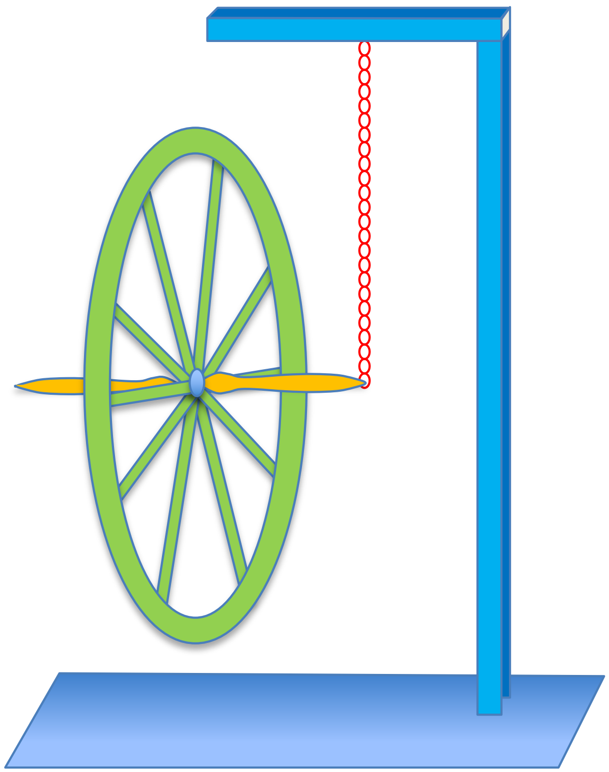
Change your orientation, you change your motion . . .
(courtesy of Yulia Lipnitskaya

A Little Non-AP Tom-Foolery



A wheel suspended from the ceiling is held motionless in the position shown.

What happens when the wheel is released?

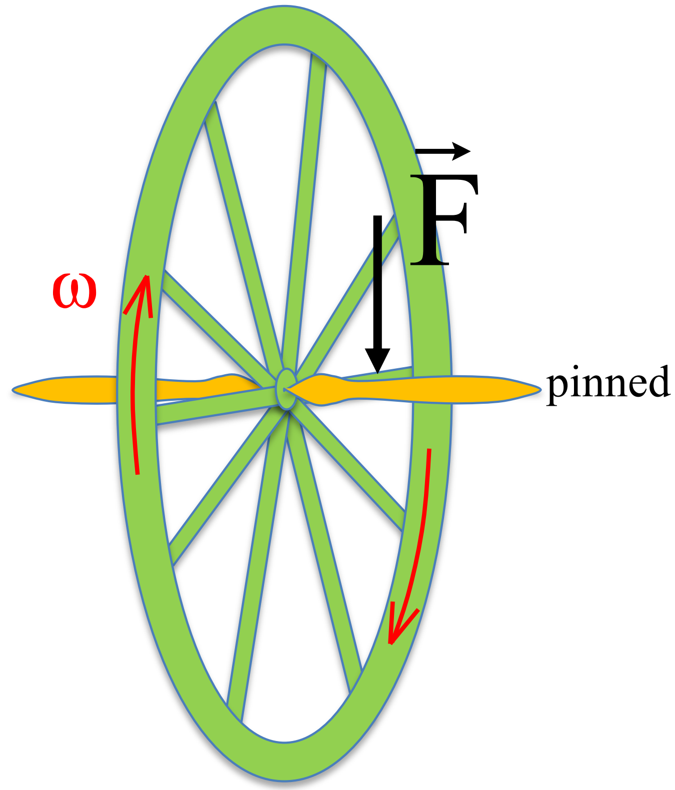
It flops down . . .

Now the same wheel is made to spin before it is released – will anything change?

It doesn't flops down . . . rather, it stays upright and begins to precess about its hold-point . . .

In the same vein:

As demonstrated in class, a force quickly applying a torque to the pinned axle of a rotating wheel will not motivate the wheel to follow the direction of the applied force. Instead, it will jerk the wheel to the right or left, depending upon the direction of the wheel's rotation.



By the end of this unit, you will have the tools to predict that this should happen!

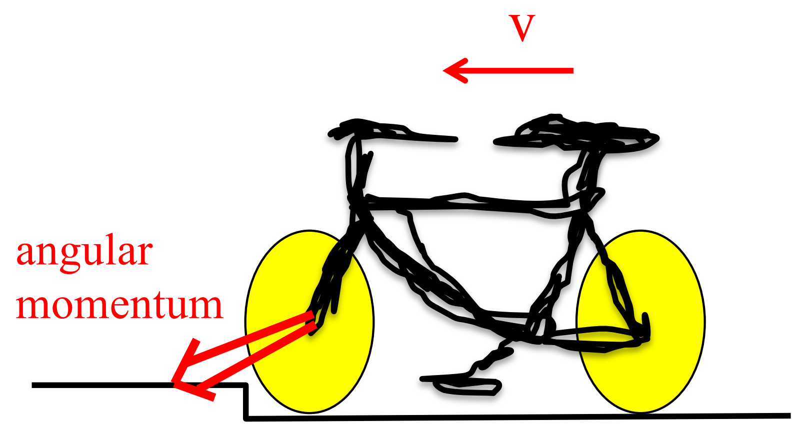
The Island Series:

You have been kidnapped by a crazed physics nerd and left on an island with twenty-four hours to solve the following problem. Solve the problem and you get to leave. Don't solve the problem and you don't.

The problem: You are on a street walking a bicycle toward a curb. You want to lift the bike by its seat up and over the curb without having the front tire flopping around as you do it. Your task is to divine a maneuver that will insure the wheel stays straight.

Solution to Island Problem

Just before you get to the curb, accelerate the bike so the front wheel is rotating as fast as possible. The front wheel's *large angular momentum* vector, directed along the wheel's axle, will keep the wheel orientated as set. That is, as long as there are no *external torques* acting, the wheel will keep its orientation just as a gyroscope keeps its.



CHAPTER 11:
Angular Momentum

We began the Momentum Chapter with an observation. If you want to stop an object in a given amount of time, the size of force required to make the stop will be governed by two parameters: the *mass* of the object (i.e., a *relative measure of the body's resistance to changing its motion*) and its *velocity*. From that observation, the idea of *MOMENTUM* ($\vec{p} = m\vec{v}$) was born.

In the same sense, if you want to stop an object's *rotational motion* in a given amount of time, the size of *torque* required to make the stop will be governed by two parameters: a relative measure of the body's resistance to changing its *rotational* motion (i.e., it's *moment of inertia*) and its *angular* velocity. From this we define *ANGULAR MOMENTUM* ($\vec{L} = I\vec{\omega}$).

And just as we developed theory that led us to relationships between *momentum* and *force* through *impulse,* and to the *conservation of momentum*, all of those avenues can be followed in an exact parallel in the world of rotation.

Summary *(with new stuff in red)*

translational world | **rotational world**

translational parameters: $x,\ v,\ a$

force: $\vec{F}$

Newton's Second Law: $F_{net,x} = ma_x$

momentum: $p_x = mv_x$

impulse: $\vec{F}_{net} = \frac{d\vec{p}}{dt} \Rightarrow \vec{F}_{net}dt = d\vec{p}$

$\Rightarrow \vec{F}_{net}\Delta t = \Delta\vec{p}$

conservation of energy

rotational parameters: $\theta,\ \omega,\ \alpha$

torque: $\vec{\tau}_F = \vec{r}x\vec{F}$

Newton's Second Law: $\tau_{net} = I\alpha$

angular momentum: $L = I\omega$

(or for a point mass about a point): $\vec{L} = \vec{r}x\vec{p}$

angular impulse: $\vec{\tau}_{net} = \frac{d\vec{L}}{dt} \Rightarrow \vec{\tau}_{net}dt = d\vec{L}$

$\Rightarrow \vec{\tau}_{net}\Delta t = \Delta\vec{L}$

conservation of energy

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

conservation of momentum

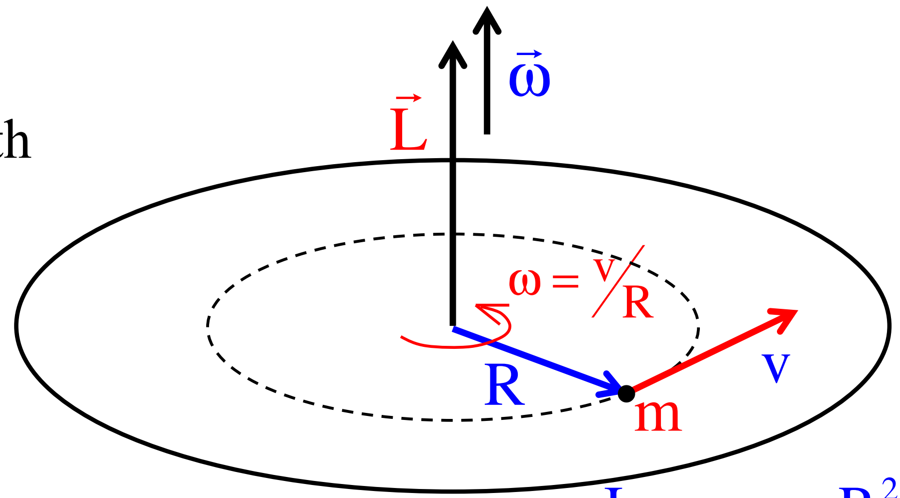
$$\sum p_{1,x} + \sum F_{ext,x}\Delta t = \sum p_{2,x}$$

conservation of angular momentum

$$\sum L_1 + \sum \Gamma_{ext}\Delta t = \sum L_2$$

Angular Momentum: There's a lot here. To make life easier, we are going to go after these ideas in pieces.

Example 1: A point mass *m* moves with velocity *v* in a circular path of radius *R*. Determine its *angular momentum* using:



$$I_{\text{pt mass}} = mR^2$$

a.) Translational parameters:

$$\begin{aligned} \left|\vec{L}\right| &= \left|\vec{r} \text{ x } \vec{p}\right| \\ &= (R)(mv)\sin 90^\circ \\ &= mvR \end{aligned}$$

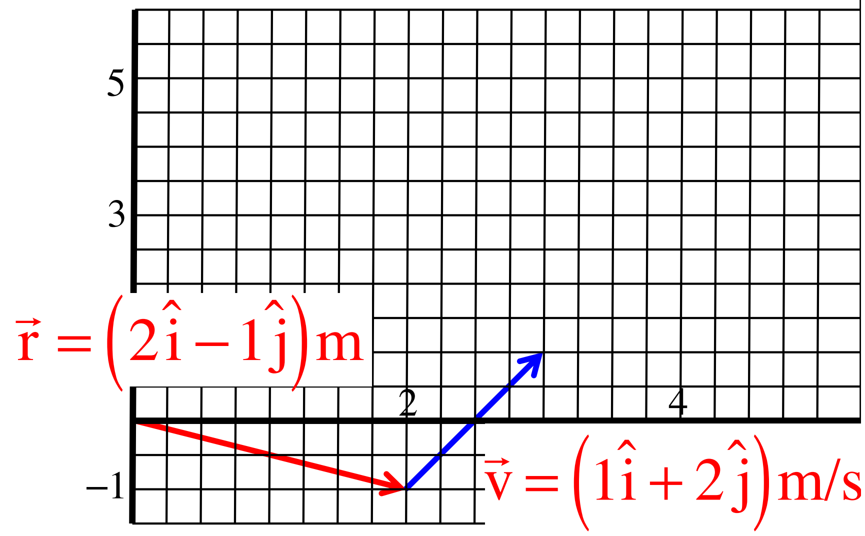
Note the direction of the cross product!

b.) Rotational parameters:

$$\begin{aligned} L &= I\omega \\ &= \left(mR^{\cancel{2}}\right)\left(\frac{v}{\cancel{R}}\right) \\ &= mvR \end{aligned}$$

Note: the *direction* of the *angular momentum* is *perpendicular to the plane of the motion* ($+\hat{k}$ direction), as would be expected of a counterclockwise rotation.

