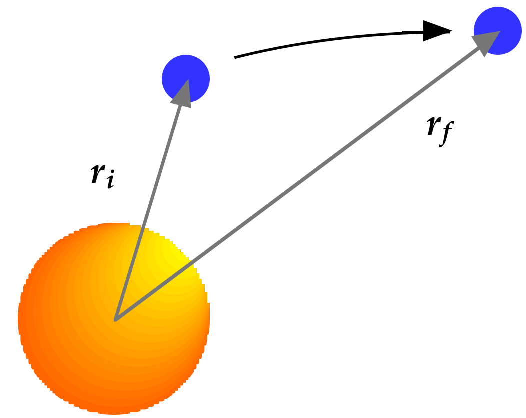
$$U = -\frac{GM_{Earth}m}{r}$$

$$U = \sum U_i = \frac{-Gm_1m_2}{r_{12}} + \frac{-Gm_2m_3}{r_{23}} + \frac{-Gm_1m_3}{r_{13}}$$

Satellites & Energy

$$E_{total} = K + U$$

$$E_{total} = \frac{1}{2}mv^2 + -G\frac{Mm}{r}$$



This is an interesting result, because it shows that

1. greater *r* causes smaller *v* for elliptical orbits (as Kepler observed, and;

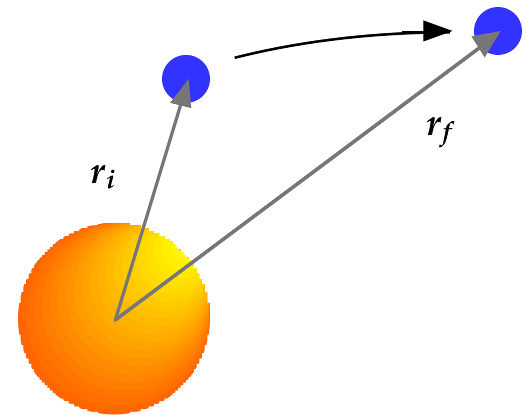
2. The energy of a system can be negative. (???!)

Satellites & Energy

Is E_{total} positive, negative, or 0?

It depends on *v*.

$$E_{total} = \frac{1}{2}mv^2 + -G\frac{Mm}{r}$$



Energy Negative?

For a bound, circular system, $F_c = F_g$.

$$\frac{mv^2}{r} = G\frac{Mm}{r^2}$$

$$\left(\frac{r}{2}\right)\frac{mv^2}{r} = G\frac{Mm}{r^2}\left(\frac{r}{2}\right)$$

$$\frac{1}{2}mv^2 = \frac{1}{2}G\frac{Mm}{r} \qquad (Note: K = -\frac{1}{2}U)$$

$$E = \frac{1}{2}mv^2 - G\frac{Mm}{r}$$

$$E = \frac{1}{2}G\frac{Mm}{r} - G\frac{Mm}{r}$$

$$E = -\frac{1}{2}G\frac{Mm}{r} = \frac{1}{2}U$$

Example 5

What minimum "escape velocity" does a satellite need to have to escape Earth's gravity completely?

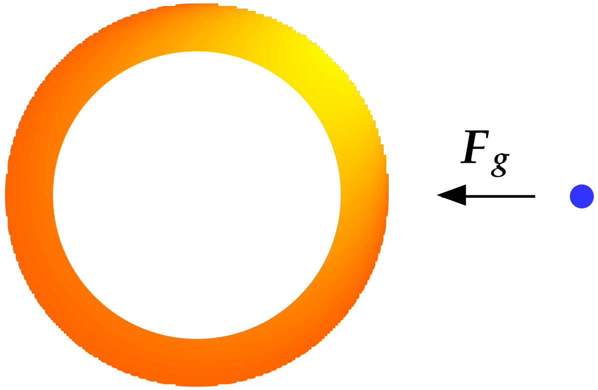
$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2} m v_{esc}^{2} - G\frac{Mm}{r_i} = 0 - 0$$

$$v_{esc} = \sqrt{\frac{2GM}{r_i}}$$

Gravity between a Particle & a Larger Mass

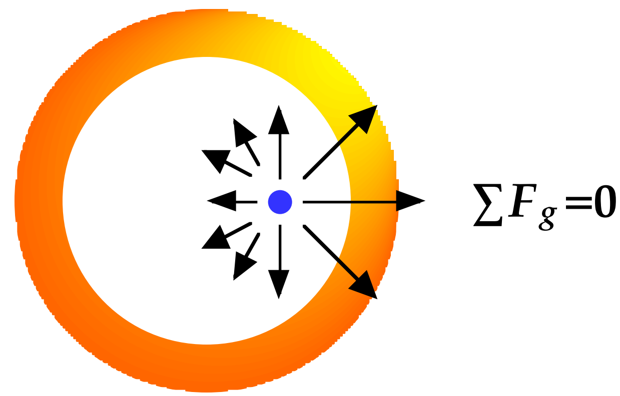
Case 1. **Spherical *Shell*, with particle outside the shell**



