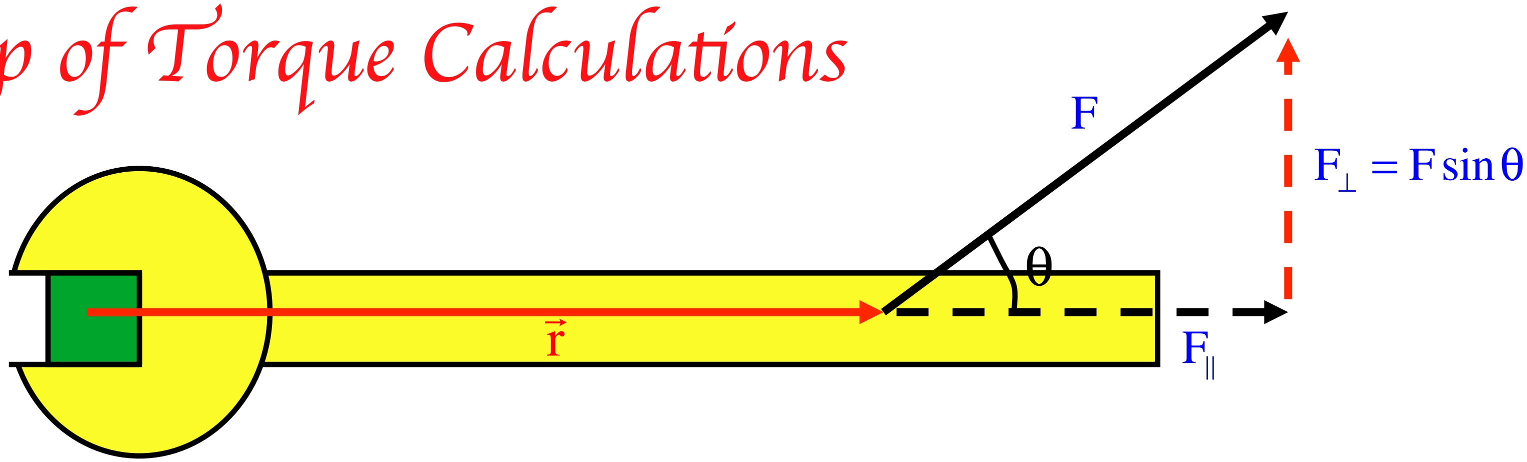
CHAPTER 12:
Rigid Bodies and Static Equilibrium

This chapter is amazingly out of place. Back when you were learning about *torques* and *Newton's Second Law*, it would have made sense to do some problems that required you to calculate torque quantities without having to deal with *angular accelerations* (i.e., problems in which the right side of the N.S.L. equation was zero). That is exactly what *rigid body/static equilibrium* problems require. In short, we should have dealt with the material in this chapter two chapters ago (I have no idea what the authors of our book were thinking). Having fomented that rant:

> *For a body* to be in static equilibrium, the sum of the forces on the body in any direction must add to zero (that is, it must not be accelerating in those directions), and the sum of the torques about any point on the body must add to zero (no angular acceleration).

A Recap of Torque Calculations



Formally defined, the MAGNITUDE of the torque $|\vec{\tau}|$ generated by the force on the wrench is mathematically equal to: $|\vec{\tau}| = |\vec{r}|\ F_\perp$

As can be seen in the graphic, the perpendicular component of the force is equal to:

$$F_\perp = F\sin\theta$$

where θ is defined as the angle between *the line of the force* and *the line of the position vector* $\vec{r}$. (This definition is going to be important later.)

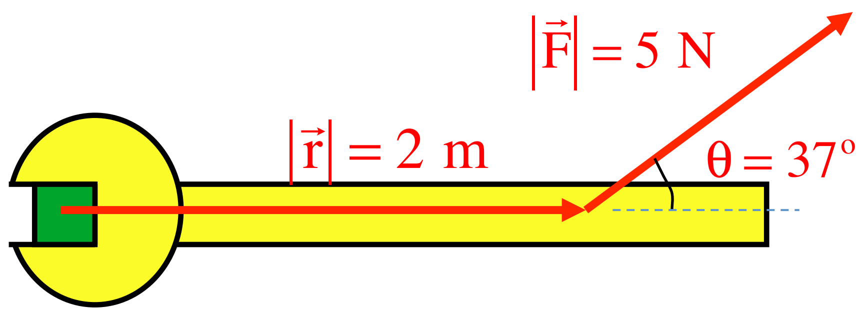
This means the torque can also be written as:

$$|\vec{\tau}| = |\vec{r}|\ F_\perp$$
$$= |\vec{r}||\vec{F}|\sin\theta$$

Three ways to calculate a torque using polar information:

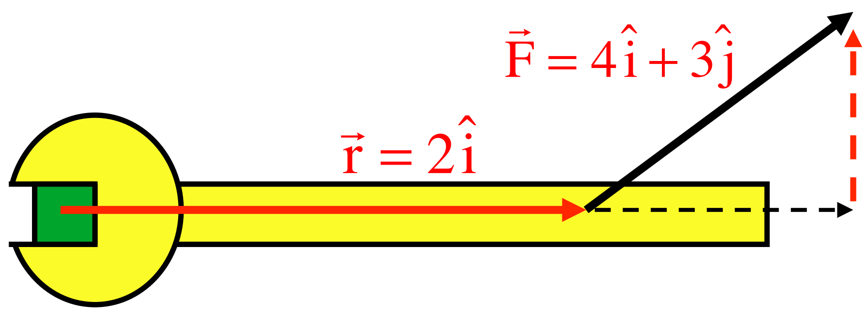
Definition approach:

$$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\theta$$
$$= (2\text{ m})(5\text{ N})\sin 37^\circ$$
$$= 6\text{ N}\bullet\text{m}$$