Example 2: What is the magnitude and direction of the *angular momentum*, relative to the origin, for a 2 kg particle moving at constant velocity $\vec{v} = (1\hat{i} + 2\hat{j})\,\text{m/s}$ at the instant it is at $\vec{r} = (2\hat{i} - 1\hat{j})\,\text{m}$?



$$\left|\vec{L}\right| = \left|\vec{r} \times \vec{p}\right|$$

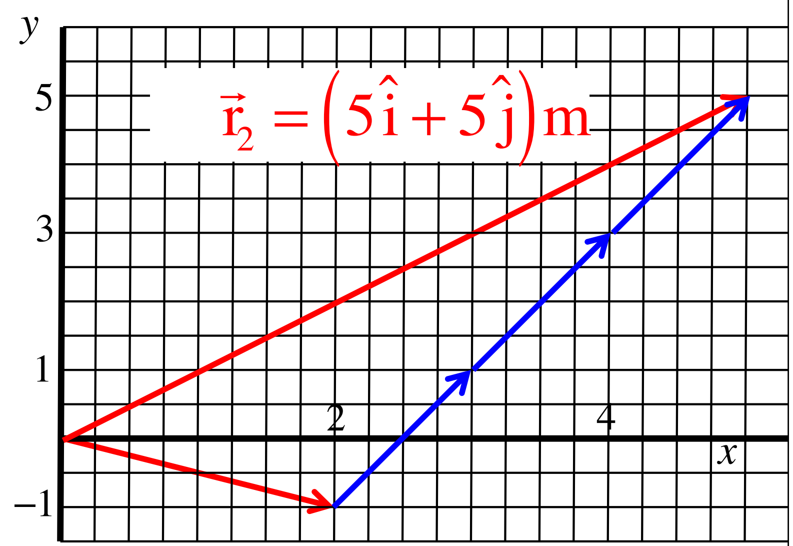
Evaluating a cross product when given vectors in *unit vector notation* requires matrix manipulation (think back to torque calculations). Noting that the momentum is

$$m\vec{v} = 2(1\hat{i} + 2\hat{j}) = (2\hat{i} + 4\hat{j})\,\text{m/s}$$

We can write:

$$\vec{r} \times (m\vec{v}) = \left|\begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ 2 & -1 & 0 & 2 & -1 \\ 2 & 4 & 0 & 2 & 4 \end{array}\right| \quad \Rightarrow \quad (8 - (-2))\hat{k} = (10\hat{k})\,\text{kg} \bullet \text{m}^2/\text{s}$$

Example 3: Our 2 kg particle (the one moving at constant velocity $\vec{v} = (1\hat{i} + 2\hat{j})$ m/s when at $\vec{r} = (2\hat{i} - 1\hat{j})$ m at $t = 0$), what is its *angular momentum* at $t = 3$ seconds?



after 3 seconds of constant velocity

Using kinematics, at $t = 3$ seconds, the body's position will be:

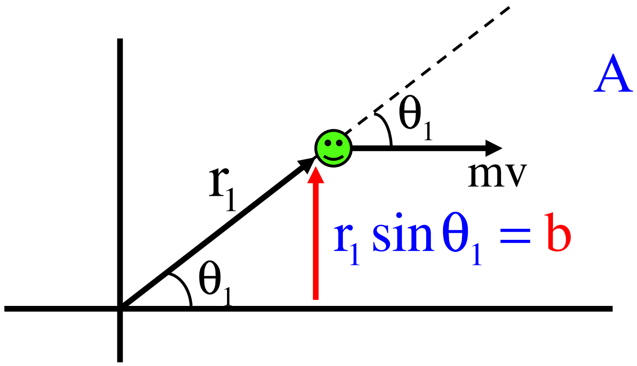
$$\begin{aligned}\vec{r}_2 &= \vec{r}_1 + \vec{v}\Delta t \\ &= (2\hat{i} - 1\hat{j}) + (1\hat{i} + 2\hat{j})(3 \text{ sec}) \\ &= (5\hat{i} + 5\hat{j})\end{aligned}$$

Momentum is still $m\vec{v} = (2\hat{i} + 4\hat{j})\,\text{kg} \bullet \text{m/s}$, so

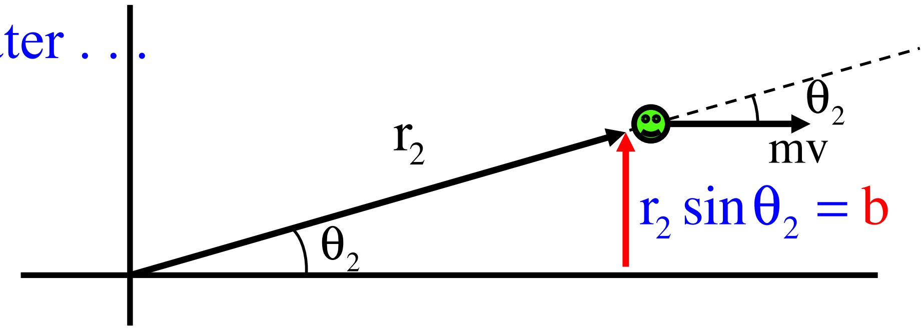
$$\vec{r}\text{x}(m\vec{v}) = \left|\begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ 5 & 5 & 0 & 5 & 5 \\ 2 & 4 & 0 & 2 & 4 \end{array}\right| \quad \Rightarrow \quad \left(20 - (10)\right)\hat{k} = (10\hat{k})\,\text{kg} \bullet \text{m}^2/\text{s}$$

From this, a HUGE observation: Objects moving in a straight line HAVE *angular momentum*, and the *angular momentum* is CONSTANT!

This seemingly insane bit of amusement is actually grounded in intuitively sound reasoning. To see how, consider the simplest motion possible, a mass moving with constant velocity *parallel to the x-axis*. How does the *angular momentum* calculate out at several points in that case?



A little bit later . . .



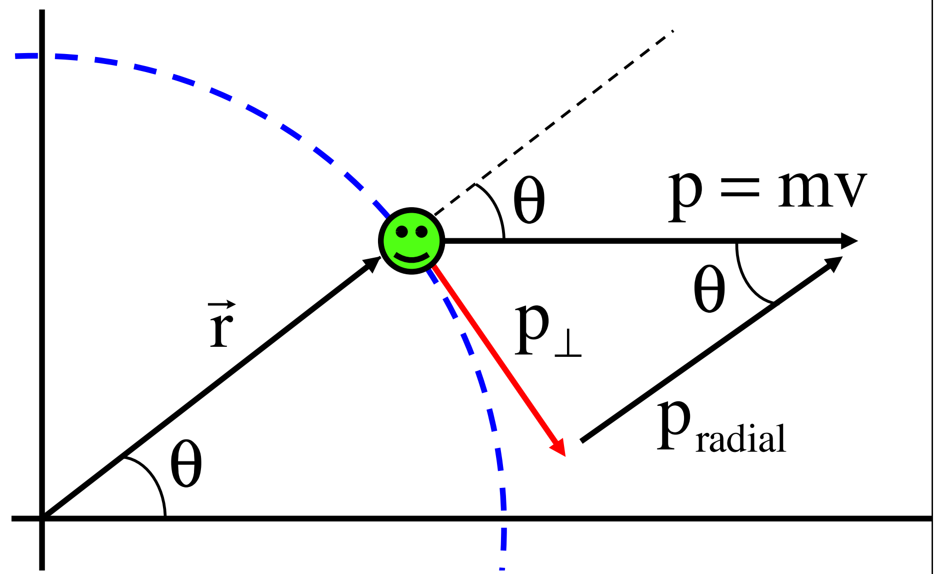
$$\begin{aligned}\left|\vec{L}\right| &= \vec{r} \text{ x } \vec{p}\\ &= r_1(mv)\sin\theta_1\\ &= mv(r_1\sin\theta_1)\\ &= mv(b)\end{aligned}$$

$$\begin{aligned}\left|\vec{L}\right| &= \vec{r} \text{ x } \vec{p}\\ &= r_2(mv)\sin\theta_2\\ &= mv(r_2\sin\theta_2)\\ &= mv(b)\end{aligned}$$

Same angular momentum! But how can a body moving STRAIGHT have *angular* momentum?

Consider a body moving in circular motion. Its *angular momentum* will equal:

$$|L| = \vec{r} \times \vec{p}$$
$$= p_{\perp} r$$



where $p_{\perp}$ is the body's momentum. This momentum will be *perpendicular* to the position vector and tangent to the path. Nobody would argue that this body's motion was without *angular momentum*, as its motion is *circular*!

Now consider a particle moving parallel to the x-axis with momentum *mv*, as shown in the sketch.

It will have a momentum component that is radial and outward from the origin, and a tangential component ($p_{\perp}$) that is perpendicular to the $\vec{r}$. In other words, one of its components will be exactly like the momentum involved in the object that was executing a pure rotation, that had *angular momentum*. Conclusion, *this body, moving in a straight line, will also have angular momentum* (assuming its velocity vector doesn't go thru the reference point).

Conservation of Angular Momentum

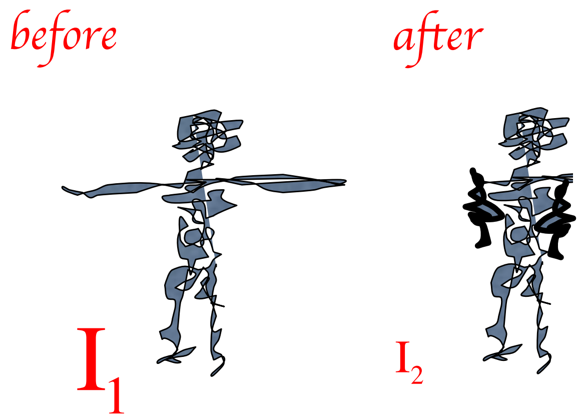
Remember how we proceeded through derivation of the *conservation of momentum* (start with the impulse relationship, apply it to two bodies that collide with one another where one has an external force via a jet pack applied to it, add the impulse relationships written for each body over the course of the collision, then leave the *external impulse terms* on the lefts side of the equation accompanied by the *beginning of the interval* variables with the *end of the interval* variables on the right side of the equation).