$$F_g = -G\frac{Mm}{r^2}\hat{\mathbf{r}} \quad \text{for } r \geq R$$

Gravity between a Particle & a Larger Mass

Case 2. **Spherical *Shell*, with particle inside the shell**

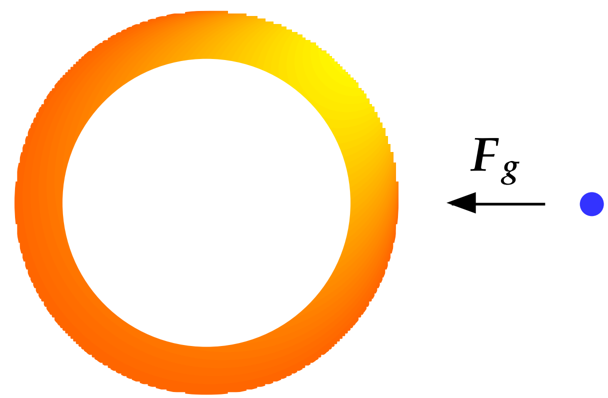


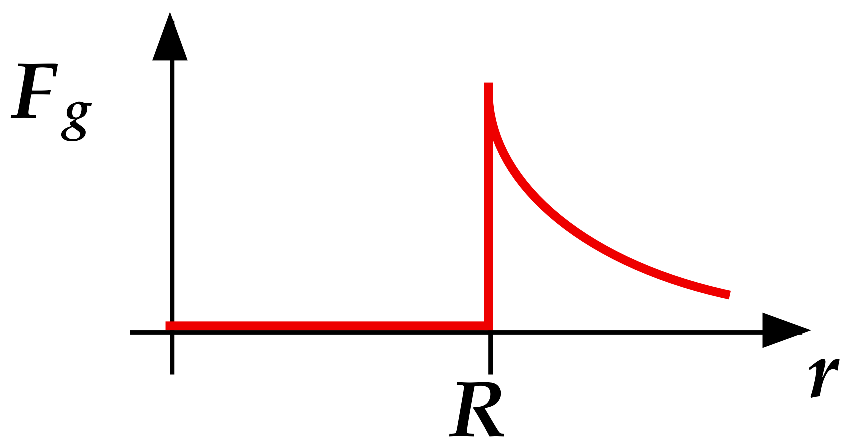
$$F_g = 0 \quad \text{for r < R}$$

Note that the shells is *not* acting as some sort of "gravity shield"--it's just that the sum of all the attractive forces balances out to zero.

Graphs

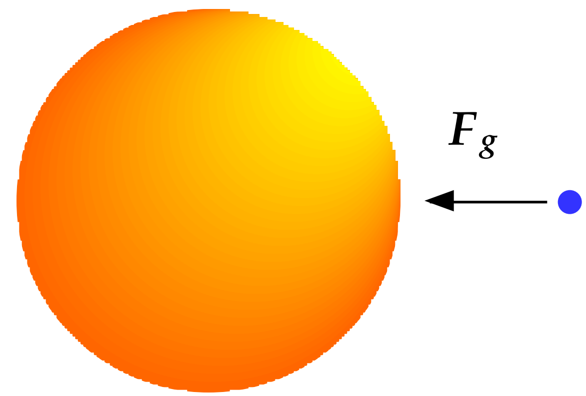
You should be able to predict what a graph of *F* vs. *r* looks like for a spherical shell.





Gravity between a Particle & a Larger Mass

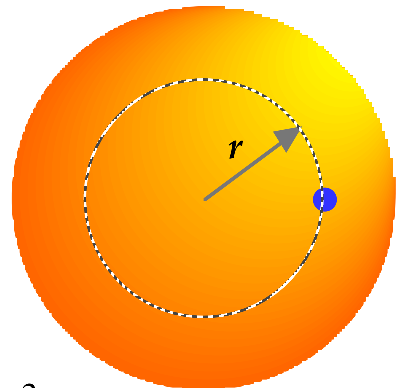
Case 3. **Spherical *Solid*, with particle outside the sphere**



$$F_g = -G\frac{Mm}{r^2}\hat{\mathbf{r}} \quad \text{for } r \geq R$$

Gravity between a Particle & a Larger Mass

Case 4. **Spherical *Solid*, with particle inside the sphere**



$$F_g = -G\frac{Mm}{r^2}\hat{\mathbf{r}}$$

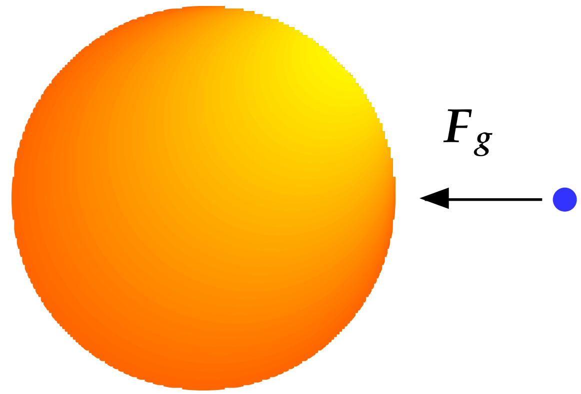
$$\frac{M_r}{M} = \frac{V_r}{V}, \text{ so } \frac{M_r}{M} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}$$

