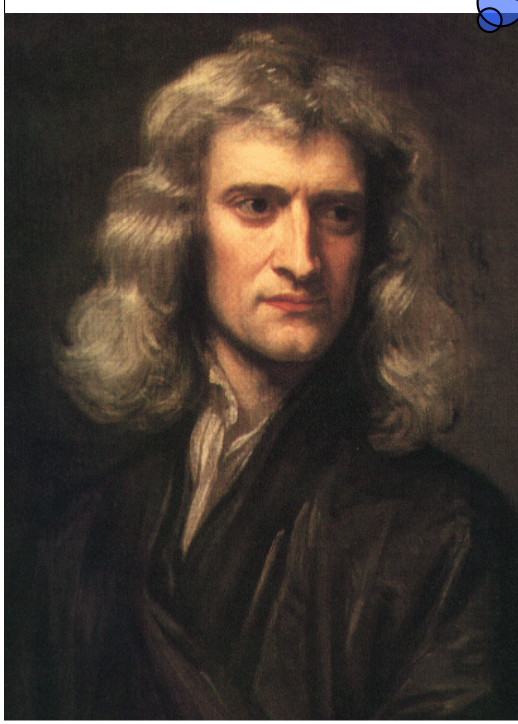
Earth from space . . .

CHAPTER 13:
Universal Gravitation

"Why do celestial objects move the way they do?"



- **Kepler (1561-1630)** Tycho Brahe's assistant, analyzed celestial motion mathematically
- **Galileo (1564-1642)** Made celestial observations by telescope
- **Newton (1642-1727)** Developed Law of Universal Gravitation

courtesy of Richard White

Newton observed that there exists a force between any two masses that is proportional to the product of the masses and inversely proportional in some way to the distance between the *center of masses* of the bodies (he didn't originally know if it was $1/r$ or $1/r^2$ or $1/r^3$ or what). The proportionality constant was/is called the universal gravitational constant G. In *polar-spherical* notation, this attractive, radial force is denoted as:

$$F_{grav} = G\frac{m_1 m_2}{r^n}(-\hat{r})$$

To determine the power n, he made another interesting observation. The same force that accelerates an object like an apple close to the surface of the earth accelerates the moon in its path around the earth. Taking that path to be circular, and noting that it takes 27.3 days for the moon to orbit the earth once, he wrote:

acceleration of apple using the theory:

$$G\frac{\cancel{m_{apple}}m_{earth}}{\left(r_{earth}\right)^n} = \cancel{m_{apple}}a_{apple}$$

$$\Rightarrow \quad a_{apple} = G\frac{m_{earth}}{\left(r_{earth}\right)^n}$$

acceleration of moon using the theory:

$$G\frac{\cancel{m_{moon}}m_{earth}}{\left(r_{to\ moon}\right)^n} = \cancel{m_{moon}}a_{moon}$$

$$\Rightarrow \quad a_{moon} = G\frac{m_{earth}}{\left(r_{to\ moon}\right)^n}$$

ratio of two accelerations yields:

$$\frac{a_{apple}}{a_{moon}} = \frac{\cancel{G}\frac{\cancel{m_{earth}}}{\left(r_{earth}\right)^n}}{\cancel{G}\frac{\cancel{m_{earth}}}{\left(r_{moon}\right)^n}}$$

$$\Rightarrow \quad \frac{a_{apple}}{a_{moon}} = \left(\frac{r_{moon}}{r_{earth}}\right)^n$$

acceleration at earth:

$$a_{apple} = 9.8\ \text{m/s}^2$$

centripetal acceleration at moon:

$$a_{moon} = \frac{(v_{moon})^2}{R}$$

$$= \frac{\left(\frac{2\pi \cancel{R}}{T}\right)^{2}}{\cancel{R}} = \frac{4\pi^2 R}{T^2}$$

$$= \frac{4\pi^2 (3.85\text{x}10^8\ \text{m})}{[(27.3\ \text{days})(24\ \text{hr/day})(3600\ \text{sec/hr})]^2}$$

$$= 2.73\text{x}10^{-3}\ \text{m/s}^2$$

using calculated accelerations:

$$\frac{a_{apple}}{a_{moon}} = \frac{9.8}{2.73\text{x}10^{-3}} = 3600$$

using the radii:

$$\left(\frac{r_{moon}}{r_{earth}}\right)^n = \left(\frac{3.85\text{x}10^8\ \text{m}}{6.38\text{x}10^6\ \text{m}}\right)^n$$

$$= (60)^n$$

putting everything together:

$$\left(\frac{a_{apple}}{a_{moon}}\right) = \left(\frac{r_{moon}}{r_{earth}}\right)^n$$

$$(3600) = (60)^n$$

$$\Rightarrow \quad n = 2$$

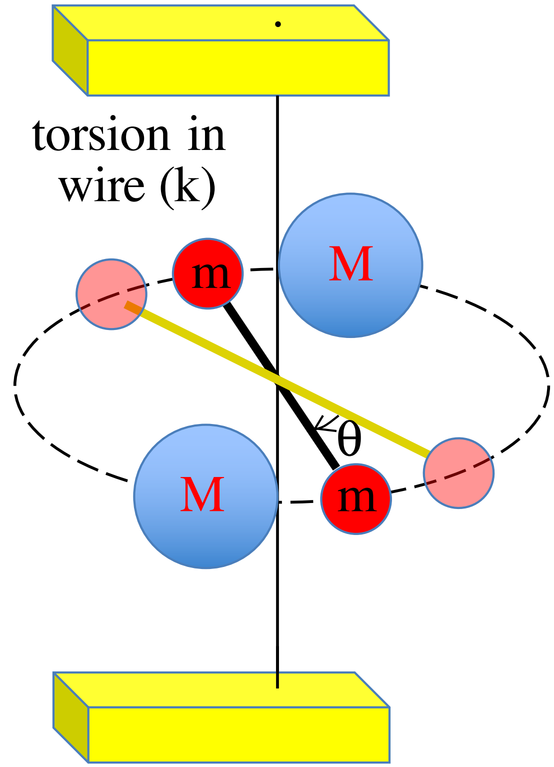
In other words:

$$F_{grav} = G\frac{m_1 m_2}{r^2}(-\hat{r})$$

Note: Newton shelved this theory for several years because the accepted distance to the moon was off and the exponent he originally calculated was something like 1.7—not something he thought nature would do.

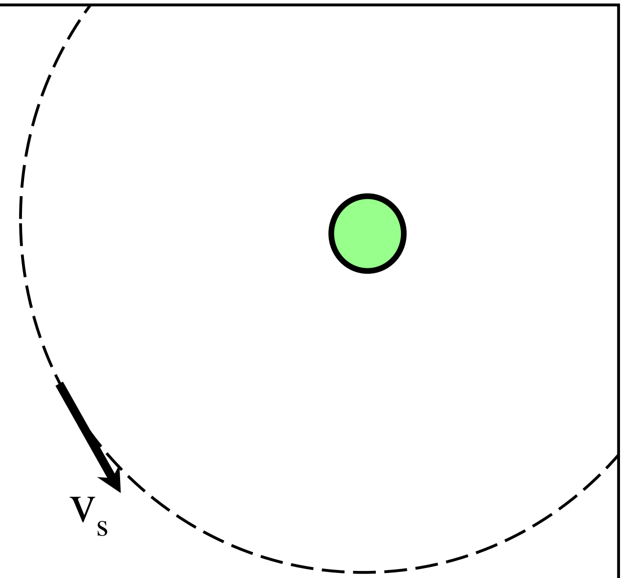
After initial experiments were done by Charles Coulomb, Henry Cavendish used a *torsion balance* in a vacuum to measure the attraction between two masses *m* and M to one another. The calculated value of *G* was:

$$G = 6.672\text{x}10^{-11}\,\text{N} \bullet \text{m}^2 / \text{kg}^2$$



Example 1: A 2000 kg space shuttle orbits the earth at a distance of 13,000 km above the earth's surface. You know:

$$r_{earth} = 6.38x10^{6} \text{ meters},\ G = 6.67x10^{-11} N \bullet m^2/kg^2,$$
$$m_{earth} = 5.98x10^{24} \text{ kg} \qquad m_{astronaut} = 60.0 \text{ kg}$$



a.) Derive an expression for, then determine the acceleration of the satellite in its orbit.

$$\sum F_{satellite}$$
$$G\frac{m_e \cancel{m_s}}{(r_e + r_{to\ orbit})^2} = \cancel{m_s} a$$
$$\Rightarrow a = (6.67x10^{-11} N \bullet m^2/kg^2)\frac{(5.98x10^{24} \text{ kg})}{(6.38x10^{6} \text{ meters} + 13.0x10^{6} \text{ meters})^2}$$
$$= 1.06 \text{ m/s}^2$$