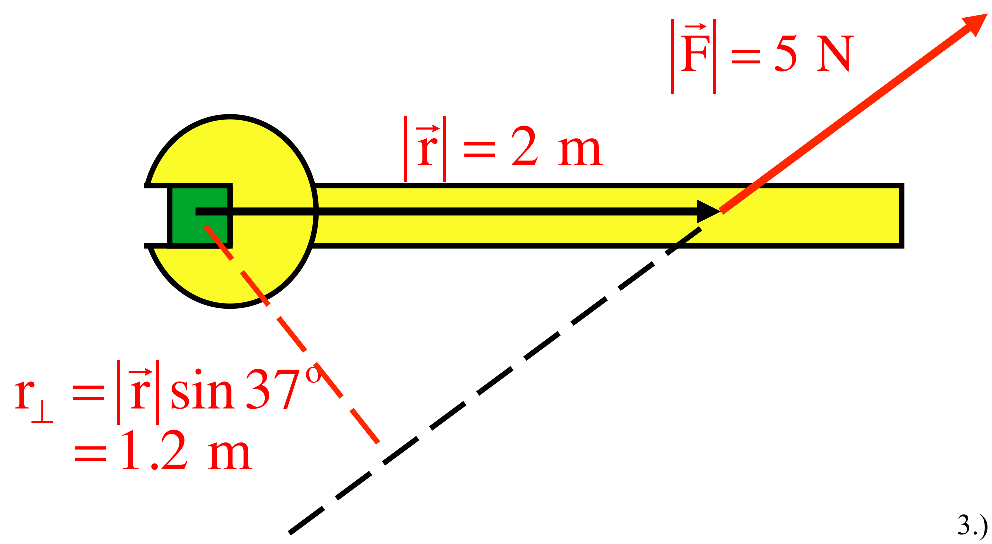
F-perpendicular approach:

$$|\vec{\tau}| = F_\perp |\vec{r}|$$
$$= (3\text{ N})(2\text{ m})$$
$$= 6\text{ N}\bullet\text{m}$$



r-perpendicular approach:

$$|\vec{\tau}| = r_\perp |\vec{F}|$$
$$= (1.2\text{ m})(5\text{ N})$$
$$= 6\text{ N}\bullet\text{m}$$



Why is the *r-perpendicular approach* so powerful (and mostly preferred)?

Can you find the shortest distance between a point and a line? If so, you can find $r_\perp$ for any situation. Called the moment arm, that is what $r_\perp$ is, the shortest distance between the *point about which you are taking the torque* and *the line of the force*. With it, $|\vec{\tau}| = r_\perp |\vec{F}|$.

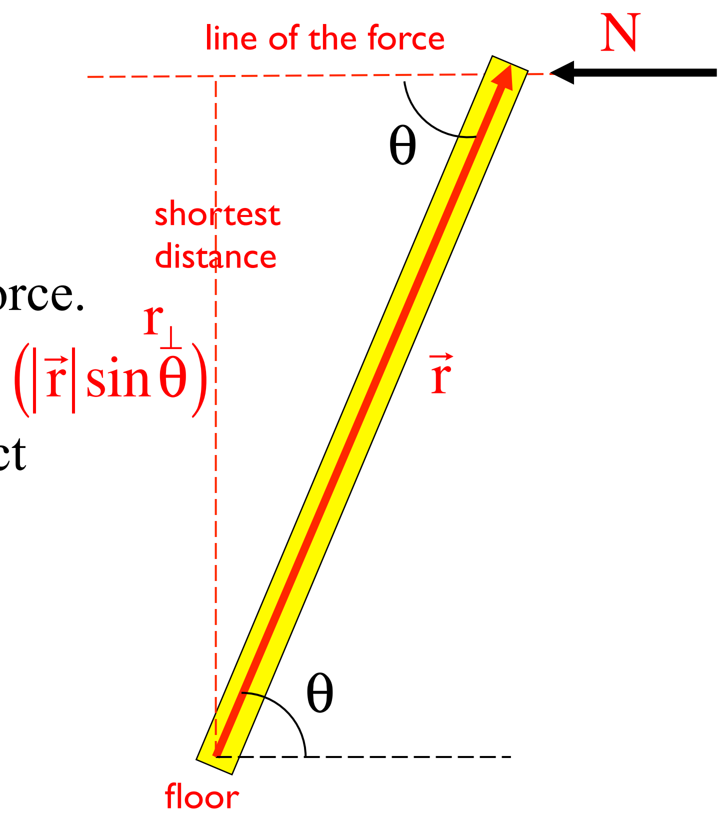
Example: A ladder sits against a wall. Using the $r_\perp$ approach, determine the torque generated by the wall's normal force *N* about the ladder's contact with the floor.