If we did a *rotational analogue* to that with two rotating objects interacting with an external torque acting in the mix, acknowledging that everything would be happening in "one dimension" (i.e., along the $\hat{k}$-direction), we could rearrange the impulse relationships likewise yielding:

$$\sum L_1 + \sum \Gamma_{ext} \Delta t = \sum L_2$$

This relationship essentially says that as long as there are not external *torque-related impulses* acting, the *angular momentum* of a system will not change over time. This is the *conservation of angular momentum* theorem.

Example 4: An ice skater with arms out has an angular speed of ω_1 and a moment of inertia I_I. She pulls her arms in.



a.) What happens to her *moment of inertia* as she pulls her arms in?

(it decreases)

b.) What is her new *angular momentum*?

(no external torques, so it doesn't change)

c.) What was her new angular speed?

There are *no* external torques acting on the woman, so *conservation of angular momentum* yields:

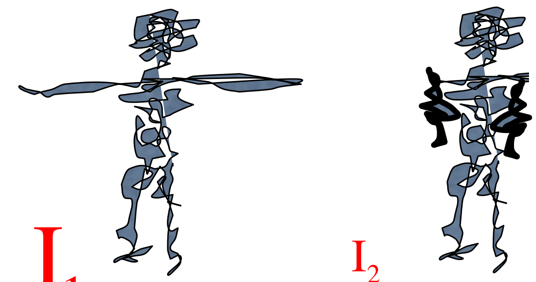
$$\sum L_1 + \sum \cancelto{0}{\Gamma_{ext}} \Delta t = \sum L_2$$

$$I_1\omega_1 + 0 = I_2\omega_2$$

$$\Rightarrow \quad \omega_2 = \frac{I_1\omega_1}{I_2}$$

. . . And as $I_2 < I_1$, her angular velocity increases.

Example 4 (cont'): An ice skater with arms out has an angular speed of ω_1 and a moment of inertia I_1. She pulls her arms in.



d.) Is mechanical energy conserved during this action? Justify and comment.

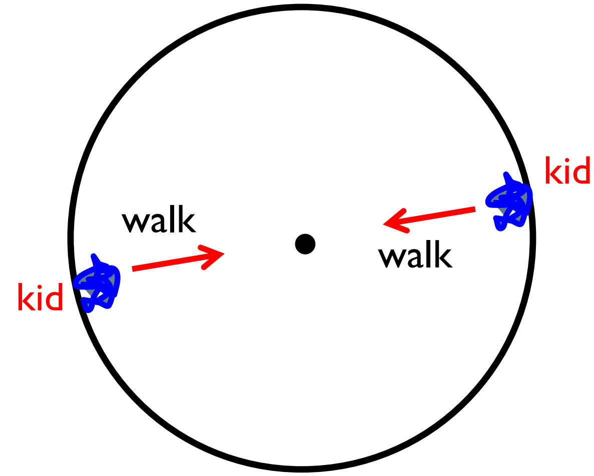
Just by using your head, chemical energy in your muscles must be burned to force your arms inward, so you might expect that the mechanical energy in the system would *not* be conserved. Looking at the math, though:

$$E_o = \frac{1}{2} I_1 (\omega_1)^2 \quad \text{for the initial mechanical energy}$$

$$E_2 = \frac{1}{2} I_2 (\omega_2)^2$$

$$= \frac{1}{2} I_2 \left(\frac{I_1}{I_2} \omega_1 \right)^2 = \frac{1}{2} \cancel{I_2} \left(\frac{I_1^{\,2}}{I_2^{\cancel{2}}} \omega_1^{\,2} \right) = \left(\frac{1}{2} I_1 \omega_1^{\,2} \right) \left(\frac{I_1}{I_2} \right) = E_O \left(\frac{I_1}{I_2} \right)$$

But $I_1/I_2 > 1$, so $E_2 > E_o$ and mechanical energy is NOT conserved. This should not be a surprise. When the *moment of inertia* diminishes and *angular velocity* gets proportionally LARGER due to *conservation of angular momentum*, KE must go UP as it is governed by velocity (remember, $\frac{1}{2} I \omega^2$).



Example 5: Another standard problem is the *merry-go-round* problem. A *merry-go-round*, assumed to be a disk, has mass M and radius R. It also has two kids who push the m.g.r.'s outer edge by running along side of it to get it up to an angular speed of ω_1. The kids, each of which have a mass of m_k, then jump on and start to walk toward the center of the m.g.r. When they get to within $R/3$ units from the center, they stop. What is their speed at that point?

As they walk inward, each kid applies a force, hence torque, to the m.g.r., that changes the m.g.r.'s *angular velocity*. As a Newton's Third Law action/action pair, the m.g.r. applies a force, hence torque, to the kids changing *their* angular velocity. As these are all internal impulses, *conservation of angular momentum* is applicable.

Interestingly, there are TWO ways we can go here with the kids:

treating the two kids as rotating point masses:

$$\sum L_1 + \sum \Gamma_{ext}\Delta t = \sum L_2$$

$$\left(2 \quad I_{kid,1} \quad \omega_1 + \quad I_{mgr} \quad \omega_1\right) + \quad 0 \quad = \left(2 \quad I_{kid,2} \quad \omega_2 + \quad I_{mgr} \quad \omega_2\right)$$

$$\left(2\left(m_k \cancel{R^2}\right)\omega_1 + \left(\frac{1}{2}M\cancel{R^2}\right)\omega_1\right) + \quad 0 \quad = \left(2\left(m_k\left(\frac{\cancel{R}}{3}\right)^2\right)\omega_2 + \left(\frac{1}{2}M\cancel{R^2}\right)\omega_2\right)$$

$$\Rightarrow \qquad \omega_2 = \frac{2m_k + M/2}{2m_k/9 + M/2}\omega_1 = \frac{18(4m_k + M)}{4m_k + 9M}\omega_1$$

the kid's velocities:

$$v = \left(\frac{R}{3}\right)\omega_2 = \frac{\cancelto{6}{18}(4m_k + M)}{(4m_k + 9M)}\left(\frac{R}{\cancel{3}}\right)\omega_1$$

$$= \frac{6(4m_k + M)}{(4m_k + 9M)}R\omega_1$$

treating the kids as point masses:

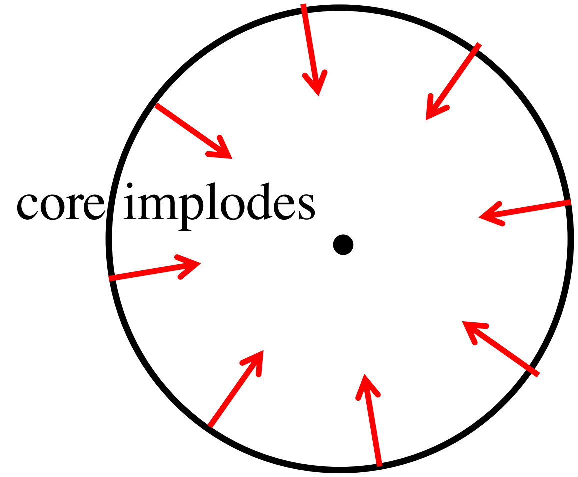
$$\begin{array}{ccccc} \sum L_1 & + & \sum \Gamma_{ext}\Delta t & = & \sum L_2 \\ \left(2\;\vec{r}_1 x\vec{p}_1 + I_{mgr}\;\omega_1\right) & + & 0 & = & \left(2\;\vec{r}_2 x\vec{p}_2 + I_{mgr}\;\omega_2\right) \end{array}$$

$$\left(2\left(m_k v_1 R\right) + \left(\frac{1}{2}MR^2\right)\omega_1\right) + 0 = \left(2\left(m_k v_2\left(\frac{R}{3}\right)\right) + \left(\frac{1}{2}MR^2\right)\frac{v_2}{\left(R/3\right)}\right)$$

$$\left(2\left(m_k \left(\cancel{R}\omega_1\right)\cancel{R}\right) + \left(\frac{1}{2}M\cancel{R^2}\right)\omega_1\right) + 0 = \left(2\left(m_k v_2\left(\frac{\cancel{R}}{3}\right)\right) + \left(\frac{1}{2}M\cancel{R^2}\right)\frac{v_2}{\left(\cancel{R}/3\right)}\right)$$

$$\Rightarrow \qquad v_2 = \frac{2m_k + M/2}{2m_k/3 + 3M/2}R\omega_1 = \frac{6\left(4m_k + M\right)}{4m_k + 9M}R\omega_1$$

Same solution either way. What's important is the set-up, not all the nasty math.



Example 6: In 1967 as a graduate student, Jocelyn Bell (aca Dame Jocelyn Bell Burnett) observed, in the face of scant support from her advisor, Antony Hewish, the first pulsar. In 1974, in a classic "keep 'em barefoot and pregnant" move, the all male, presumably all white Nobel committee gave Hewish the Nobel Prize in Physics for the discovery while ignoring Bell altogether. With that monumental injustice in mind, consider the lowly pulsars:

When a star with a core between 1.4 and 1.8 solar masses dies, it explodes spectacularly in what is called a supernova. (Example: In 1054, a supernova occurred that was observed by the Chinese and was visible *during the day* for two weeks.) When a supernova happens, the outer part of the star blows outward creating what is called a nebulae (the supernova in 1054 created the *Crab Nebulae*) and the core is blown inward. The *implosion* is so violent that it forces electrons into the nuclei of their atoms (removing all the space in the atoms in the process) where they combine with the protons there to produce neutrons that stop the implosion by literally jamming up against one another. With all that space removed, the resulting structure is incredibly dense (think *a thousand Nimitz class aircraft carriers* compressed into the size of a marble) and small (think 10 to 15 kilometers across).

(con't) The significance of all of this is that nature provides us with a WICKED example of *conservation of angular momentum*.