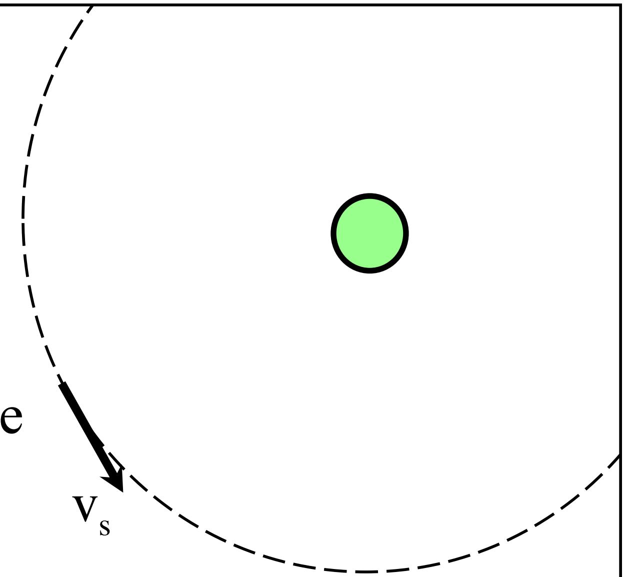
b.) What is the acceleration of an astronaut inside the orbiting satellite?

It will be the same!

c.) What is her weight?

$$F_g = ma_g$$
$$= (60 \text{ kg})(1.06 \text{ m/s}^2)$$
$$= 63.6 \text{ N} \qquad \text{(around 15 pound)}$$

$r_{earth} = 6.38x10^{6}$ meters, $G = 6.67x10^{-11} N \bullet m^{2}/kg^{2}$,
$m_{earth} = 5.98x10^{24}$ kg, $r_{to\ orbit} = 13.0x10^{6}$ meters, $m_s = 2000$kg



d.) Derive an expression for, then determine the velocity the satellite must maintain to keep its orbit.

Important point: Orbital motion has two masters. There is Newton's general gravitational force $G\frac{m_e m_s}{r^2}(-\hat{r})$, and there is the centripetal acceleration $\frac{(v_s)^2}{r}$ required to keep the body from plummeting into the celestial body it is traveling around. What that means is that for a given orbital radius, there is only ONE speed that will hold the body in orbit!

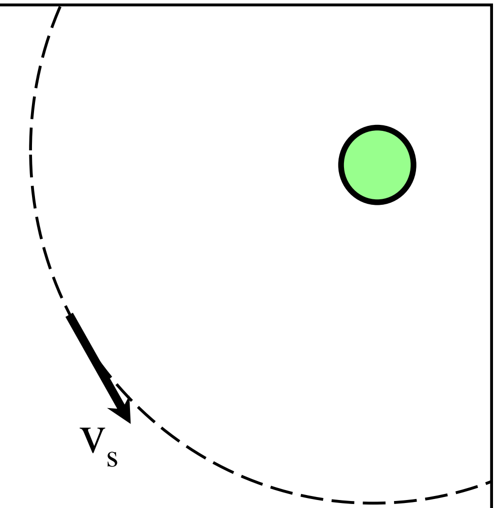
$$\sum F_{satellite}$$

$$G\frac{m_e \cancel{m_s}}{(r_e + r_{to\ orbit})^{\cancel{2}}} = \cancel{m_s}\left(\frac{v^2}{\cancel{(r_e + r_{to\ orbit})}}\right)$$

$$\Rightarrow v = \sqrt{\left(6.67x10^{-11} N \bullet m^2/kg^2\right)\frac{\left(5.98x10^{24}\ kg\right)}{\left(6.38x10^{6}\,meters + 13.0x10^{6}\,meters\right)}}$$

$$= 4537\ m/s$$

e.) What is the radius of a geosynchronous orbit?



A geosynchronous orbit is a circular orbit in which the satellite is always over the same point on the earth. In other words, the satellite's *angular velocity* is the same as the earth's *angular velocity*.

We know: $\omega_s = \omega_{earth} = \dfrac{2\pi \text{ rad}}{(24 \text{ hr})\left(\dfrac{60 \text{ min}}{\text{hr}}\right)\left(\dfrac{60 \text{ sec}}{\text{min}}\right)} = 7.27\text{x}10^{-5} \text{ rad/sec}$

and $v_s = (r_e + r_{orbit})\omega$

$$\sum F_{satellite}:$$

$$G\frac{m_e \cancel{m_s}}{(r_e + r_{\text{to orbit}})^2} = m_s\left(\frac{v_s^{\,2}}{(r_e + r_{\text{to orbit}})}\right)$$

$$= \cancel{m_s}\left(\frac{(r_e + r_{\text{to orbit}})^{\cancel{2}}\omega^2}{\cancel{(r_e + r_{\text{to orbit}})}}\right)$$

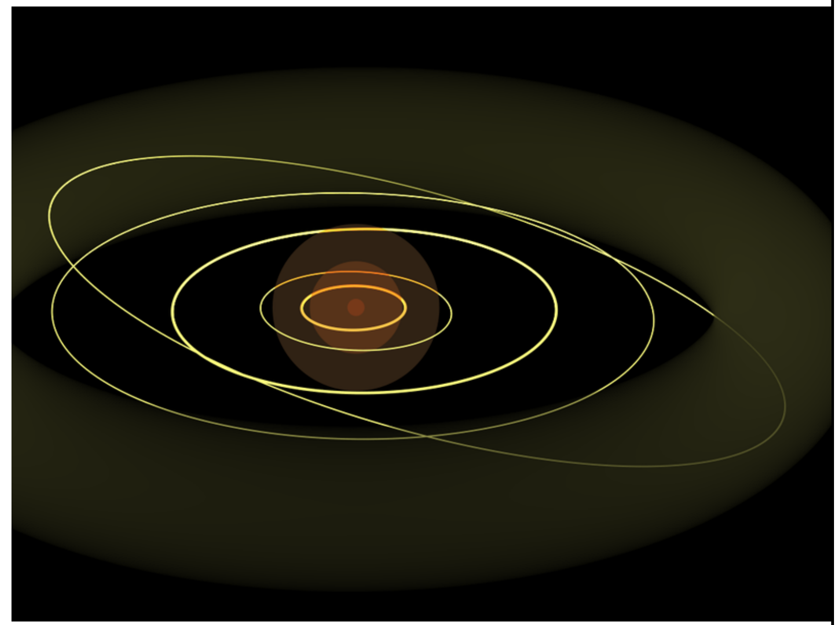
$$\Rightarrow \quad (r_e + r_{\text{to orbit}})^3 = {Gm_e}/{\omega^2}$$

$$\Rightarrow \quad r_{\text{to orbit}} = \left({Gm_e}/{\omega^2}\right)^{1/3} - r_e$$

$$= \left(\frac{(6.67\text{x}10^{-11} \text{N}\bullet\text{m}^2/\text{kg}^2)(5.98\text{x}10^{24} \text{ kg})}{(7.27\text{x}10^{-5})^2}\right)^{1/3} - 6.38\text{x}10^6$$

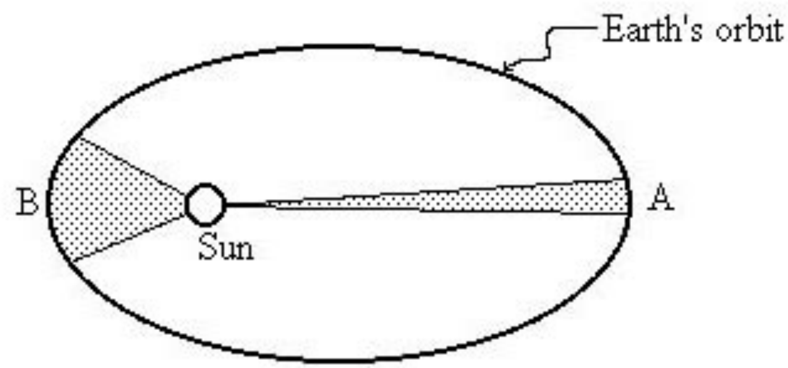
$$\Rightarrow \quad r_{geo} = 3.59\text{x}10^7 \text{ m}$$

Kepler's Laws



As a consequence of data taken by Tycho Brahe, a Danish astronomer (last of the naked-eye celestial observers), Kepler was able to deduce three "laws" about planetary motion. They are:

Kepler's First (called the Law of orbits): Planets move in *elliptical orbits* with the Sun at one focal point.