1. Identify the position vector $\vec{r}$ for the normal force.

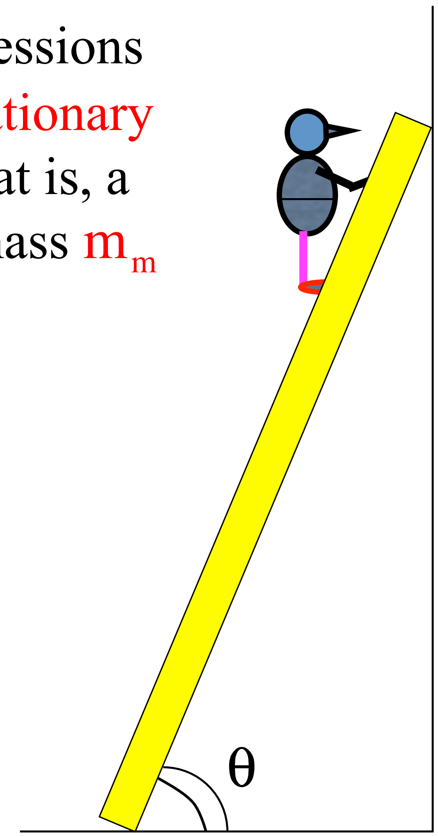
2. Identify the *line of N*.

3. Identify shortest distance between floor contact and line of N, then express in terms of $|\vec{r}|$.

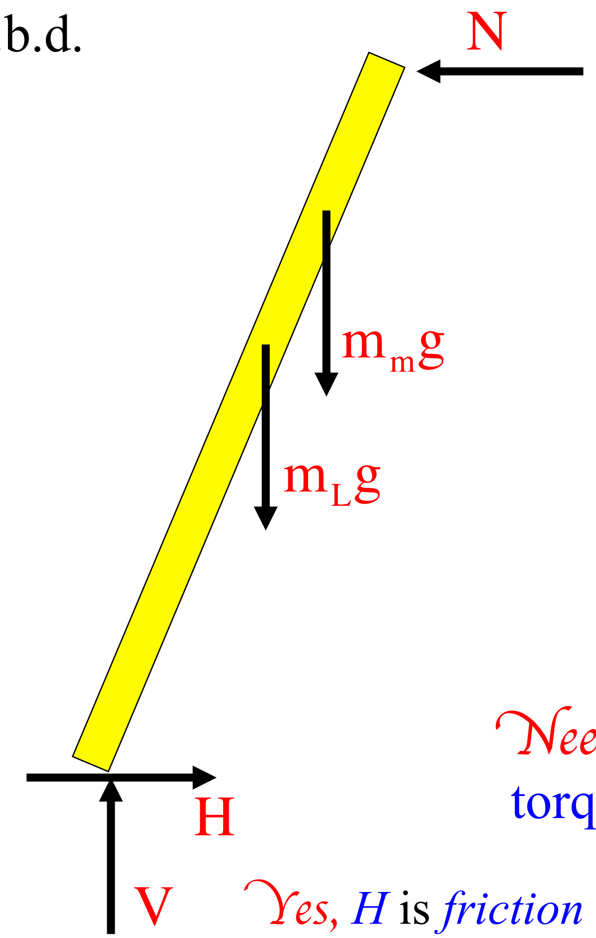
4. Determine the torque with $|\vec{\tau}| = r_\perp |\vec{F}|$.



Example 1: A standard rigid body problem: Derive expressions for all the forces acting at the floor and wall of an upright, stationary ladder of length L and mass m_L against a frictionless wall (that is, a wall that provides no vertical force component) if a man of mass m_m stands a distance 3L/4 from the floor.



f.b.d.



$\sum F_x$:

$$N - H = m_L a_x \quad (a_x = 0)$$

$$\Rightarrow \quad N = H$$

$\sum F_y$:

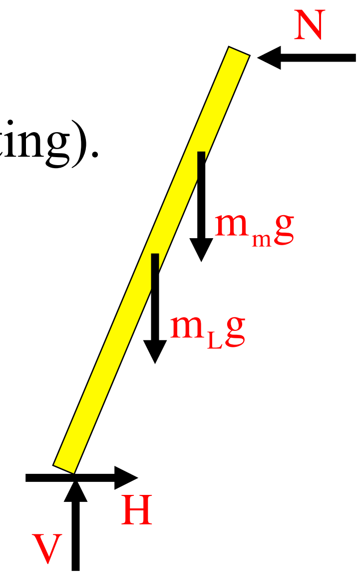
$$V - m_L g - m_m g = m_L a_x \quad (a_x = 0)$$

$$\Rightarrow \quad V = m_L g + m_m g$$

Need one more equation. Where to get it? By summing torques about ANY point and putting the sum equal to zero!

Yes, H is *friction* and *V* is a *normal*, but conventionally, this is the way they are denoted.

To get a third equation, you could sum the torques ABOUT ANY POINT and put that sum equal to zero (nothing is angularly accelerating). To start with, we'll sum about the ladder's *center of mass*.