How so? There are no external torques acting during the supernova, so angular momentum is conserved. The enormously massive structure spread out over hundreds of thousands of kilometers starts out with a HUGE RADIUS and *angular momentum* even though its *angular speed* is low (the sun takes 25 days to rotate once about its axis). In other words, its *angular momentum* looks like:

$$L = I_{before}\ \omega_{before}$$

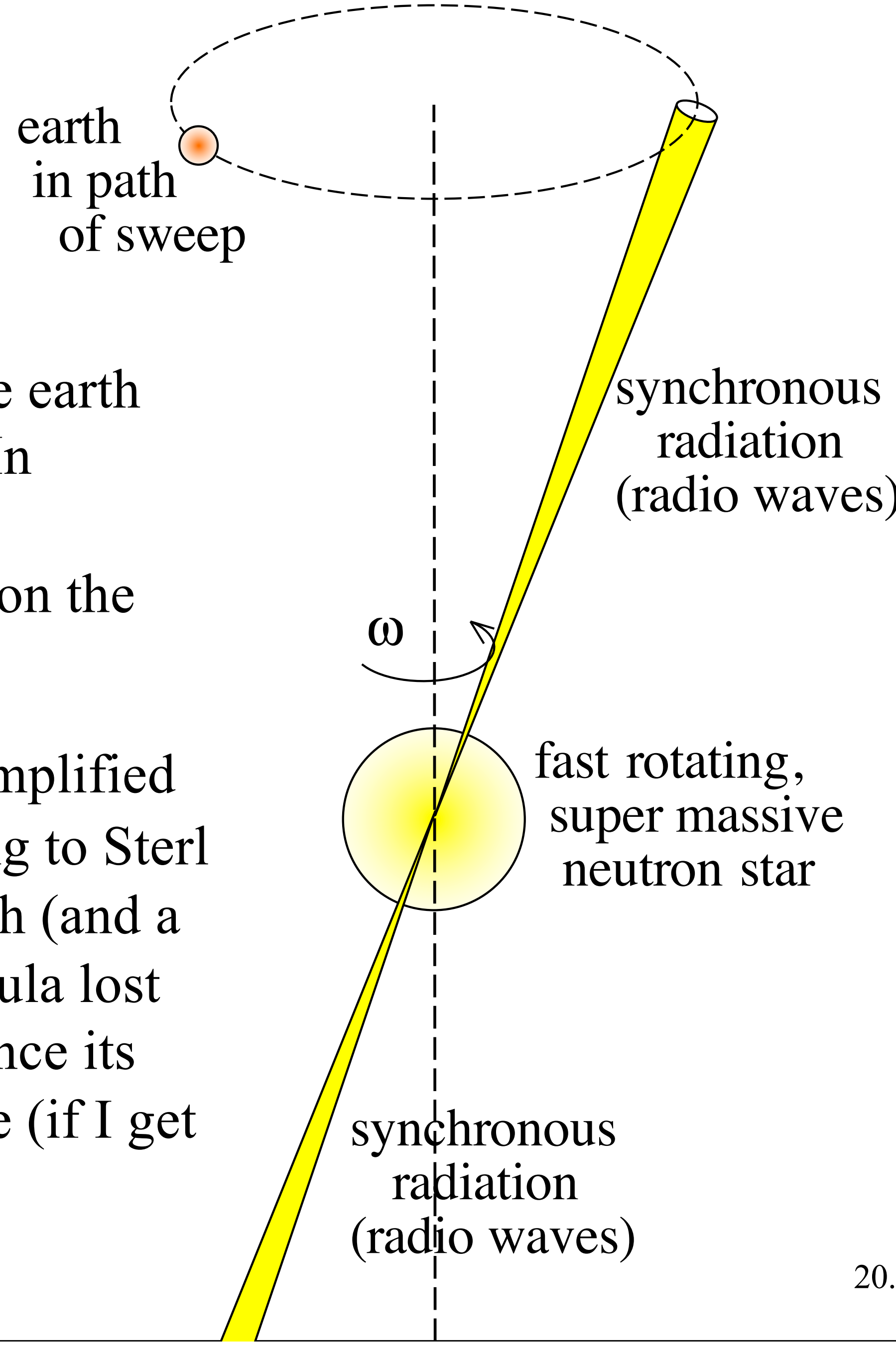
After the supernova, the *moment of inertial* drops precipitously because the radius goes from *several hundred thousand kilometers* to, maybe, *15 kilometers* during the explosion, BUT THE ANGULAR MOMENTUM STAYS THE SAME which means the angular velocity skyrockets. In other words, the *final* angular momentum relationship will look like:

$$L = I_{after}\,\omega_{after}$$

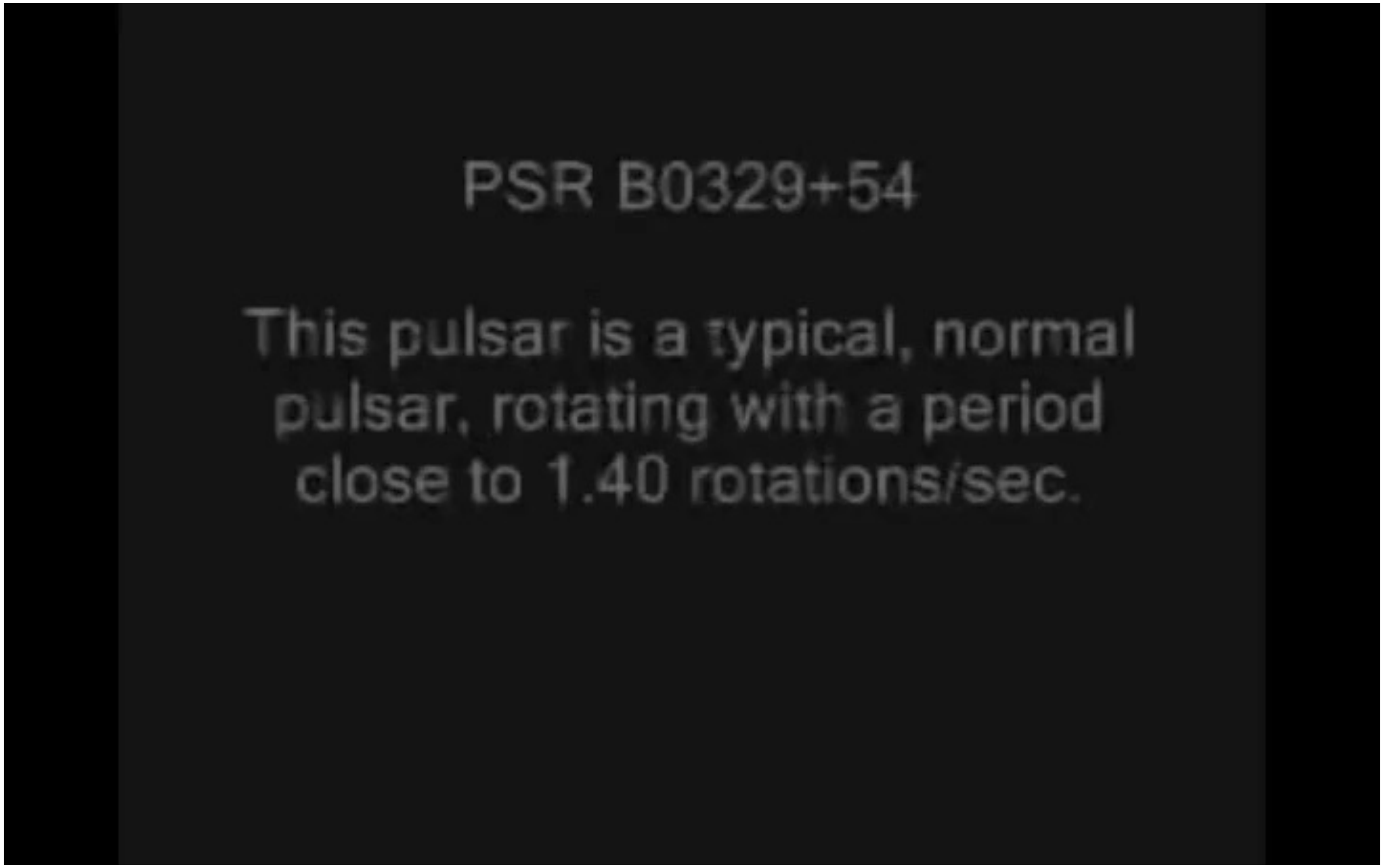
In short, pulsars (neutron stars) are super dense structures that rotate anywhere from a *few cycles per second* all the way up the *several hundred cycles per second*, all as a consequence of *conservation of angular momentum*.

But what's really cool is that they put out what is called *synchronous radiation*—radiation that is very directional and that is in the radio frequency range. So if the sweep of radiation of one of these fast rotating objects just happens to cross the earth's path, a blast of radio wave will hit the earth *every time the star completes one rotation*. In other words, we can *hear* them using a radio telescope. This is what you will experience on the next slide. Pretty amazing!

And as a small side-point, I've REALLY simplified what's going on with these things. According to Sterl Phinney, Professor of Astrophysics at Caltech (and a Poly parent), the progenitor of the Crab Nebula lost 99% of its angular momentum during and since its supernova. More about this on the next slide (if I get the time to generate it).

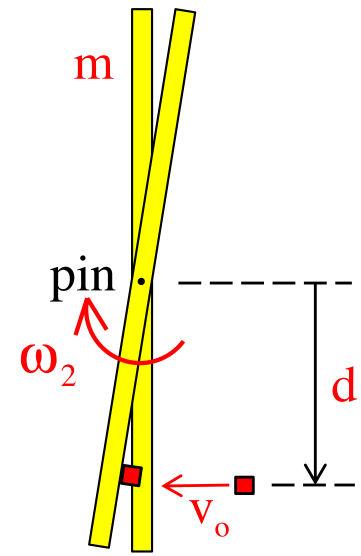


Non-AP minutia about neutron stars: According to Dr. Phinney, *Remembering that* these are super dense (density of 1000 Nimitz-class aircraft carriers compressed to the size of a marble) stars that are, maybe, 15 km across, and that each rotation produces one beat, here is what a pulsar sounds like as observed by a radio telescope.



AMAZING!!!

Example 7: Consider a *meter stick* of mass M pinned at its center. A puck of mass m moving with velocity v_o strikes the *meter stick* a distance d units from the center and sticks to it.



a.) Is momentum conserved through the collision? If so, write out the *conservation of momentum* relationship thru the collision. If not, justify your response.

Momentum is NOT CONSERVED as the pin provides an *external impulse*.

b.) Is angular momentum conserved through the collision? If so, write out the *conservation of angular momentum* relationship. If not, justify.

Angular momentum IS conserved as all the *torques about the pin are internal*.

$$\sum L_1 + \sum \Gamma_{ext}\Delta t = \sum L_2$$

$$mv_o d + 0 = I_m\omega_2 + I_{ms}\omega_2$$

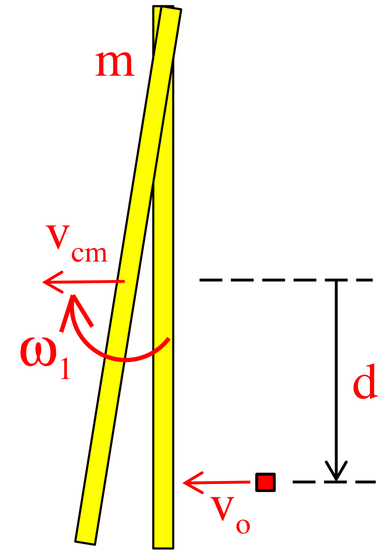
$$mv_o\cancel{d} + 0 = \left(md^{\cancel{2}}\right)\omega_2 + \left(\tfrac{1}{12}Md^{\cancel{2}}\right)\omega_2$$

$$\Rightarrow \quad \omega_2 = \frac{mv_o}{(md)+\left(\tfrac{1}{12}Md\right)}$$

c.) Is energy conserved through the collision?

Nope! IT'S A COLLISION!!!!

Example 8: Consider a stationary *meter stick* of mass *M* sitting freely on a frictionless surface. A puck of mass *m* moving with velocity v_o strikes the *meter stick* a distance *d* units from the center and stops dead.



a.) Is momentum conserved through the collision?