Kepler's Second (called the Law of areas): A radius drawn from the Sun to any planet sweeps out *equal areas* in *equal time intervals*.

Kepler's Third (called the Law of periods): The square of a planet's period is proportional to the cube of its semi-major axis.

Kepler generated his Laws by using observational data from Brahe. Each of the laws do, though, have a theoretical justification. To wit:

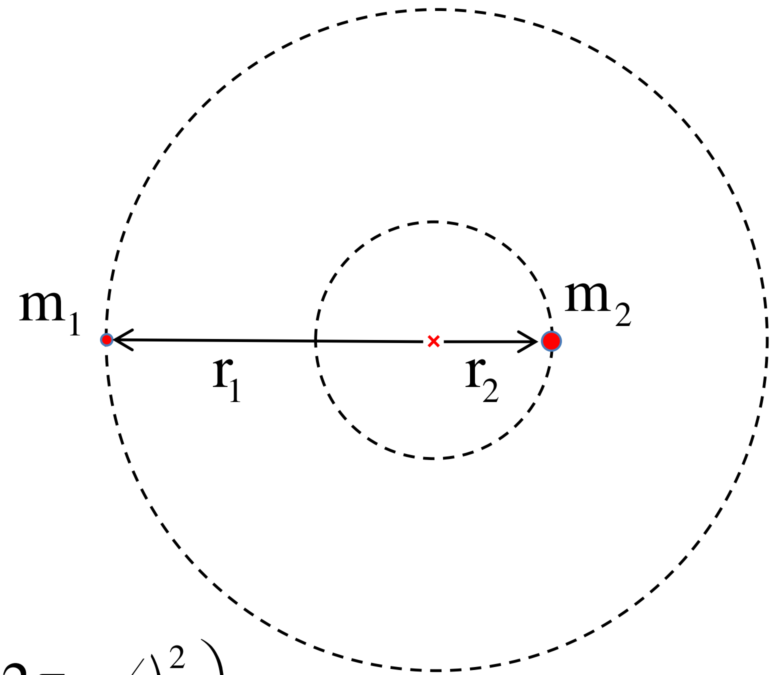
Kepler's First (called the Law of orbits): Planets move in *elliptical orbits* with the Sun at one focal point.

Using conservation of angular momentum and *conservation of energy*, it is possible to derive an expression for the radial position of a planet as a function of its angular position in the orbit (i.e., $r(\theta)$). The derived expression is that of an ellipse.

Kepler's Second (called the Law of areas): A radius drawn from the Sun to any planet sweeps out *equal areas* in *equal time intervals*.

The derivable expression for a planet's area-sweep with time (i.e., dA/dt) looks just like the derived expression for a planet's *angular momentum* (give or take a constant). As the *angular momentum* of a torque-free body is constant, dA/dt must also be constant.

Example 2: Derive an expression for the *period of motion* of a planet of mass m_1 as it orbits the *center of mass* of a two planet system, where the mass of the second planet is m_2 (see sketch).



This is a Newton's Second Law problem. Summing the forces on m_1:

$$\sum F_{radial}:$$

$$G\frac{m_1 m_2}{(r_1+r_2)^2} = m_1\left(\frac{v_1^{\ 2}}{r_1}\right)$$

The velocity in terms of the period is: $v_1 = \frac{2\pi r_1}{T}$

$$G\frac{\cancel{m_1} m_2}{(r_1+r_2)^2} = \cancel{m_1}\left(\frac{\left(2\pi r_1/T\right)^2}{r_1}\right)$$

Substituting this into our relationship yields:

$$\Rightarrow \quad G\frac{m_2}{(r_1+r_2)^2} = \frac{4\pi^2 r_1^{\cancel{2}}}{\cancel{r_1}T^2}$$

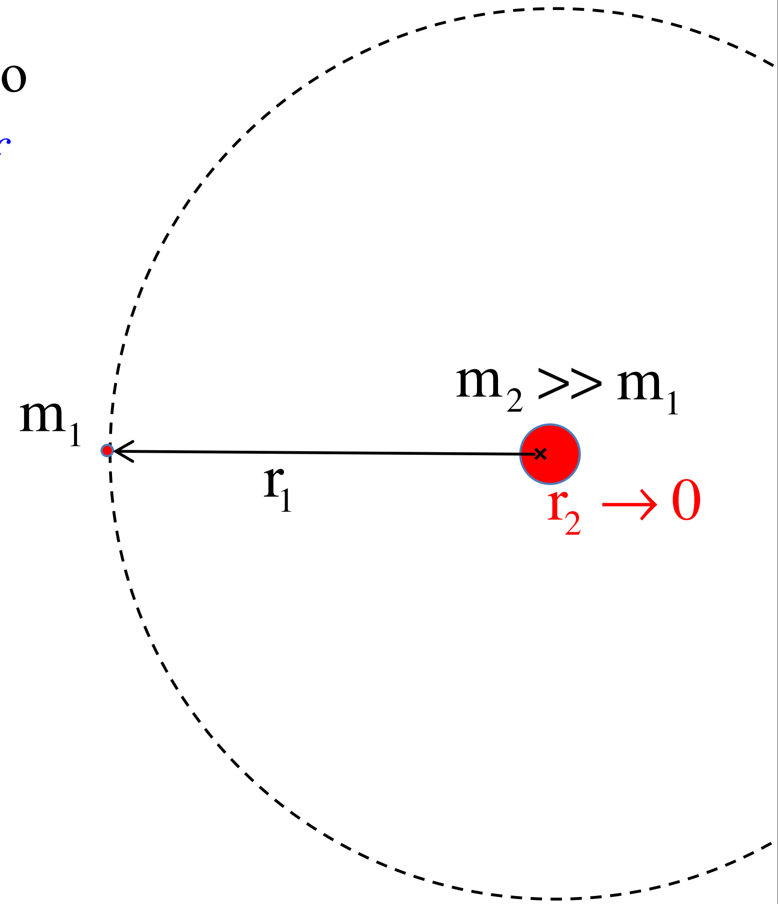
$$\Rightarrow \quad T^2 = \left(\frac{4\pi^2}{Gm_2}\right) r_1 (r_1+r_2)^2$$

Notice: If we let the planet become very small so that $m_2 >> m_1$, then $r_2 \rightarrow 0$, the system's *center of mass* migrates toward the center of m_2 and we can write:

$$T^2 = \left(\frac{4\pi^2}{Gm_2}\right) r_1 \left(r_1 + \cancel{r_2}^{\,0}\right)^2$$

$$\Rightarrow \quad T^2 = \left(\frac{4\pi^2}{Gm_2}\right) r_1^{\,3}$$

$$r_1\left(r_1 + r_2\right)^2 \Rightarrow r_1^{\,3}$$

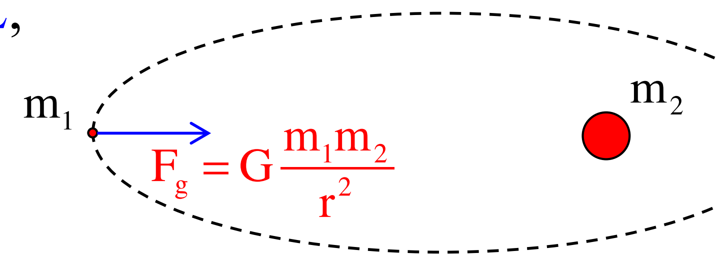


This is Kepler's Third Law, which is to say, the *square of the period* is proportional to the *cube of the radius* (or semi-major axis).

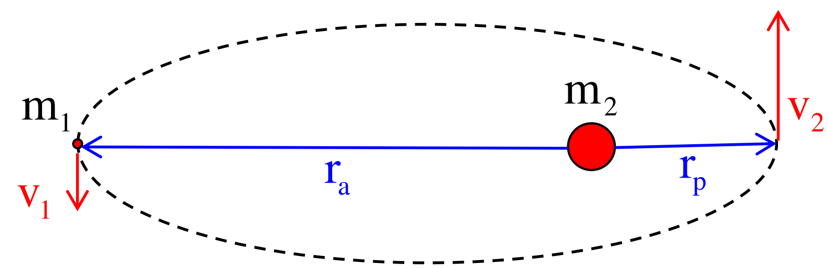
Note that of Kepler's three laws, this is the only one that is an approximation.

Conservation of Angular Momentum

Because gravitational forces are RADIAL, they *don't produce torques*. With no external torques acting, planetary motion adheres to *conservation of angular momentum*. With that in mind:



Example 3: If a planet moving in an elliptical orbit has a velocity at its farthest point (its aphelion) of v_1, what is its velocity at its closest point (its perihelion)? Assume you know the distances identified on the sketch.)