$\sum \Gamma_{cm}$:

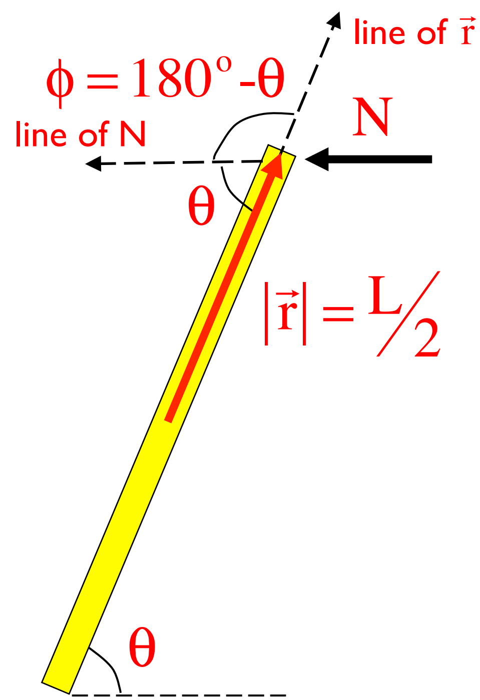
$$\tau_N + \tau_{m_L g} + \tau_{m_m g} + \tau_H + \tau_V = 0$$

$$N\left(\frac{L}{2}\right)\sin\left(180^\circ - \theta\right) + 0 - m_m g\left(\frac{L}{4}\cos\theta\right) + H\left(\frac{L}{2}\sin\theta\right) - \left(\frac{L}{2}\right)V\cos\theta = 0$$

Justification for for normal force *N* using definition approach:

$$\tau_N = |\vec{r}||\vec{F}|\sin\phi$$
$$= (N)\left(L/2\right)\sin\left(180^\circ - \theta\right)$$

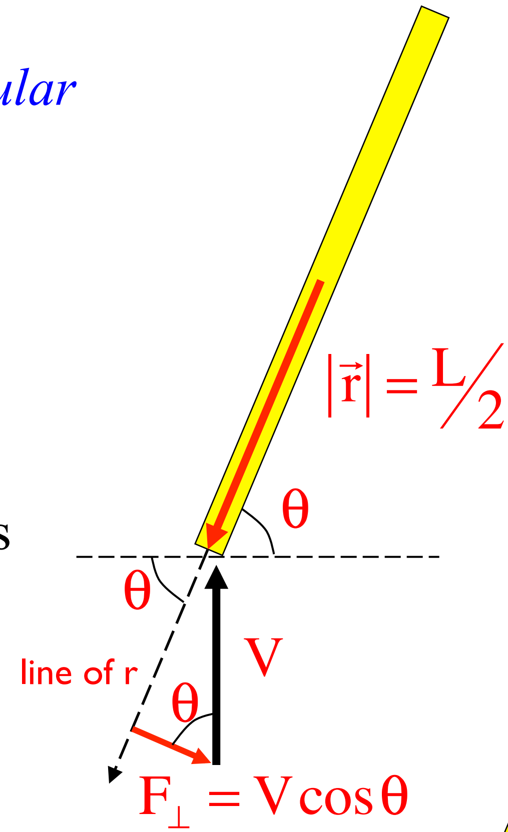
N motivates the ladder to angular accelerate about the c. of m. *counterclockwise,* so the torque is *positive* (as prescribed by the right-hand rule).



Justification for for vertical force *V* using *F-perpendicular* approach:

$$\tau_V = |\vec{r}| F_\perp = \left(\frac{L}{2}\right)(V\cos\theta)$$

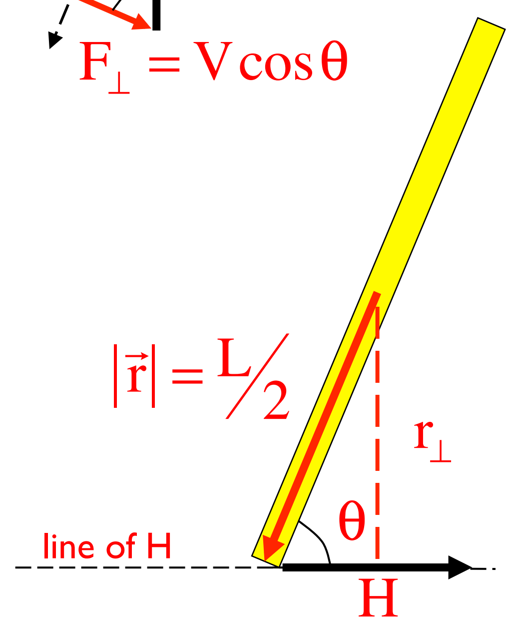
V motivates the ladder to angular accelerate about the c. of m. *clockwise,* so the torque is *negative* (as prescribed by the right-hand rule).



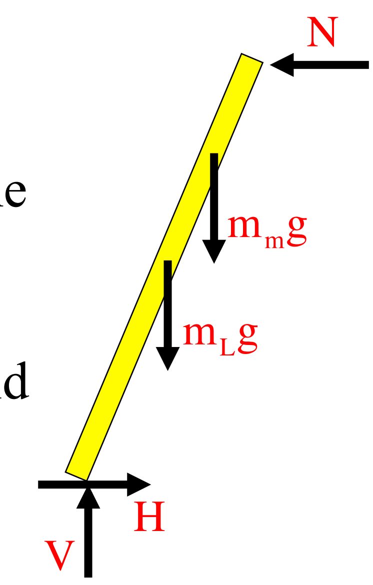
Justification for for horizontal force *H* using *r-perpendicular* approach (shortest distance between c. of m. and *line of H*):

$$\tau_H = r_\perp |\vec{F}| = \left(\frac{L}{2}\sin\theta\right)H$$

H motivates the ladder to angular accelerate about the c. of m. *counterclockwise,* so the torque is *positive* (as prescribed by the right-hand rule).