Momentum IS conserved as there are no *external impulses*.

$$\sum p_{1,x} + \sum F_{ext,x}\Delta t = \sum p_{2,x}$$
$$mv_o + 0 = Mv_{cm}$$
$$\Rightarrow \quad v_{cm} = \frac{mv_o}{M}$$

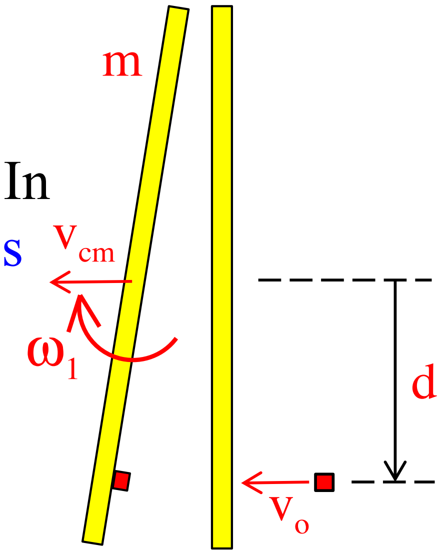
b.) Is angular momentum conserved through the collision?

Angular momentum IS conserved as all the *torques are internal*.

$$\sum L_1 + \sum \Gamma_{ext}\Delta t = \sum L_2$$
$$mv_o d + 0 = I_{ms}\,\omega_1$$
$$mv_o d + 0 = \left(\tfrac{1}{12}ML^2\right)\omega_1$$
$$\Rightarrow \omega_1 = \frac{mv_o d}{\left(\tfrac{1}{12}ML^2\right)} = \frac{12md}{ML^2}v_o$$

NOTICE: Momentum and *angular momentum* are both conserved but are *INDEPENDENT of one another*, and the *center of mass* velocity and angular velocity about the center of mass are *NOT related* by $v = r\omega$.

Example 9: A considerably more complex problem takes form if the mass sticks to the meter stick from the last problem. In that case: the *meter stick* of mass *M* sitting freely on a frictionless surface. A puck of mass *m* moving with velocity v_o strikes the *meter stick* a distance *d* units from the center and sticks to it.



a.) What does conservation of momentum through the collision tell us?

Momentum IS STILL conserved as there are no *external impulses*, so.

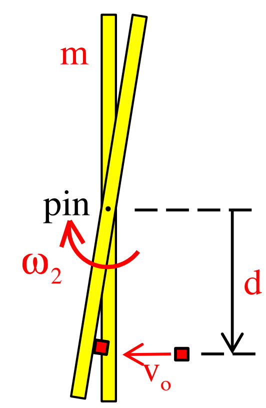
$$\begin{array}{ccccc} \sum p_{1,x} & + & \sum F_{ext,x}\Delta t & = & \sum p_{2,x} \\ mv_o & + & 0 & = & (M+m)v_{cm} \end{array}$$

$$\Rightarrow \quad v_{cm} = \frac{mv_o}{M+m}$$

b.) Is angular momentum conserved through the collision?

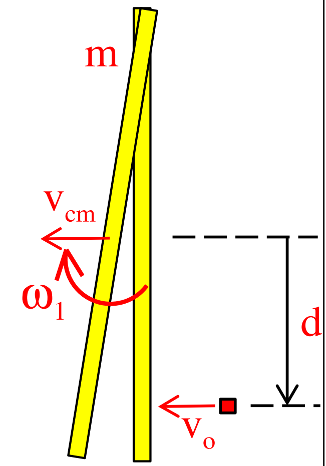
Angular momentum IS conserved as all the *torques are internal*, but this is where things get sticky. What we've done so far is to determine the angular momentum about the AXIS OR ROTATION of the system. Think about it:

b.) (con't.)



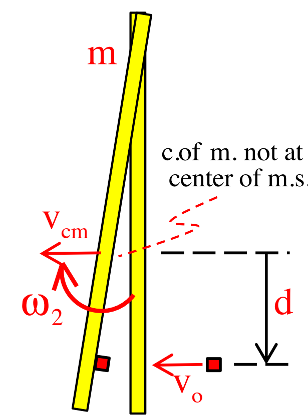
For the pinned beam situation, we calculated the angular momenta about the pin (i.e., at the *center of mass* of the meter stick).

$$mv_o d = \left(md^2\right)\omega_2 + \left(\tfrac{1}{12}Md^2\right)\omega_2$$



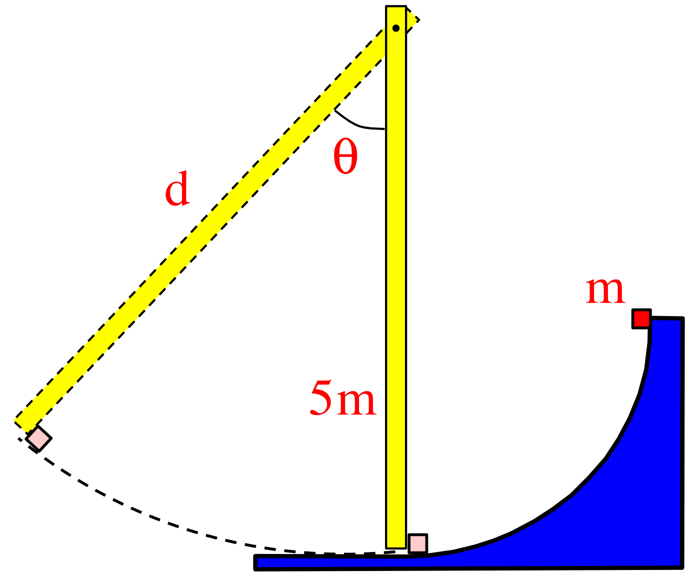
For the unpinned situation in which the puck stopped and all the after-collision motion was around the meter stick's *center of mass*, we calculated the angular momenta about the the *center of mass* of the meterstick (that was where the rotation was centered).

$$mv_o d = \left(\tfrac{1}{12}ML^2\right)\omega_1$$



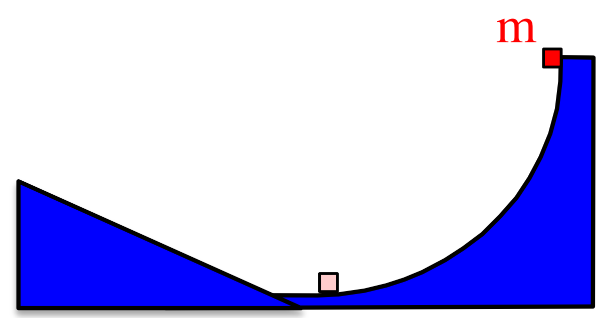
For the unpinned situation in which the puck STICKS, the rotation is NOT around the meter stick's *center of mass*. It is around the *center of mass* of the two-body system. That means to do this problem, we have to find the system's new *center of mass*, determine the meter stick's "I" about that new axis (parallel axis theorem), calculate the puck's initial angular momentum relative to that new *center of mass*, then put that equal to the angular momentum of the two bodies rotating around that new *center of mass* after the collision. Nasty . . .

Example 10: A mass m sits at the top of a curved, frictionless incline of radius R. It slides down the incline and executes a perfectly inelastic collision with the end of a pinned rod of mass $5m$ and length d. The two rotate up to some angle θ before coming to rest. If $R = .4d$, derive an expression for θ. You know:



$$m, d, R, g, \text{ and } I_{cm,rod} = \frac{5}{12} md^2$$

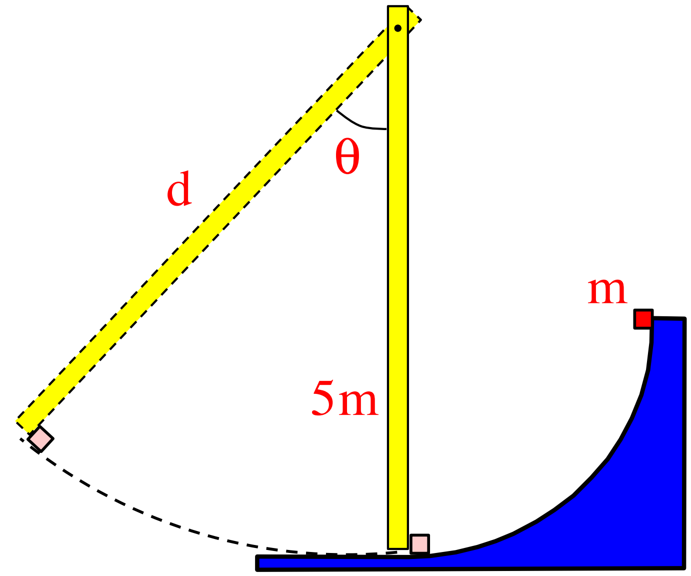
This is the rotational analogue to the problem shown to the right: A mass m sitting at the top of a curved incline of radius R slides down the incline, executes a perfectly inelastic collision with a $5m$ mass, and proceeds up a ramp. How high up the ramp does it go?



So how would you do *this* problem?

(Energy up to the collision, momentum through the collision, energy after the collision!)

$$m, d, R, g, \text{ and } I_{cm,rod} = \frac{5}{12} md^2$$



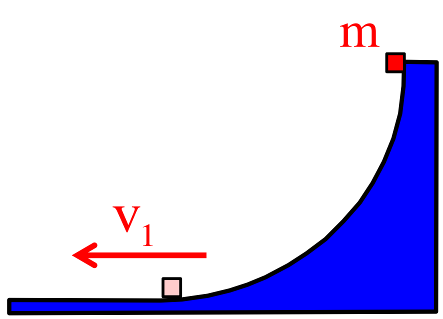
How are the problems different as far as solving goes? That is, why can't we just use *conservation of momentum* when the two masses collide in the *pinned beam* problem?