This is a conservation of angular momentum problem:

$$\sum L_1 + \sum \tau_{ext}\Delta t = \sum L_2$$
$$\vec{r}_1 x \vec{p}_1 + \quad 0 \quad = \vec{r}_2 x \vec{p}_2$$
$$\Rightarrow (\cancel{m}_1 v_1) r_a = (\cancel{m}_1 v_2) r_p$$
$$\Rightarrow \quad v_2 = \frac{r_a}{r_p} v_1$$

OR

$$\sum L_1 + \sum \tau_{ext}\Delta t = \sum L_2$$
$$I_1\omega_1 + \quad 0 \quad = I_2\omega_2$$
$$\Rightarrow (\cancel{m}_1 r_a^{\cancel{2}})\left(\frac{v_1}{\cancel{r_a}}\right) = (\cancel{m}_1 r_p^{\cancel{2}})\left(\frac{v_2}{\cancel{r_p}}\right)$$
$$\Rightarrow \quad v_2 = \frac{r_a}{r_p} v_1$$

Point of Order about FIELDS

For all sorts of reasons, Newton didn't really like his *theory of gravity*. . . even though it does do a great job of predicting how the real world acts (we put a man on the moon using it). First, why should two objects be attracted to one another simply by virtue of each having mass (that's what his universal gravitational force equation $F_{\text{on } m_1} = G \frac{m_1 m_2}{r^2}$ suggests). And second, contact forces makes sense (push on something, it pushes back), but forces acting at a distance . . . how does that work?

An alternate view is to think of a mass as creating a *disturbance in the region around it* (called it a gravitational field), and define it as the amount of *force per unit mass* AVAILABLE at a point due to the presence of that field-producing mass. Such quantities would be field-producing, force-related, but would be independent of any mass feeling the affect. That is, they would exist whether a mass resided at a point of interest or not.

The math for the *gravitational field* due to *any mass* would look like:

$$\frac{F_g}{m_2} = \left(\frac{\left[G \frac{\left(m_{\text{fld producing}} \cancel{m_2} \right)}{r^2} \right]}{\cancel{m_2}} \right)$$

$$= \left(\frac{G}{r^2} \right) m_{\text{fld producing}}$$

Not so important here, but the idea will become more important in *E&M*.

Gravity Near and Far

Close to the surface of the earth, the magnitude of the force on a mass m_1 due to the presence of the earth's mass m_e is

$$F_{m_1} = G\frac{m_e m_1}{{r_e}^2}$$

where denominator is the *square* of the *distance between* the *center of mass* of the two objects or, in this case, the *radius of the earth*.

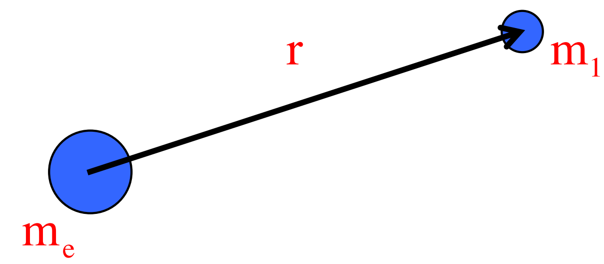
Putting numbers into this expression for *G*, the *mass* and *radius* of the earth, that relationship becomes:

$$F_{m_1} = \left(9.8\ \text{m/s}^2\right)m_1 = m_1 g$$

with a *potential energy function* that we've derived as:

$$U_{\text{grav near earth}} = m_1 g y$$

Far from the earth (or any celestial body), the magnitude of the force on a mass m_1 due to the presence of the earth's mass m_e is



$$F_{m_1} = G\frac{m_1 m_e}{r^2}$$

where denominator is the *square* of the *distance* between the *center of mass* of the two objects which, in this case, is NOT the radius of the planet.

Noting that there is a preferred *F = 0* point *at infinity*, which will be our *zero potential energy point*, the *potential energy function* for this far-field gravitational force derives as:

$$\Delta U = -\int \vec{F} \bullet d\vec{r}$$

$$\Rightarrow \quad U(r) - \cancelto{\equiv 0}{U(r=\infty)} = -\int_{r=\infty}^{r} \left(-G\frac{m_1 m_2}{r^2}\vec{r}\right) \bullet (dr\hat{r})$$

$$\Rightarrow \quad U(r) = -\int_{r=\infty}^{r} \left(G\frac{m_1 m_2}{r^2}\right) dr \cancelto{-1}{\cos(180^\circ)}$$

$$= -Gm_1 m_2 \left(\frac{1}{r}\Big|_{r=\infty}^{r}\right)$$

$$= -\left(G\frac{m_1 m_2}{r} - \cancelto{0}{G\frac{m_1 m_2}{\infty}}\right)$$

$$= -G\frac{m_1 m_e}{r}$$

Example 4: Derive an expression for the *escape velocity* required for a mass to free itself from the earth's gravitational field.

This is an energy problem. Realizing that for an object to become completely free of the earth, it must move to infinity, and remembering that the gravitational potential energy for far-field situations is *not zero* at the earth's surface, *conservation of energy* yields:

$$\sum KE_1 \quad + \quad \sum U_1 \quad + \sum W_{ext} = \sum KE_2 + \quad \sum U_2$$

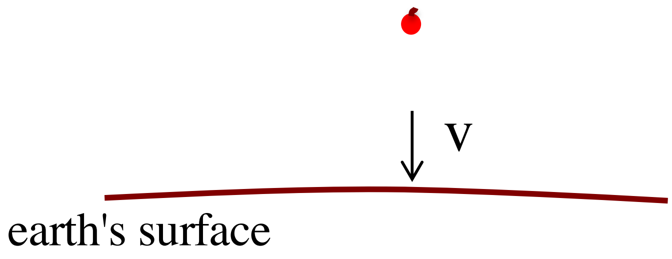
$$\frac{1}{2}\cancel{m_1}\left(v_{escape}\right)^2 + \left(-G\frac{\cancel{m_1}m_e}{r_e}\right) + \quad 0 \quad = \quad 0 \quad + \cancelto{0}{\left(-G\frac{m_1 m_e}{\infty}\right)}$$

$$\Rightarrow \qquad v_{escape} = \sqrt{2G\frac{m_e}{r_e}}$$

$$\Rightarrow \quad v_{escape} = \sqrt{2\left(6.67\text{x}10^{-11}\text{N}\bullet\text{m}^2/\text{kg}^2\right)\frac{\left(5.98\text{x}10^{24}\ \text{kg}\right)}{\left(6.38\text{x}10^{6}\ \text{m}\right)}}$$

$$= 1.12\text{x}10^4\,\text{m/s}$$ (this is approximately 7 mi/sec)

Example 5: An apple is released from a height of 10,000 meters above the earth’s surface. How fast is it moving just before it hits the earth, assuming no air friction.



Again, a classic *conservation of energy* problem.

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

$$0 + \left(-G\frac{\cancel{m_a} m_e}{(r_e + 10{,}000)}\right) + 0 = \frac{1}{2}\cancel{m_a} v^2 + \left(-G\frac{\cancel{m_a} m_e}{r_e}\right)$$

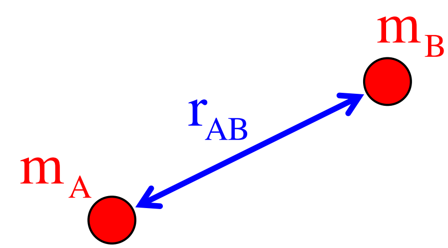
$$\Rightarrow \quad v = \sqrt{2Gm_e\left(\frac{1}{r_e} - \frac{1}{(r_e + 10{,}000)}\right)}$$

$$\Rightarrow \quad v_{escape} = \sqrt{2\left(6.67x10^{-11} N \bullet m^2/kg^2\right)\left(5.98x10^{24}\ kg\right)\left(\frac{1}{\left(6.38x10^6\ m\right)} - \frac{1}{\left(6.38x10^6 + 1x10^4\right)m}\right)}$$

$$= 442.4\ m/s$$

Note that using *mgy* as your *potential energy function*, the velocity comes out to be 442.7 m/s, so even at 10,000 meters up, the *mgy* approximation is a fairly good one.

Potential Energy of Multiple Mass Systems



It takes *no energy* to draw a mass m_A from infinity to some position in space. Once there, that mass will produce a *potential energy* field around that point.