Point of order: When a force acts through the point about which a torque is taken, its torque about that point will be ZERO. Summing the torques about the *center of mass*, then, eliminated the torque due to $m_L g$. *If we had been clever*, though, we'd have summed the torques about contact point at the floor. In doing so, the torque due to both *H* and *V* would have been zero, which would have left us with only *one unknown*, the one we were looking to determine, *N*. Using the r-perpendicular approach on everything, *that* torque summation yields:



$$\sum \Gamma_{floor}:$$

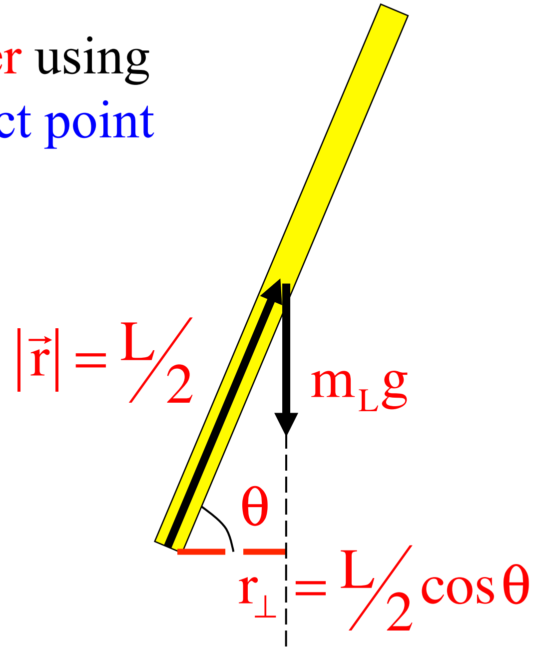
$$\tau_N \quad + \quad \tau_{m_L g} \quad + \quad \tau_{m_m g} \quad + \tau_H + \tau_V = 0$$

$$N(L)\sin\theta - m_L g\left(\frac{L}{2}\cos\theta\right) - m_m g\left(\frac{3L}{4}\cos\theta\right) + 0 + 0 = 0$$

$$\Rightarrow \quad N = \frac{(2m_L + 3m_m)gL\cos\theta}{4\sin\theta}$$

Justification for for the gravitational force *mg on the ladder* using r-perpendicular approach (shortest distance between contact point and *line of mg*):

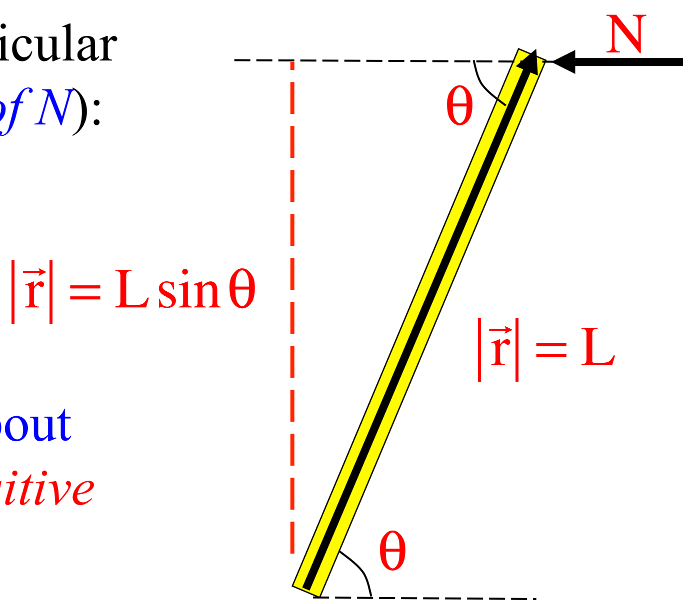
$$\tau_{mg} = r_{\perp}\left|\vec{F}\right|$$
$$= \left(L/_2 \sin\theta\right) m_L g$$



mg motivates the ladder to angularly accelerate about the floor *clockwise*, so the torque is *negative* (as prescribed by the right-hand rule).

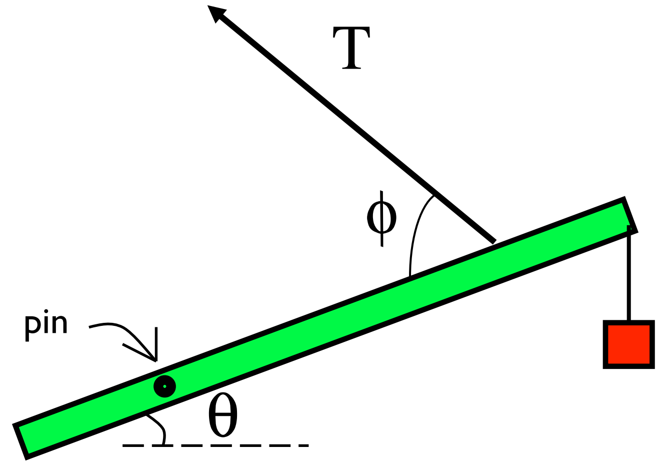
Justification for for normal force *N* using r-perpendicular approach (shortest distance between floor and *line of N*):