To use conservation of momentum, we need a system in which there are *no external impulses*. The pin provides an *external impulse* (it keeps the rod from accelerating en-mass to the left through the collision), so *conservation of momentum* won't work for this collision. There *are* no *external torques* acting about the pin, though, so conservation of *angular* momentum IS applicable.

$$m, d, R, g, \text{ and } I_{cm,rod} = \frac{5}{12}md^2$$

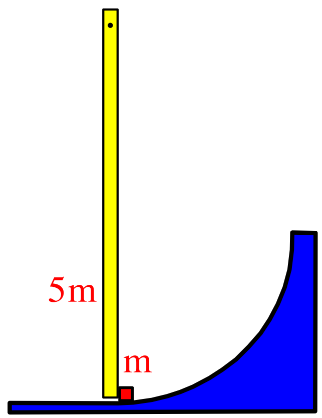
To begin, the velocity v_1 of the mass *m* just before the collision can be determine using *conservation of energy*:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$
$$0 \quad + mgR \quad + \quad 0 \quad = \frac{1}{2}mv_1^{\ 2} + \quad 0$$
$$\Rightarrow \quad v_1 = (2gR)^{1/2}$$



Because there are no external torques acting about the pin, *conservation of angular momentum* is the key to the collision. Taking a *time interval* through the collision, and summing the angular momenta *about the pin*, we can write:

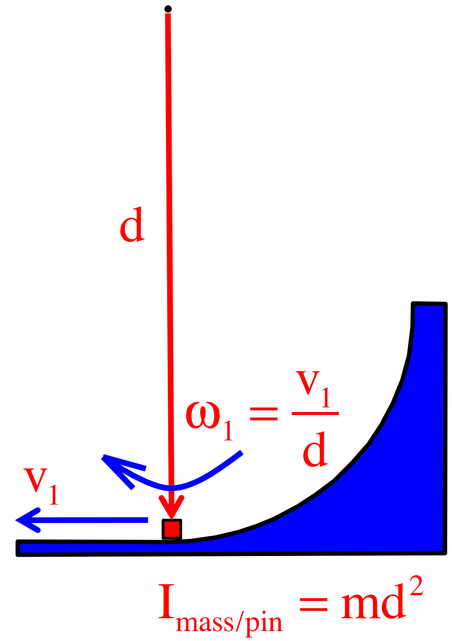
$$\sum L_{1,pin} + \sum \tau_{ext}\Delta t = \sum L_{2,pin}$$
$$L_{1,mass} + \quad 0 \quad = L_{2,mass} + L_{2,rod}$$
$$\Rightarrow \quad \vec{r}x\vec{p}_1 = I_{mass}\omega_2 + I_{pin}\omega_2$$



We need those angular momentum quantities.

$$\text{m, d, R, g, and } I_{cm,rod} = \frac{5}{12}md^2$$

To determine the *angular momentum* of the mass about the pin, we can go two ways. We can either treat the mass as a translating point mass and using $\vec{r}x\vec{p}$, or we can use rotational parameters. I'll show both (assuming the velocity of the mass at the bottom of the incline is v_1):



translating point mass:

$$\left|L_{1,m}\right| = \vec{r}x\vec{p}_1$$
$$= d(mv_1)$$

OR

rotational parameters:

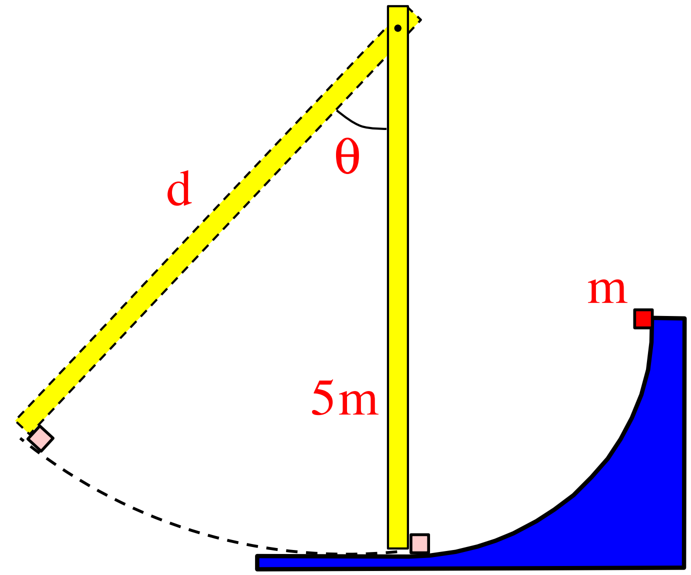
$$\left|L_{1,m}\right| = I_{mass/pin}\,\omega_1$$
$$= \left(md^{\cancel{2}}\right)\left(\frac{v_1}{\cancel{d}}\right)$$
$$= (md)v_1$$

Same either way.

After the collision, the mass's *angular momentum* in terms of *angular velocity*:

$$\left|L_{2,m}\right| = \vec{r}x\vec{p}_2$$
$$= d(mv_2)$$
$$= d(m(d\omega_2)) = md^2\omega_2$$

$$m, d, R, g, \text{ and } I_{cm,rod} = \frac{5}{12} md^2$$



We need the *moment of inertia* of the rod about the pin. We'll use the *parallel axis theorem* for that:

$$I_{cm} = I_{cm} + Md^2$$
$$= \tfrac{5}{12} md^2 + (5m)\left(\tfrac{d}{2}\right)^2$$
$$= \frac{5}{3} md^2$$

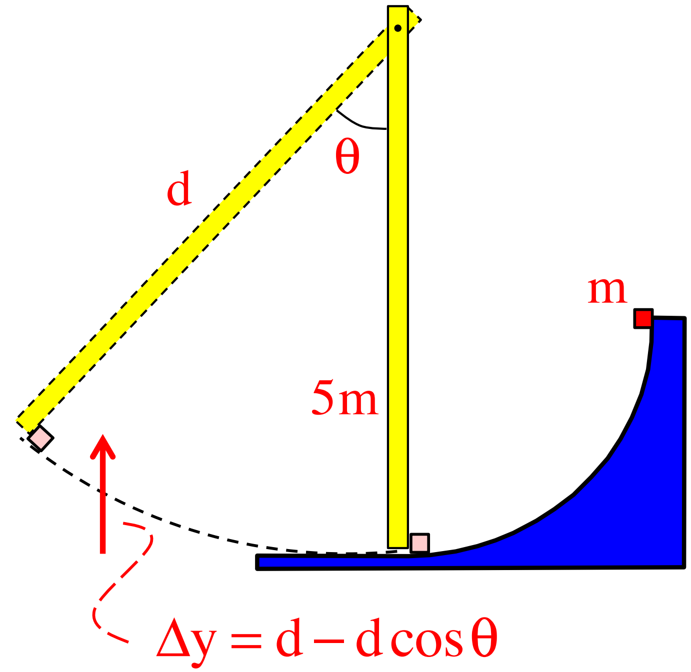
Putting everything together through the collision yields:

$$\sum L_{1,pin} + \sum \tau_{ext}\Delta t = \sum L_{2,pin}$$
$$\vec{r} x \vec{p}_{1,mass} \quad\quad 0 \quad\quad = I_{mass}\omega_2 + I_{pin}\omega_2$$
$$\Rightarrow \quad mdv_1 = (md^2)\omega_2 + \left(\frac{5}{3} md^2\right)\omega_2$$
$$\Rightarrow \quad \omega_2 = \frac{v_1}{d + \tfrac{5}{3}d} = \frac{3v_1}{8d}$$

As

$$\omega_2 = \frac{v_2}{d}, \; v_2 = \omega_2 d = \left(\frac{3v_1}{8d}\right)d$$
$$\Rightarrow \quad v_2 = \frac{3}{8}v_1 = \frac{3}{8}\sqrt{2gR}$$

$$m, d, R, g, \text{ and } I_{cm,rod} = \frac{5}{12} md^2$$



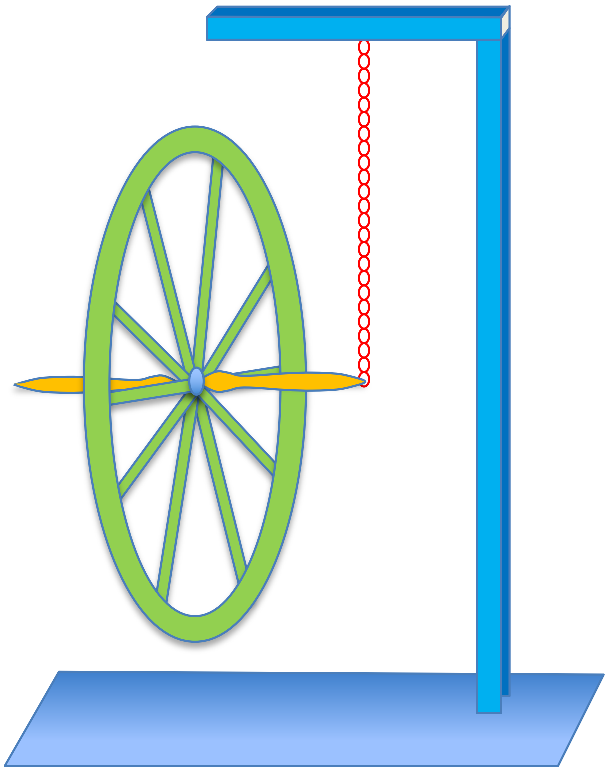
Knowing the *after-collision* velocities, we can use *conservation of energy* to determine how high the rod's *center of mass* rises, and how high up the *mass* rises, before coming to a stop. Without doing the math to its conclusion, assuming the *time interval* is from just after the collision to the stop point and noting that the mass rises a distance $(d - d\cos\theta)$ (you should understand why by now) while the rod's center of mass rises $\left(d/2 - d/2 \cos\theta\right)$, that equation looks like:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

$$\left(KE_{rod} + KE_{mass}\right) + 0 + 0 = 0 + \left(U_{mass} + U_{rod}\right)$$

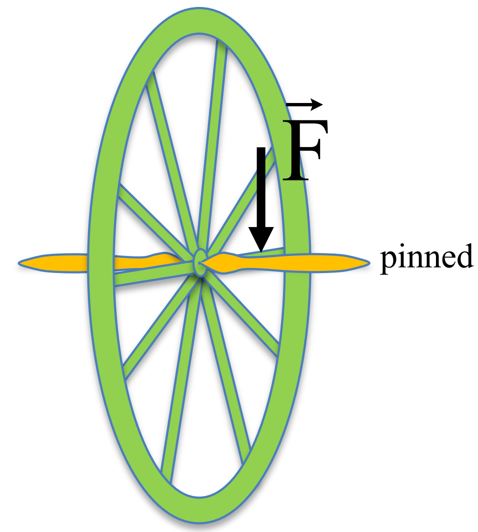
$$\left(\frac{1}{2} I_{rod/pin}\omega_2{}^2 + \frac{1}{2} m v_2{}^2\right) + 0 + 0 = 0 + \left(mg(d - d\cos\theta) + (5m)g\left(\frac{d}{2} - \frac{d}{2}\cos\theta\right)\right)$$

Back to the Non-AP Tom-Foolery



Remember the precessing wheel demonstration?

It was accompanied by a second demonstration in which a torque was quickly applied to the pinned axle of a rotating wheel, and instead of the wheel following the direction of the applied force, the wheel jerked to the right or left, depending upon the direction of the wheel's rotation.



It's time to make sense of both of these.

The mathematical key to these seemingly mysterious behaviors are wrapped up in the relationship between torque and angular momentum (or, Newton's Second Law, rotation style). That is, the relationship:

$$\vec{\tau} = \frac{\Delta\vec{L}}{\Delta t}$$

This relationship suggests one of two things.

1.) If the direction of the net torque applied to a body matches the direction of the angular momentum vector of the body (translation: it matches the direction of the body's angular velocity), an applied torque will change the magnitude of the angular momentum (that is, the body will angularly speed up or slow down). The translational parallel to this is a force along the line of motion making a body speed up or slow down.