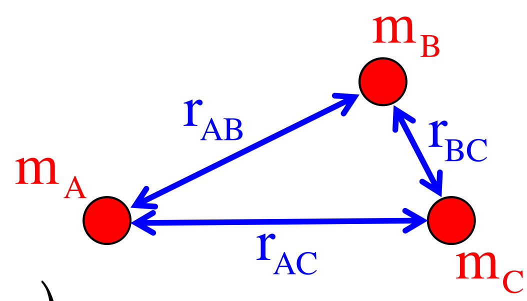
The amount of energy required to draw a second mass m_B in from infinity to a distance r_{AB} units for the first mass is equal to the amount of work *you* need to put into the system to effect the deed. This will be *minus* the amount of energy the field does (assuming you come in with constant velocity). As the amount of work the field does is equal to $-\Delta U_A$, where *U* is the fields *potential energy function* equal to $-G\, m_A m_B/r$, the amount of work *you* must do will be $+\Delta U_A$. That is:

$$W_{you} = +\Delta U_{AB} = \left(U_{final,AB} - U_{initial,AB}\right)$$

$$= \left[\left(-G\frac{m_A m_B}{r_{AB}}\right) - \cancelto{0}{G\frac{m_A m_B}{\infty}}\right] = -G\frac{m_A m_B}{r_{AB}}$$

(That's right, you need to *apply a force opposite the direction of motion*, hence the negative work, to keep the body from accelerating as it comes in.)

The amount of energy required to draw a *third* mass m_C in to a distance r_{AC} from m_A and r_{BC} from m_B will equal the amount of work *you* have to do to bring the mass in from infinity. That will be:



$$W_{you} = +\Delta U_{AC} + \Delta U_{BC} = \left(U_{final,AC} - U_{initial,AC}\right) + \left(U_{final,BC} - U_{initial,BC}\right)$$

$$= \left[\left(-G\frac{m_A m_C}{r_{AC}}\right) - \cancelto{0}{G\frac{m_A m_C}{\infty}}\right] + \left[\left(-G\frac{m_B m_C}{r_{BC}}\right) - \cancelto{0}{G\frac{m_B m_C}{\infty}}\right] = -G\frac{m_A m_C}{r_{AC}} - G\frac{m_B m_C}{r_{BC}}$$

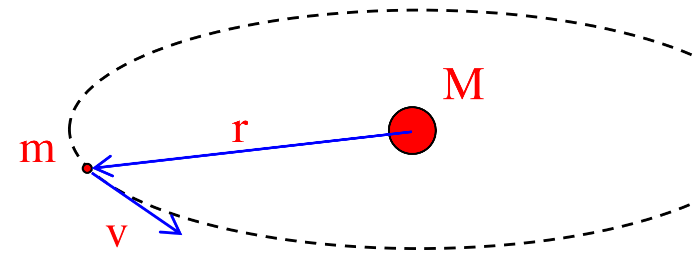
Adding this to the original bit of work done to bring in the second piece, and we get the *potential energy wrapped up in the system* as:

$$U = -G\frac{m_A m_B}{r_{AB}} + \left(-G\frac{m_A m_C}{r_{AC}} - G\frac{m_B m_C}{r_{BC}}\right)$$

Bottom line: Each added mass must interact with all the masses already present.

Total Energy in an Orbiting System Where the Orbit is Eliptical

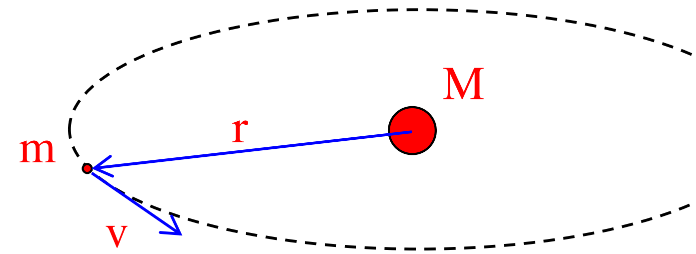
Example 6: Assuming an eliptical orbit, what is the *total mechanical energy* wrapped up in an orbiting satellite.



$$
\begin{aligned}
E_{tot} &= \sum KE + \sum U_1 \\
&= \frac{1}{2}mv^2 + \left(-G\frac{mM}{r}\right)
\end{aligned}
$$

Total Energy in an Orbiting System Where the Orbit is Circular

Example 7: Assuming an eliptical orbit, what is the *total mechanical energy* wrapped up in an orbiting satellite.



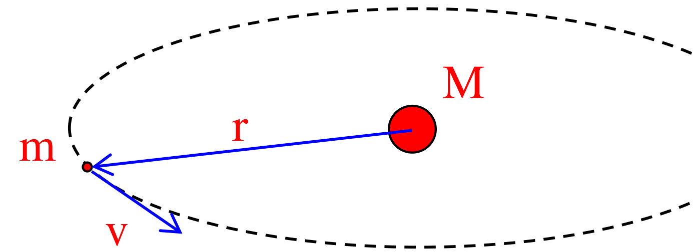
$$E_{tot} = \sum KE + \sum U_1$$
$$= \frac{1}{2}mv^2 + \left(-G\frac{mM}{r}\right)$$

To relate the velocity, look at what *Newton's Second* has to say about the system:

$$\sum F_{radial}:$$
$$G\frac{\cancel{m}M}{r^{\cancel{2}}} = \cancel{m}\left(\frac{v^2}{\cancel{r}}\right)$$
$$\Rightarrow \quad v^2 = G\frac{M}{r}$$

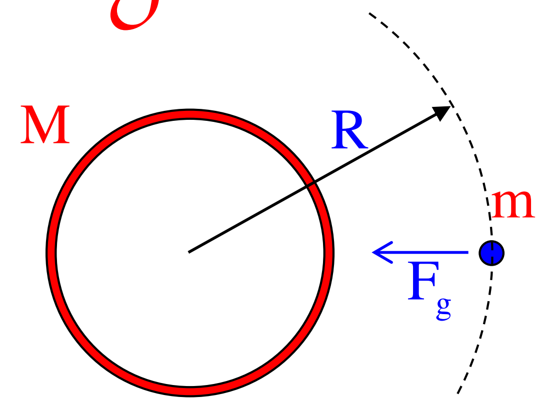
Combining:

$$
\begin{aligned}
E_{tot} &= \frac{1}{2}mv^2 + \left(-G\frac{mM}{r}\right) \\
&= \frac{1}{2}m\left(G\frac{M}{r}\right) + \left(-G\frac{mM}{r}\right) \\
&= -\frac{1}{2}m\left(G\frac{M}{r}\right) \\
&= \frac{1}{2}U
\end{aligned}
$$



Bottom line: The total amount of *mechanical energy* wrapped up in circular orbital motion is equal to half the *potential energy* in the system. This relationship is still good with elliptical orbits if you make the *r* term into the semi-major axis.

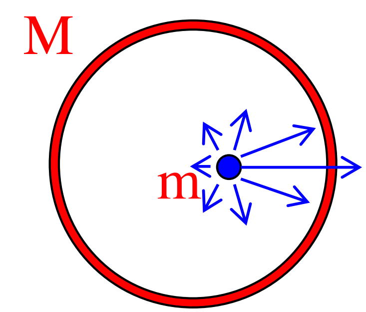
A Particle's Interaction with a Larger Mass



Case 1 (spherical shell with particle outside):

The net force on *m* is due to all the bits of mass inside the sphere of radius *R*. Newton created Calculus to justify the claim that there is no difference between this situation and the situation in which all of the mass is located at the system's *center of mass*. (as the math yields the same relationship). For the ring of mass M shown, this is:

$$\vec{F}_g = G\frac{mM}{r^2}(-\hat{r}) \qquad \text{for } r \geq R$$

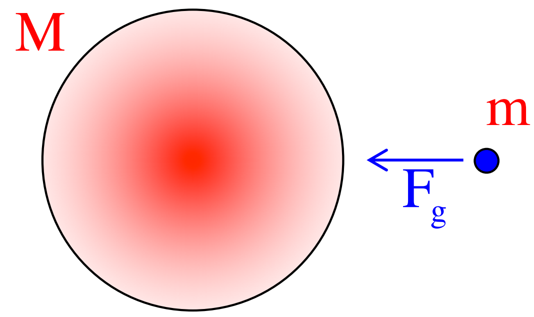


Case 2 (spherical shell with particle inside):