$$\tau_N = r_{\perp}\left|\vec{F}\right|$$
$$= \left(L \sin\theta\right) N$$



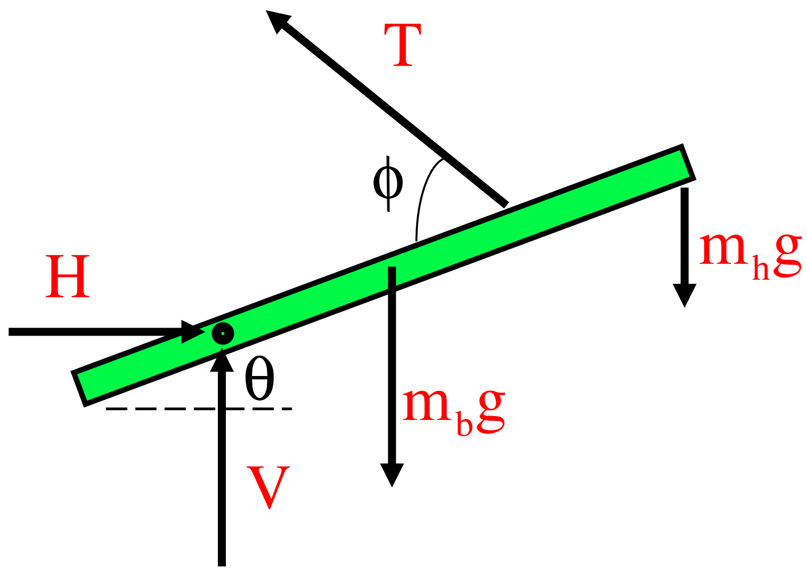
N motivates the ladder to angular accelerate about the floor *counterclockwise*, so the torque is *positive* (as prescribed by the right-hand rule).

Example 9: A beam of mass m_b and length L is pinned at an angle θ a quarter of the way up the beam (i.e., at $L/4$). A hanging mass m_h is attached at the end. Tension in a rope three-quarters of the way from the end keeps it stationary. What is known is:



$$m_b, m_h, L, g, \theta, \phi \text{ and } I_{cm,beam} = \frac{1}{12} m_b L^2$$

a.) Draw a f.b.d. identifying all the forces acting on the beam.

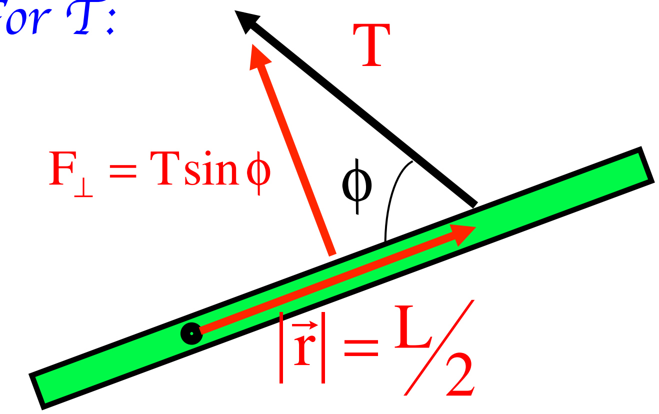


$$m_b, m_h, L, g, \theta, \phi \text{ and } I_{cm,beam} = \frac{1}{12} m_b L^2$$

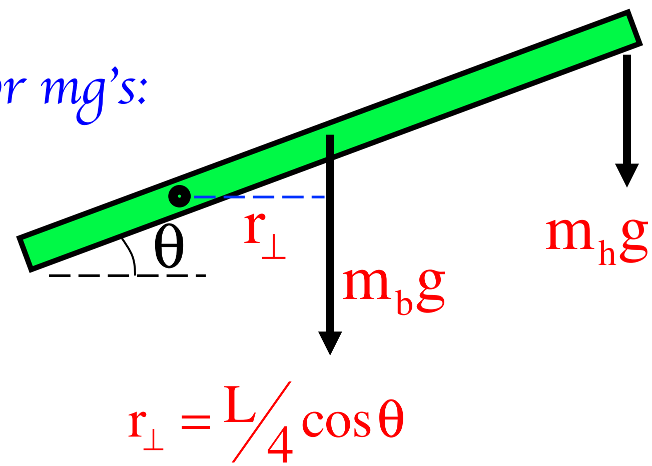
b.) *Derive* an expression for the tension in the line.

The clever thing to do here is to sum the torques about the pin. That will eliminate both *H* and *V* leaving you with only one unknown, *T*. What's even more clever is to use the *F-perpendicular approach* on *T* as that component is REALLY easy to determine (given ϕ).