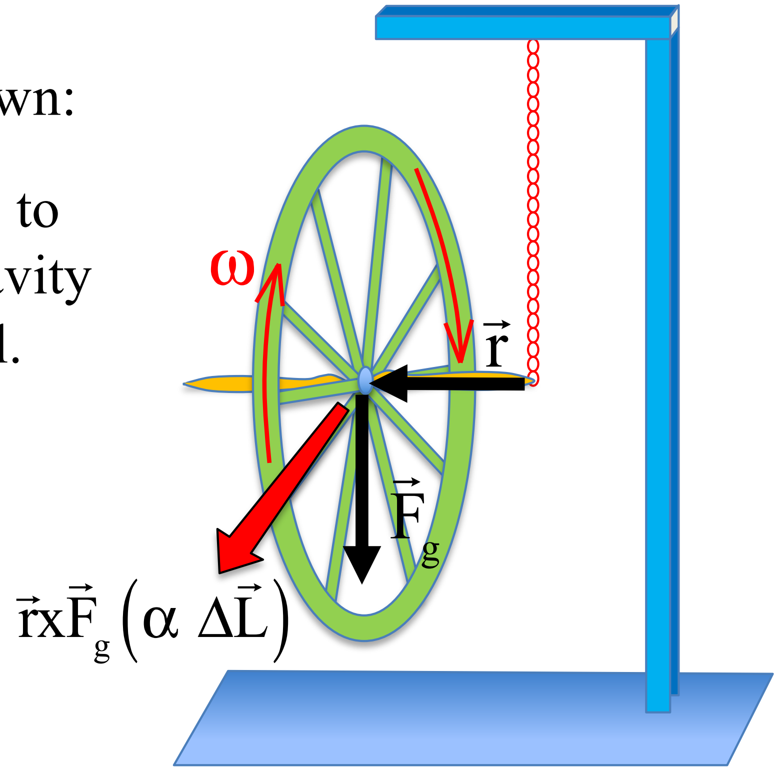
But:

2.) If the direction of the net torque applied to a body does not match the direction of a body's angular momentum vector, the body's angular momentum will still change but the change will be in the angular momentum's DIRECTION, not its magnitude. The translational parallel to this is a force that is perpendicular to the line of motion creating a centripetal situation. The precessing wheel circumstance falls into this latter category.

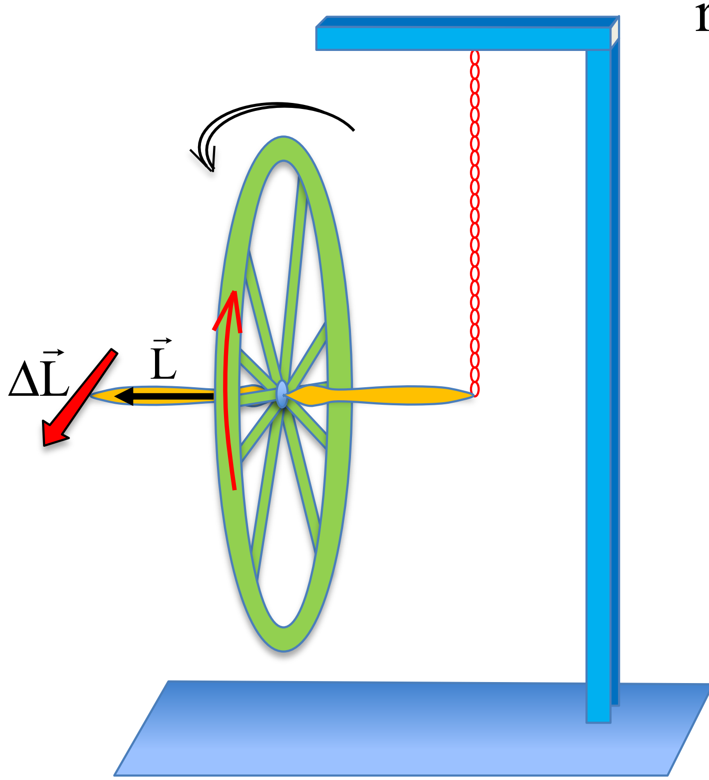
Starting with the hanging wheel rotating as shown:

The system is pinned where the chain attaches to the axle. The force being applied is due to gravity and happens at the center of mass of the wheel.

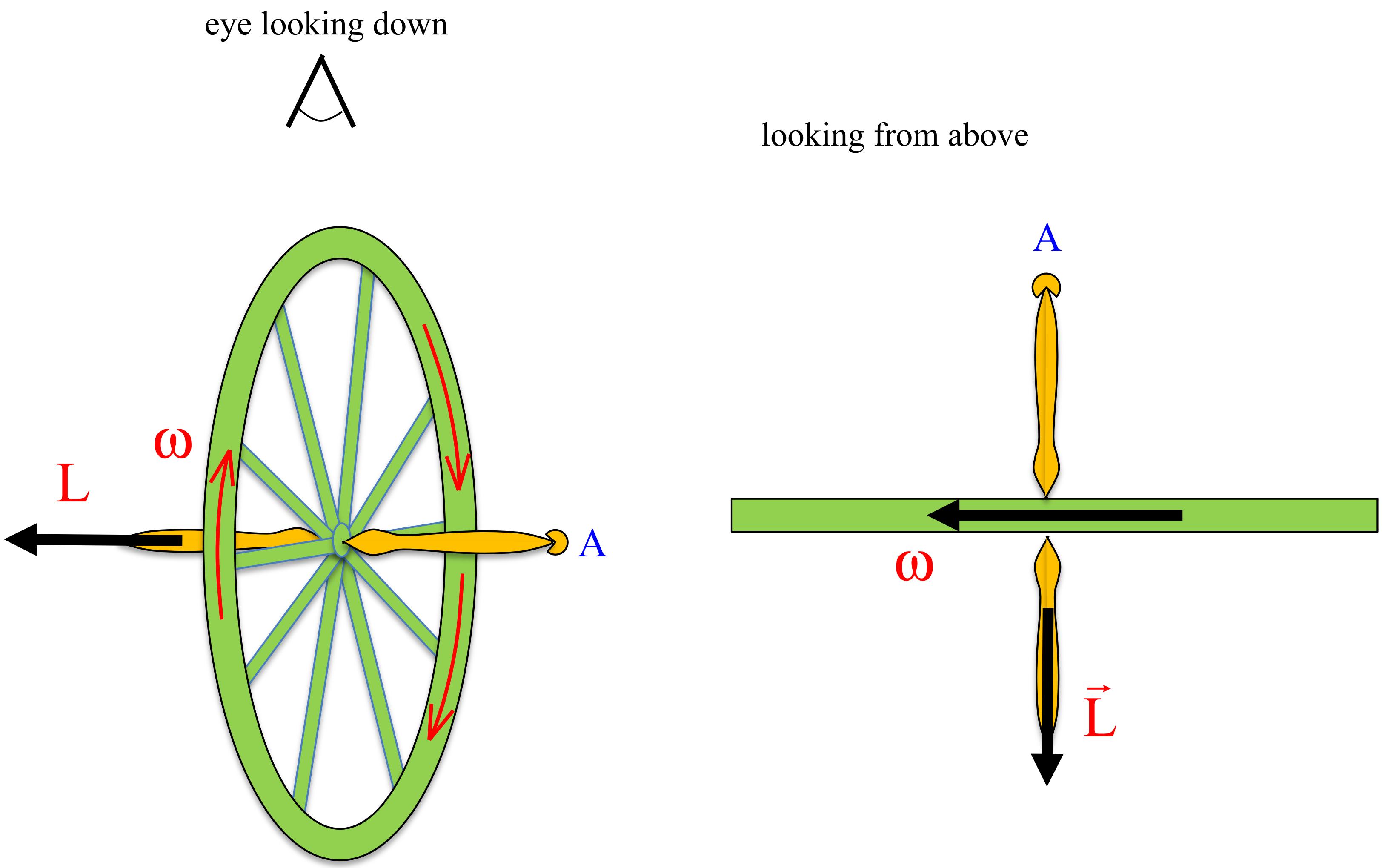
The torque produced by gravity is at right angles to the plane defined by $\vec{F}_g$ and $\vec{r}$.



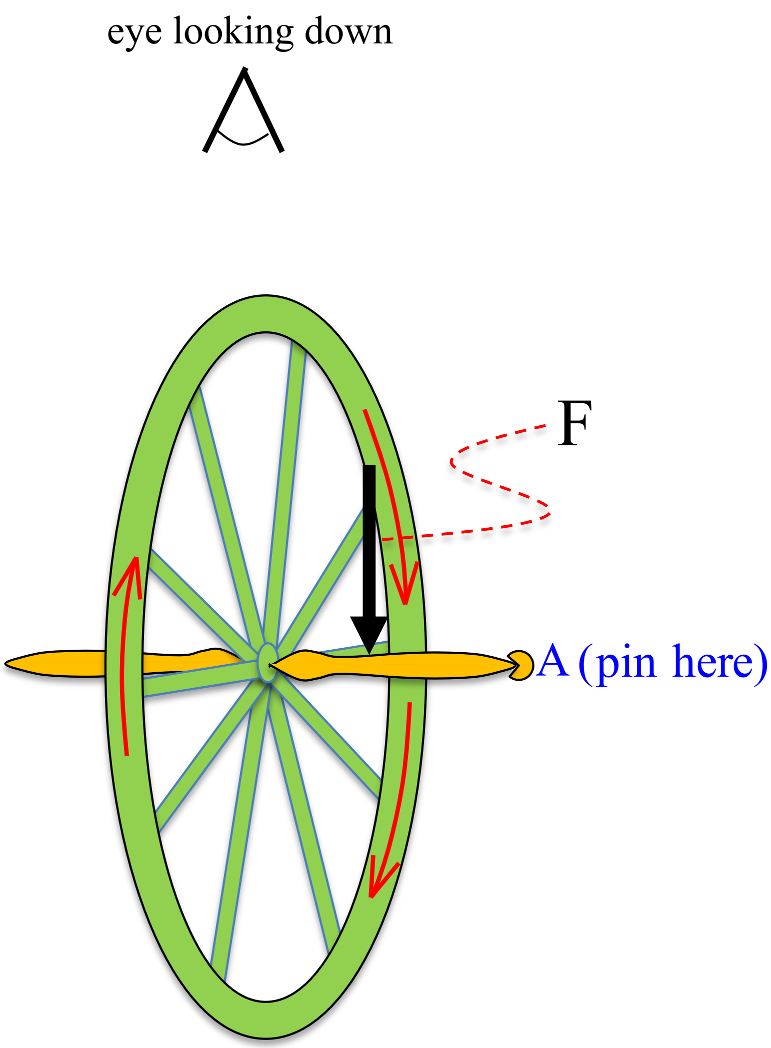
With the torque (i.e., $\Delta\vec{L}$) at right angle to $\vec{L}$, the direction of the change of angular momentum demands that the body's axle must precess!