Inside the ring, the net force on the mass *m* due to all the bits of mass encompassed in the ring will, because gravity is an inverse square function, vectorially add to zero, so:

$$\vec{F}_g = 0 \qquad \text{for } r \leq R$$

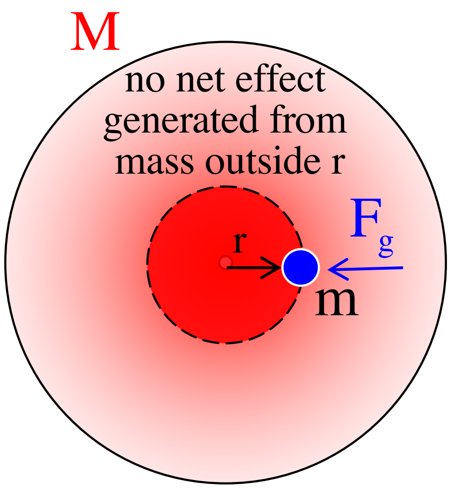
Case 3 (spherical solid with particle outside):



The net force on the mass *m* is, again, due to all the bits of mass encompassed in the ring of radius *R*. And just as before, this will equal:

$$\vec{F}_g = G\frac{mM}{r^2}(-\hat{r}) \qquad \text{for } r \geq R$$

Case 4 (spherical solid with particle inside):



This is a little trickier. Think of the mass as sitting on a sphere of radius *r*. From Case 2, all the mass outside that radius provides no *net* gravitational force, whereas from Cases 1 and 3, the gravitational force from the mass *inside that radius* will be as though all of *that* mass was located at the sphere's *center of mass*. In other words:

$$\vec{F}_g = G\frac{m\int_{r=0}^{r} dM}{r^2}(-\hat{r}) \qquad \text{for } r \geq R$$

where $\int_{r=0}^{r} dM$ is the fraction of the sphere's mass *inside the sphere of radius r*.

The mass inside r can be determine in two ways. One only works if the mass if homogeneous. The other uses differentially thin spherical shells and works whether the mass distribution is uniform or not. We'll do the easy way first, then I'll show you the more complex approach.

Assuming the mass is *uniformly distributed*, the *volume mass density function* (mass per unit volume) ρ can be written in two ways:

$$\rho = \frac{\text{total mass}}{\text{total volume}} = \frac{M}{\left(\frac{4}{3}\pi R^3\right)} \qquad \text{and} \qquad \rho = \frac{\text{mass inside r}}{\text{volume inside r}} = \frac{m_{\text{inside}}}{\left(\frac{4}{3}\pi r^3\right)}$$

Equating: $$\frac{M}{\left(\cancel{\frac{4}{3}}\cancel{\pi} R^3\right)} = \frac{m_{\text{inside}}}{\left(\cancel{\frac{4}{3}}\cancel{\pi} r^3\right)} \Rightarrow m_{\text{inside}} = \frac{r^3}{R^3}M$$

So: $$\vec{F}_g = Gm\frac{\left(m_{\text{inside}}\right)}{r^2}(-\hat{r})$$

$$= Gm\frac{\left(\frac{r^{\cancel{3}}}{R^3}M\right)}{\cancel{r^2}}(-\hat{r}) = \left(G\frac{mM}{R^3}\right)r\ (-\hat{r}) \qquad \text{for } r \leq R$$

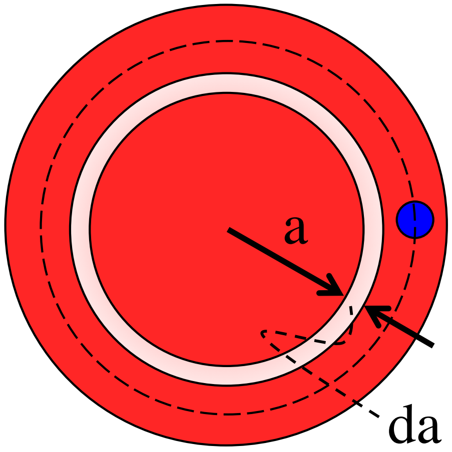
It shouldn't be terribly surprising that the force would be a function of *r*.

You would expect the force to be zero at the sphere's center (i.e., at r = 0), which this function satisfies (whereas a $1/r^2$ function wouldn't).

In any case, the more exotic way to do this would be to create a differentially thin spherical shell, determine the mass in it, then integrate to determine the total mass inside.

$$\rho = \frac{\text{total mass}}{\text{total volume}} = \frac{M}{\left(\frac{4}{3}\pi R^3\right)} = \frac{3M}{4\pi R^3}$$

The *differential volume* is the *surface area* ($dS = 4\pi a^2$) of a spherical shell times the differential thickness *da*, which means: $\rho = \frac{dm}{dV} = \frac{dm_{\text{inside}}}{\left(4\pi a^2\right)da} \quad \Rightarrow \quad dm = \rho\left(4\pi a^2\right)da$



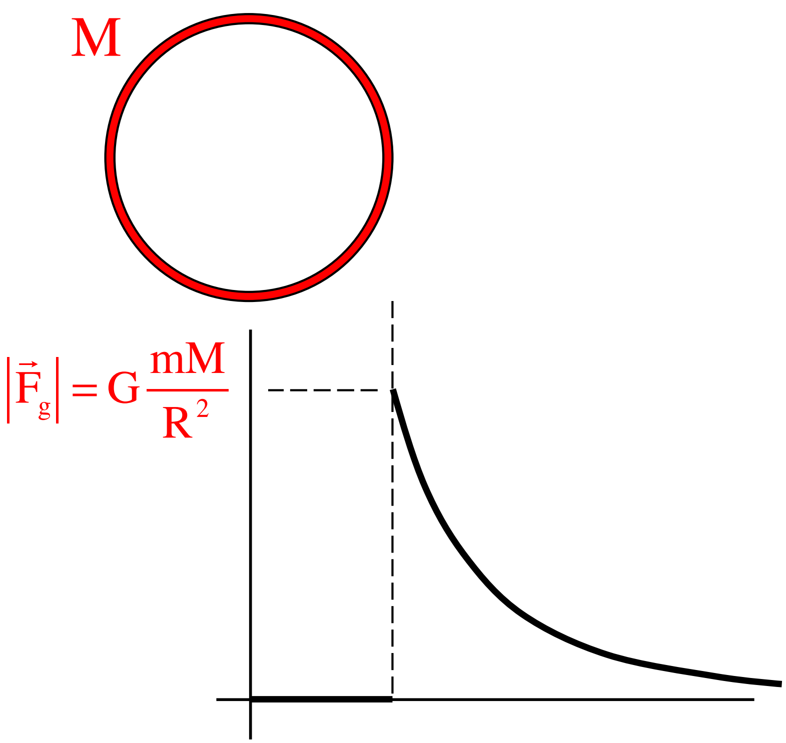
So: $\left|\vec{F}_g\right| = Gm\frac{\int_{r=0}^{r} dm}{r^2}$

$$= Gm\frac{\int_{r=0}^{r}\rho\left(4\pi a^2\right)da}{r^2} = Gm\left(\frac{3M}{\cancel{4\pi}R^3}\right)\left(\cancel{4\pi}\right)\frac{\int_{a=0}^{r} a^2\,da}{r^2}$$

$$= \left(Gm\left(\frac{3M}{R^3}\right)\frac{1}{r^2}\right)\left(\frac{a^3}{3}\Big|_{a=0}^{r}\right) = \left(Gm\left(\frac{\cancel{3}M}{R^3}\right)\frac{1}{\cancel{r^2}}\right)\left(\frac{\cancel{r^3}}{\cancel{3}}\right) = Gm\left(\frac{M}{R^3}\right)r$$

Graphs

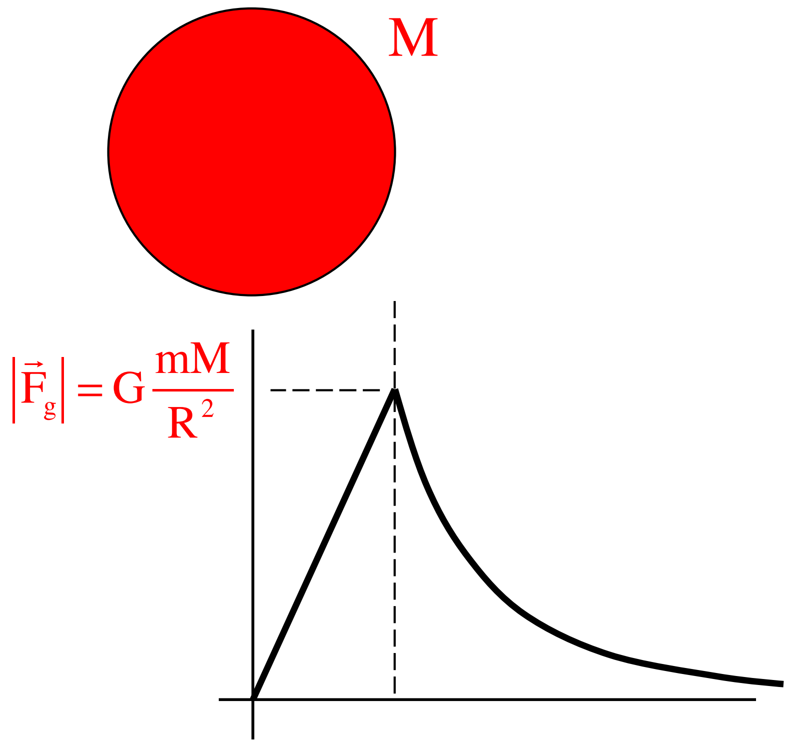
Magnitude of force on particle due to a spherical shell:



$$\left|\vec{F}_g\right| = 0 \qquad \text{for } r < R$$

$$\left|\vec{F}_g\right| = G\frac{mM}{r^2} \qquad \text{for } r \geq R$$

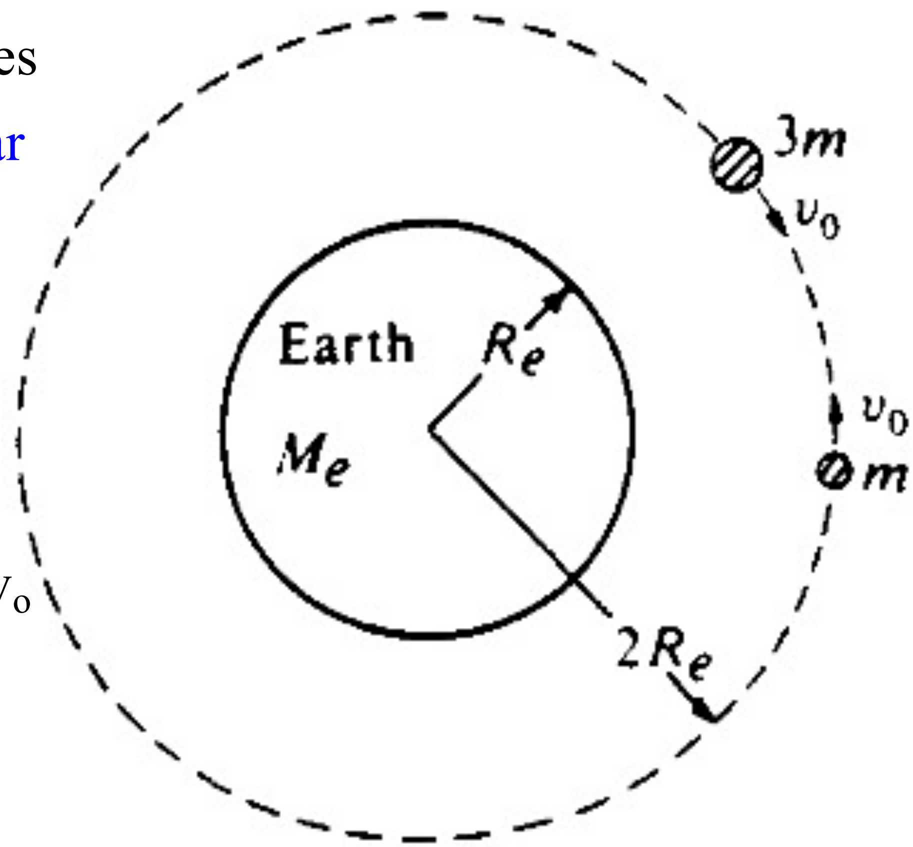
Magnitude of force on particle due to a solid shell:



$$\left|\vec{F}_g\right| = Gm\left(\frac{M}{R^3}\right)r \qquad \text{for } r < R$$

$$\left|\vec{F}_g\right| = G\frac{mM}{r^2} \qquad \text{for } r \geq R$$

AP Example 7: Two satellites of masses m and 3m, respectively, are in the same circular orbit about the Earth's center, as shown in the diagram above. The Earth has mass M_e and radius R_e. In this orbit, which has a radius of $2R_e$, the satellites initially move with the same orbital speed v_o but in opposite directions.



a.) Derive an expression for the orbital speed v_o of the satellites in terms of G, M_e, and R_e.

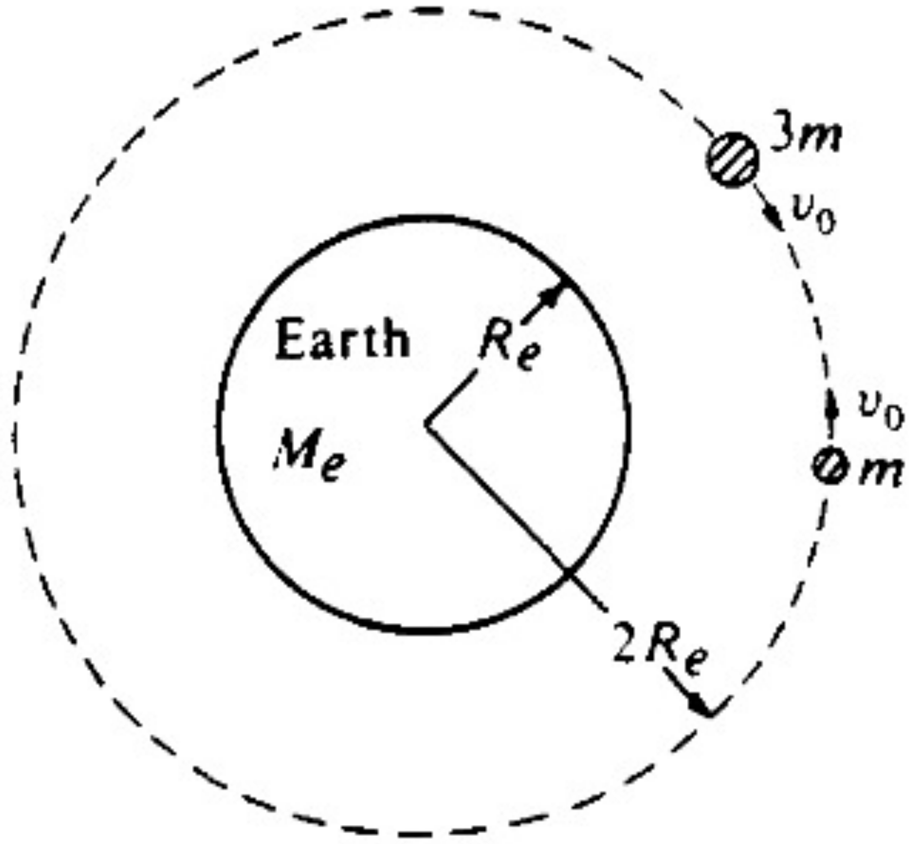
$$\sum F_{radial}:$$

$$G\frac{M_e \cancel{m_1}}{(2R_e)^2} = \cancel{m_1}\left(\frac{{v_o}^2}{2R_e}\right)$$

$$\Rightarrow \quad v_o = \left(G\frac{M_e}{2R_e}\right)^{1/2}$$

b.) *Assume* that the satellites collide head-on and stick together. In terms of v_o, determine the speed v of the combination immediately after the collision.

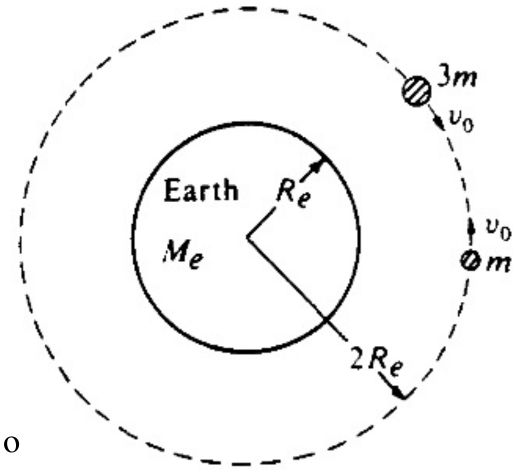
You should be able to use either *conservation of momentum* (no external impulses acting) or *conservation of angular momentum* (no external torque-related impulses). We'll start with *conservation of angular momentum*, and because these are both *point masses* (and to be a little exotic), we'll calculate the *angular momentum* using both $I\omega$ and the cross product $\vec{r}\times\vec{p}$:



Noting that:

$$v_o = \left(G\frac{M_e}{2R_e}\right)^{1/2}$$

We can write:



$$\sum L_1 + \sum \tau_{ext}\Delta t = \sum L_2$$

$$[-I_{3m}\omega_{3m} + \vec{r}_m x \vec{p}_m] + 0 = -\vec{r}_{4m} x \vec{p}_{4m}$$

$$\left[-(mr^2)\left(\frac{v_o}{2R_e}\right) + (|\vec{r}_m|)(|\vec{p}_m|)\sin 90^o\right] = -(|\vec{r}_{4m}|)(|\vec{p}_{4m}|)\sin 90^o$$

$$\left[-\left((3m)(2R_e)^{\cancel{2}}\right)\left(\frac{v_o}{\cancel{2R_e}}\right) + (2R_e)(mv_o)\cancel{\sin 90^o}^{\,0}\right] = -(2R_e)(4mv)\cancel{\sin 90^o}^{\,0}$$