For T:



For mg's:

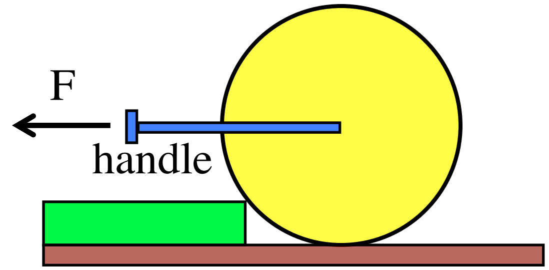


$\sum \tau_{pin}$:

$$T\sin\phi\left(\frac{L}{2}\right) - m_b g\left(\frac{L}{4}\cos\theta\right) - m_h g\left(\frac{3L}{4}\cos\theta\right) = I_{pin}\alpha^{\,0}$$

$$\Rightarrow \quad T = \frac{(m_b + 3m_h)g}{2}\left(\frac{\cos\theta}{\sin\phi}\right)$$

Example 3: A handle is attached to the central axis of a cylinder of radius *R* and mass *m*. The cylinder is rolled along a flat surface until it reaches a curb of height *R/3*. You may assume you know:

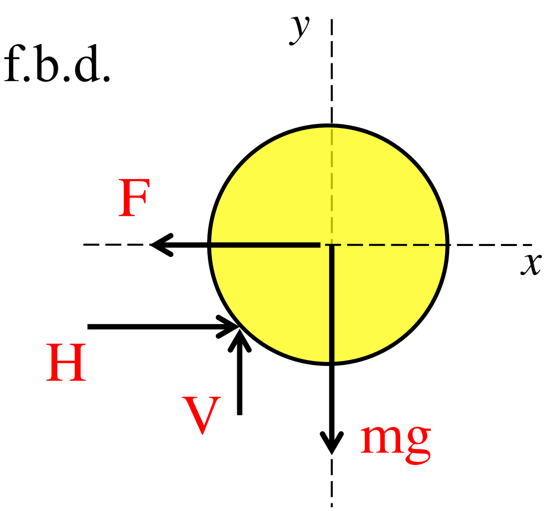


$$m, R, g, \text{ and } I_{cyl,cm} = \frac{1}{2}mR^2$$

a.) Derive an expression for the minimum force *F* the handle must apply to lift the cylinder over the curb if it is to be kept in the horizontal.

The farther the cylinder gets lifted up over the curb, the less force will be required to continue the motion, so the maximum force will be required when the cylinder just lifts off the flat surface. At that point, the normal force at the surface will go to zero, there will be a horizontal and vertical force at the curb and the f.b.d. for the situation looks like:

f.b.d.



$$m, R, g, \text{ and } I_{cyl,cm} = \frac{1}{2} mR^2$$

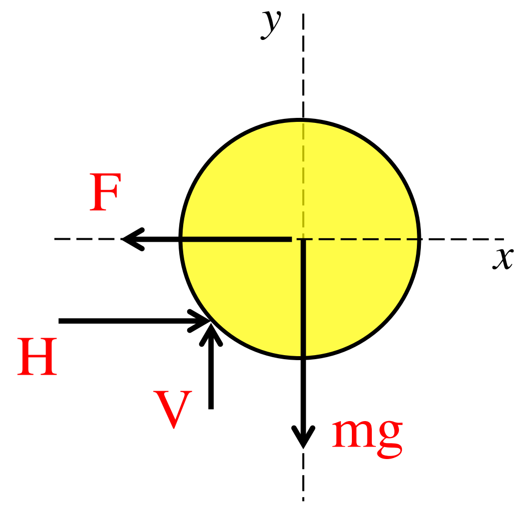
Most people start a problem like this by summing forces, which we can do.

forces in x-direction:

$$\sum F_x :$$
$$F - H = ma_x \;(a_x = 0)$$
$$\Rightarrow \quad F = H$$

forces in y-direction:

$$\sum F_y :$$
$$V - mg = ma_y \;(a_y = 0)$$
$$\Rightarrow \quad V = mg$$



That leaves us with three unknowns, *V, H,* and *F,* and the need for a third equation. That third comes from summing the torques about any point we choose.

The clever choice is to sum about the contact point between the curb and the cylinder. In doing so, we eliminate *H* and *V,* and in doing so come to a horrible realization. We *could* have solved this problem *without ever summing the forces* if we had just *start* with this step.

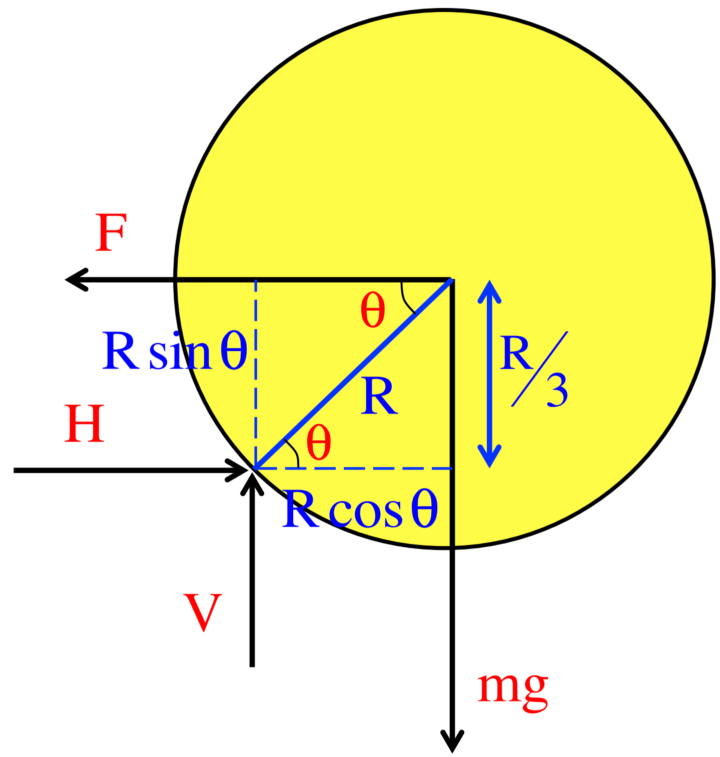
$m, R, g, \text{ and } I_{cyl,cm} = \frac{1}{2} mR^2$

torques about the curb
(using r-perpendicular approach):