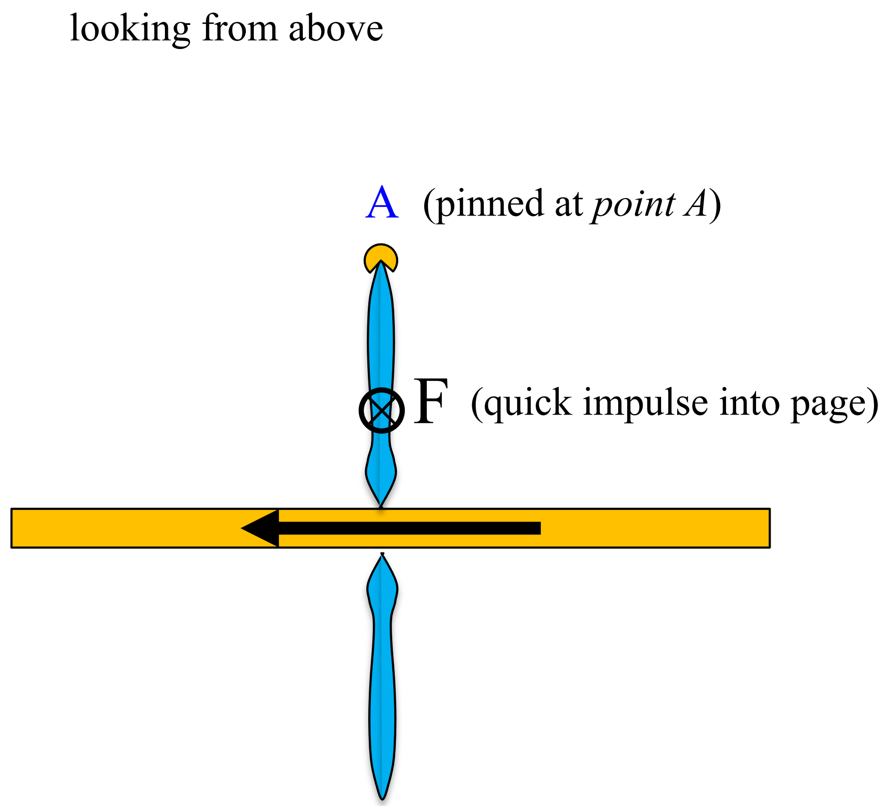


--What about the jerking wheel? Assume a clockwise rotation. The angular velocity (and, hence, angular momentum) vector is shown below in the sketch with the view from above shown to the right (I've put a nub at *point A* for reference).

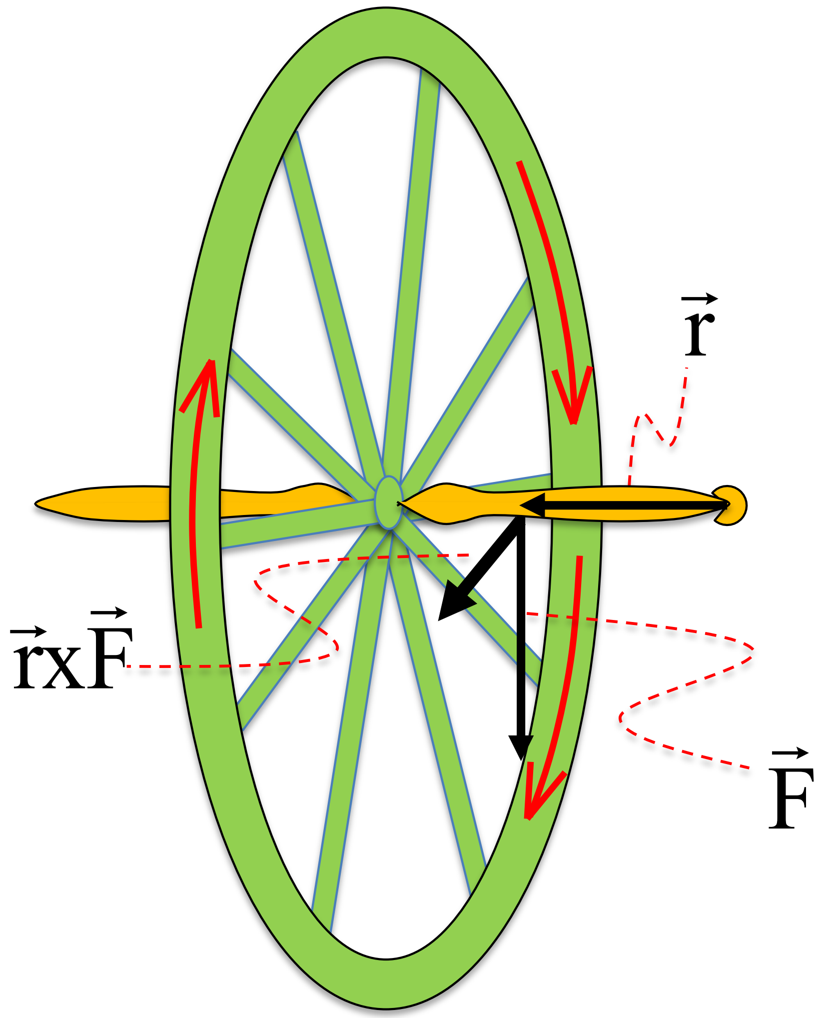


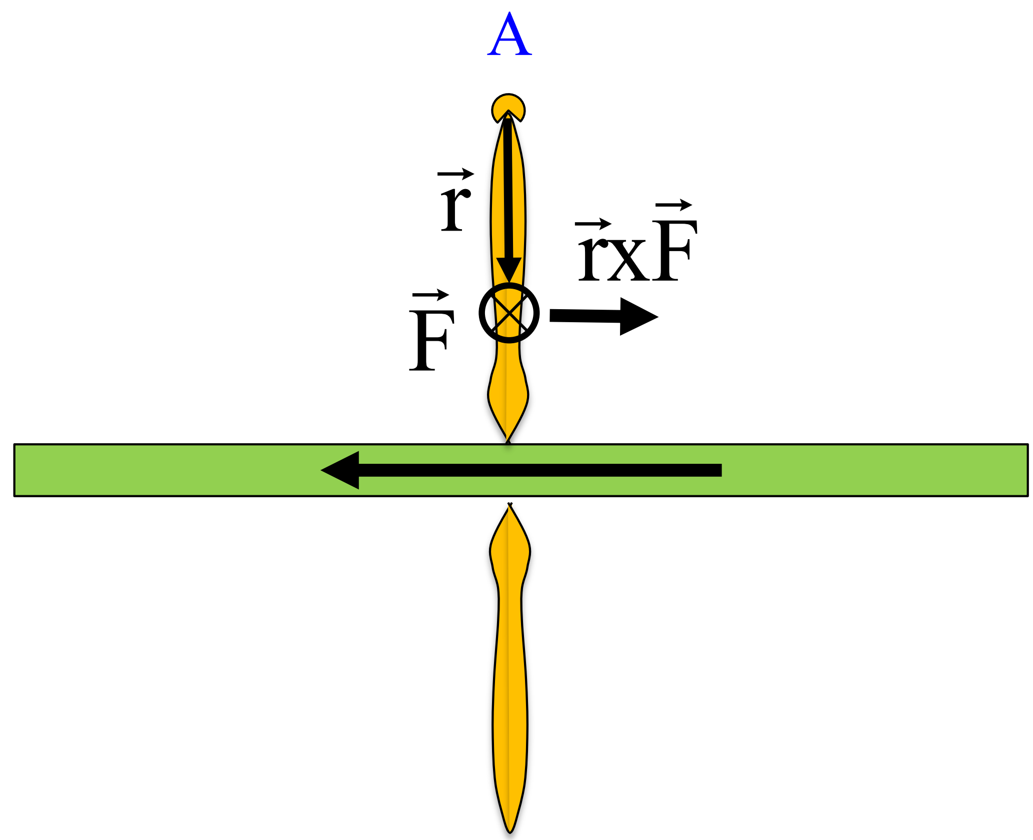
--Let's try to rotate the wheel about *point A* by applying a quick downward force F to *point A*.



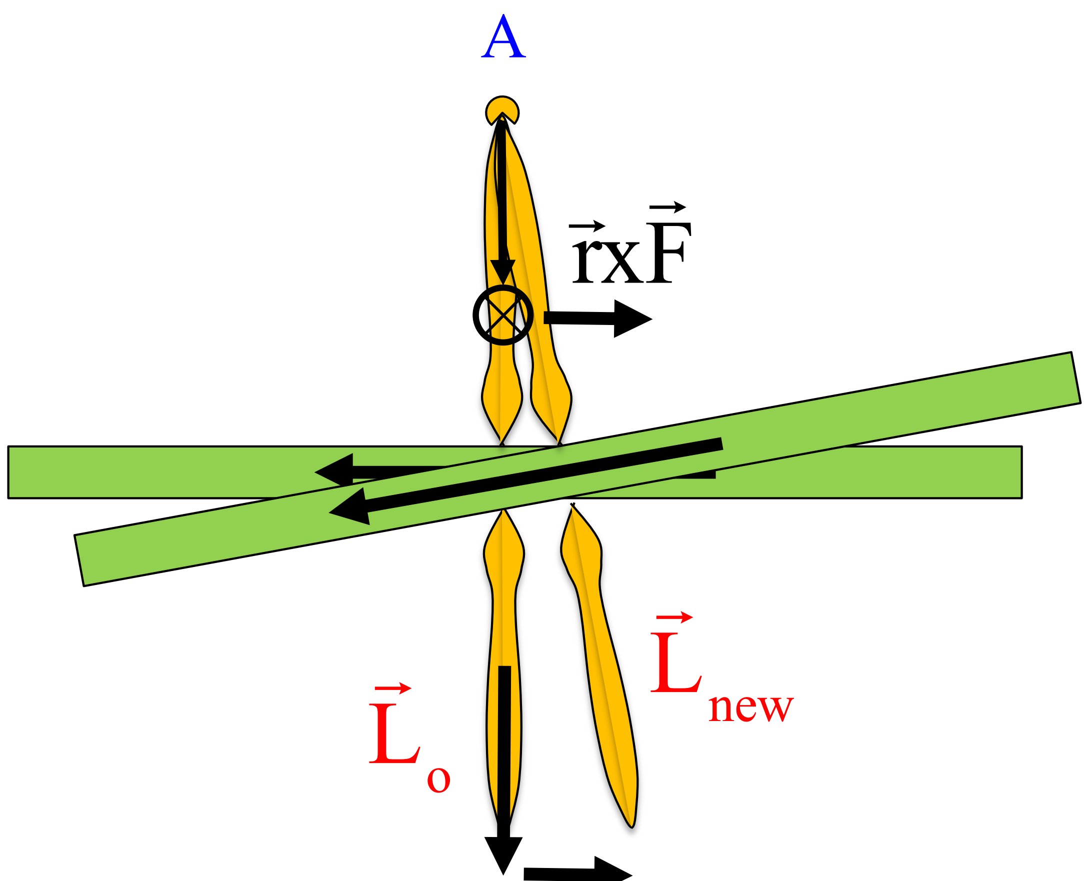


--Note that the direction of the torque applied by $\vec{F}$ is NOT in the direction of the angular momentum vector, which is along the axle.





--If the torque is in the direction of the CHANGE OF angular momentum, then the NEW angular momentum direction must be as shown below . . . and hence we predict precession!



$\vec{r}x\vec{F}\ \alpha\ \Delta\vec{L}$

$$(\text{as } \vec{\tau} = \frac{\Delta\vec{L}}{\Delta t})$$