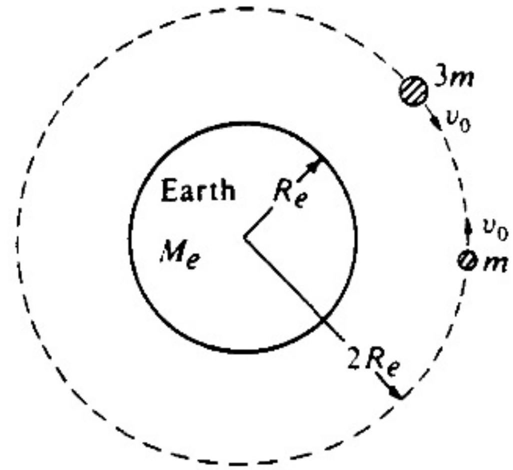
$$\Rightarrow \quad -4mR_e v_o = -8mvR_e$$

$$\Rightarrow \quad 4\cancel{m}\cancel{R}_e\left(G\frac{M_e}{2R_e}\right)^{1/2} = 8\cancel{m}v\cancel{R}_e$$

$$\Rightarrow \quad v = \frac{1}{2}\left(G\frac{M_e}{2R_e}\right)^{1/2}$$

where the negative sign suggests the final angular velocity that is clockwise, as expected.

Using conservation of momentum:



$$\sum p_1 \quad + \sum F_{ext}\Delta t = \sum p_2$$

$$\left[-(3m)v_o + mv_0\right] + \quad 0 \quad = -(4m)v$$

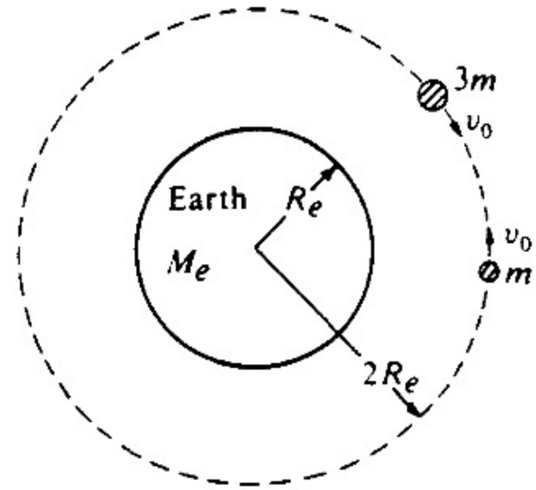
$$\Rightarrow \quad -2\cancel{m}v_o = -4\cancel{m}v$$

$$\Rightarrow \quad v = \frac{1}{2}\left(G\frac{M_e}{2R_e}\right)^{1/2}$$

where again, the negative sign suggests the final velocity is in the same direction as the original direction-of-motion of the *3m* mass.

Clearly, the *conservation of momentum* is the easier way to go here, but both are educational.

c.) Calculate the *total mechanical energy* of the system immediately after the collision in terms of G, m, M_e, and R_e.



$$E = \frac{1}{2}(4m)v^2 + \left(-G\frac{M_e(4m)}{(2R_e)}\right)$$

$$= \frac{1}{2}(4m)\left(\left(\frac{1}{2}\left(G\frac{M_e}{2R_e}\right)^{1/2}\right)^2 + \left(-G\frac{M_e(4m)}{(2R_e)}\right)\right)$$

$$= \left(\frac{1}{8}\left(G\frac{M_e(4m)}{2R_e}\right)\right) + \left(-G\frac{M_e(4m)}{(2R_e)}\right)$$

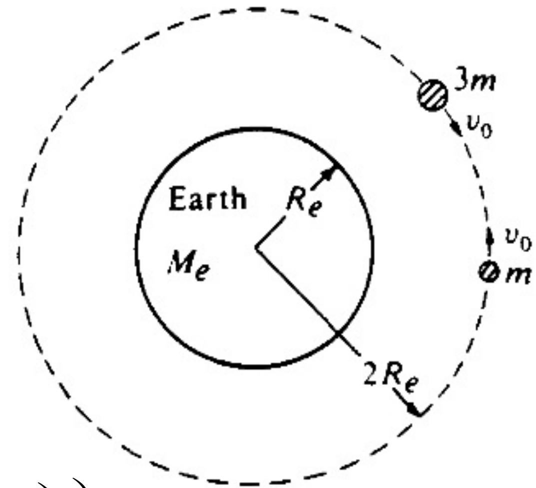
$$= -\frac{7}{8}\left(G\frac{M_e(4m)}{2R_e}\right)$$

As the gravitational potential energy is $U = \left(-G\frac{M_e(4m)}{(2R_e)}\right)$

apparently:

$$E = \frac{7}{8}U$$

Note: For this new combo-satellite to carry the new velocity in a circular orbit, its new radius would have to be:

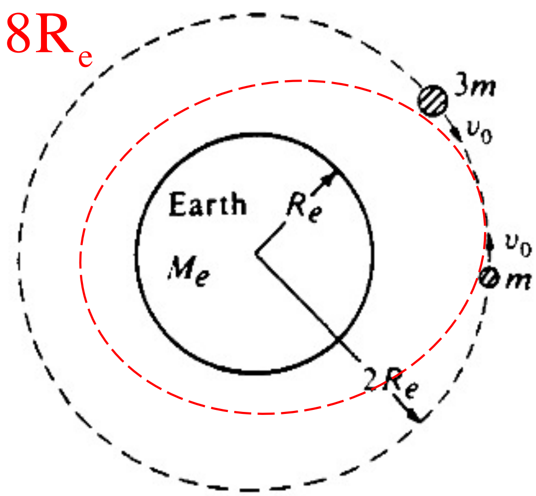


$$G\frac{M_e(4m)}{(r_{new})^2} = (4m)\left(\frac{v_{new}{}^2}{r_{new}}\right)$$

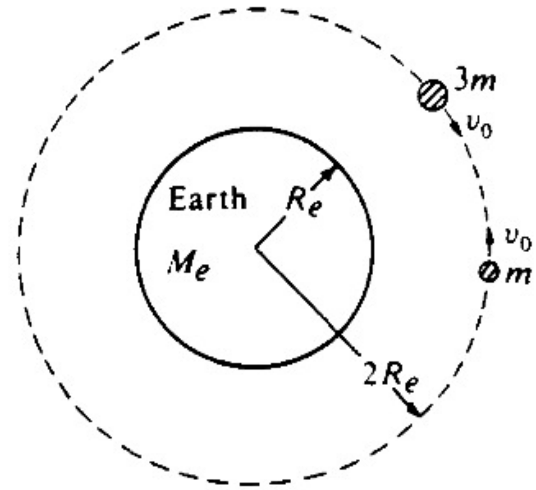
$$G\frac{M_e(4m)}{(r_{new})^2} = (4m)\left(\frac{\left(\frac{1}{2}\left(G\frac{M_e}{2R_e}\right)^{1/2}\right)^2}{r_{new}}\right) = (4m)\left(\frac{\left(\frac{1}{4}G\frac{M_e}{2R_e}\right)}{r_{new}}\right) = G\frac{mM_e}{2R_e r_{new}}$$

$$G\frac{M_e(4m)}{(r_{new})^2} = G\frac{mM_e}{2R_e r_{new}} \quad \Rightarrow \quad \frac{(4)}{(r_{new})} = \frac{1}{2R_e} \quad \Rightarrow \quad r_{new} = 8R_e$$

This wouldn't happen, though, as the new motion would become elliptical looking something like:



With the new radius, the *total mechanical energy* becomes:



$$E = \frac{1}{2}U$$

$$= \frac{1}{2}\left(-G\frac{M_e(4m)}{(r_{new})}\right)$$

$$= -2G\frac{M_e m}{(8R_e)}$$

Weird stuff to think about... Courtesy of Mr. White

It's the different escape speeds required for different planets that explains why some planets have atmospheres and others don't. Gas molecules have speeds that depend on their temperatures: the greater the temperature, the greater the average speed of the molecules, and the greater the chance is that they'll have a velocity that allows them to escape the planet.

Mercury? No atmosphere

Earth? Light molecules gone, heavier molecules remain

Jupiter? Even hydrogen can't escape!