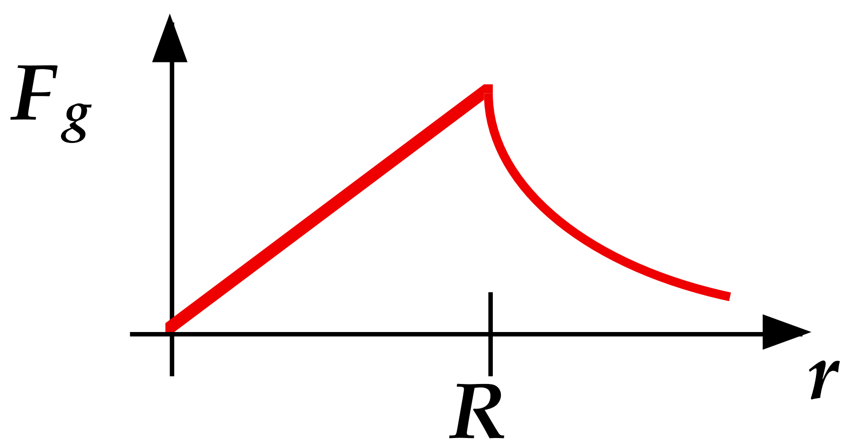
$$M_r = \frac{Mr^3}{R^3}$$

$$F_g = -G\frac{\frac{Mr^3}{R^3}m}{r^2}\hat{\mathbf{r}} = -G\frac{Mmr}{R^3}\hat{\mathbf{r}}$$

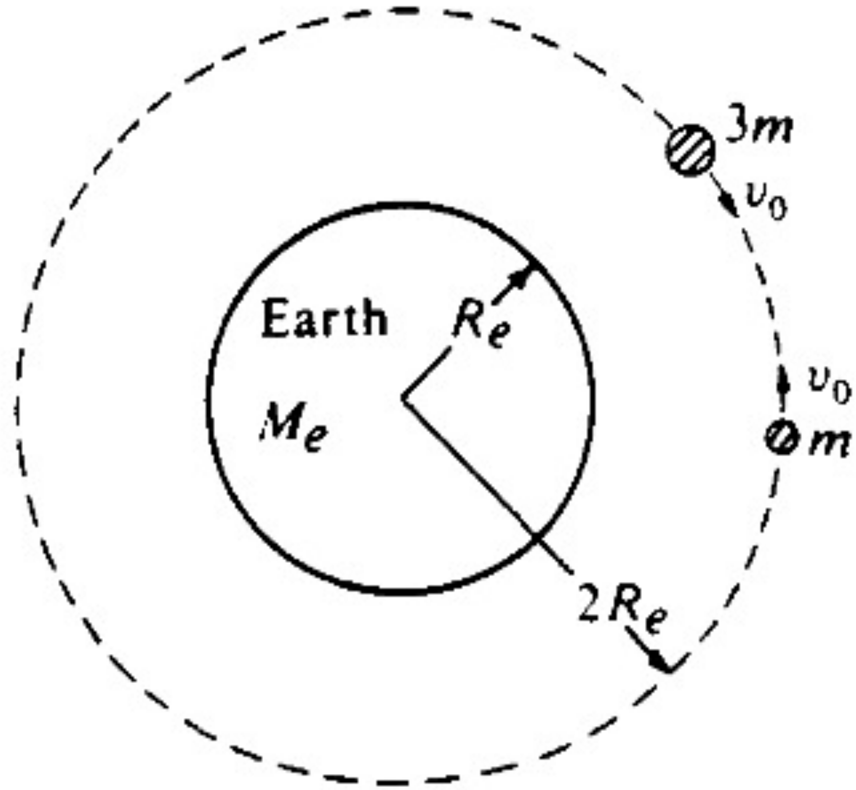
Graphs

You should be able to predict what a graph of *F* vs. *r* looks like for a solid sphere.





Example 6



Two satellites, of masses m and 3m, respectively, are in the same circular orbit about the Earth's center, as shown in the diagram above. The Earth has mass M_e and radius R_e. In this orbit, which has a radius of $2R_e$, the satellites initially move with the same orbital speed v_o but in opposite directions.

a. Calculate the orbital speed v_o of the satellites in terms of G, M_e, and R_e.

b. Assume that the satellites collide head-on and stick together. In terms of v_o find the speed v of the combination immediately after the collision.

c. Calculate the total mechanical energy of the system immediately after the collision in terms of G, m, M_e, and R_e. Assume that the gravitational potential energy of an object is defined to be zero at an infinite distance from the Earth.

Weird stuff to think about...

It's the different escape speeds required for different planets that explains why some planets have atmospheres and others don't. Gas molecules have speeds that depend on their temperatures: the greater the temperature, the greater the average speed of the molecules, and the greater the chance is that they'll have a velocity that allows them to escape the planet.

Mercury? No atmosphere

Earth? Light molecules gone, heavier molecules remain

Jupiter? Even hydrogen can't escape!