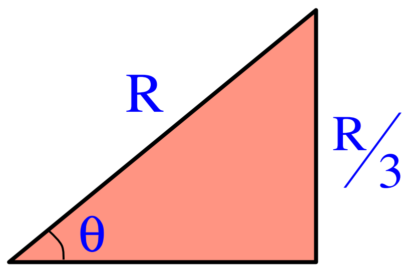
$\sum \tau_{curb}$:

$$F(R\sin\theta) - mg(R\cos\theta) = I_{cm}\alpha^{\,0}$$

$$\Rightarrow \quad F = \frac{mg\cos\theta}{\sin\theta}$$



Noting from the geometry that the angle is:

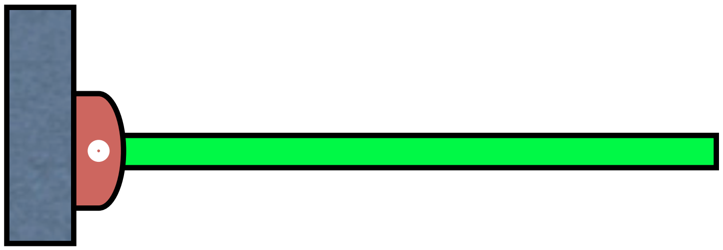


$$\Rightarrow \quad \sin\theta = \frac{R/3}{R} \quad \Rightarrow \quad \theta = 19.47^\circ$$

So

$$F = \frac{mg\cos(19.47^\circ)}{\sin(19.47^\circ)} = .35mg$$

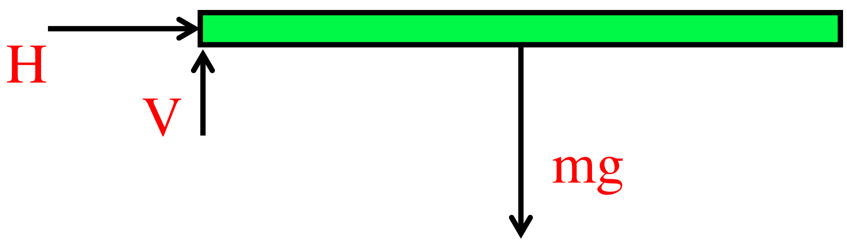
Example 3: Show that the effect of all the bits of mass along a beam are the same as thinking of all the mass centered at the beam's *center of mass*. That is, show that the torque about the pin is the same for both cases. Known is *m, L,* and *g*.



Torque, assuming all the mass is centered at the beam's center of mass:

$$\sum \tau_{pin}:$$

$$-mg\left(\frac{L}{2}\right) = -I_{pin}\alpha$$



But what's really happening?

Tiny bits of mass all along the beam feel the effect of gravity, bearing down on the beam. To justify our assumption, we need to determine how much torque each of those individual bits generate, then sum all of those differential torques to see if their magnitude comes out to be *mg(L/2)*.

As before, the torque due to *H* and *V* about the pin is zero.

Identify an arbitrary bit of mass *dm* at an arbitrary position *x*.

Define the differential gravitational force associated with *dm*, then do torque calc.

$$\sum \tau_{pin}:$$

$$|\vec{\tau}| = \int d\tau_{gravity} \qquad \left(= I_{pin}|\alpha|\right)$$

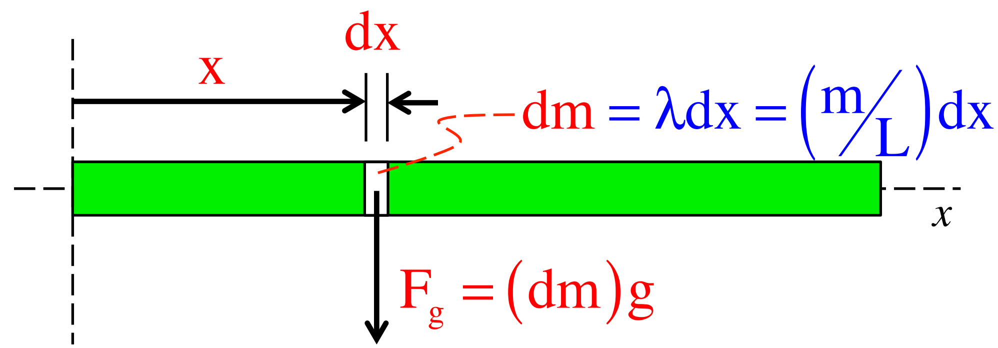
$$= \int \vec{r} \times d\vec{F}_g$$

$$= \int_{x=0}^{L} x\left[(dm)g\right]\sin 90^\circ$$

$$= \int_{x=0}^{L} x\left[(\lambda dx)g\right] = \left(\frac{m}{L}\right)g\int_{x=0}^{L} x\,dx$$

$$= \left(\frac{m}{L}\right)g\left(\frac{x^2}{2}\right)\Bigg|_{x=0}^{L} = \left(\frac{m}{L}\right)g\left(\frac{L^2}{2}\right)$$

$$= mg\left(\frac{L}{2}\right) = I_{pin}|\alpha|$$



The assumption is a good one!