Rotational Motion

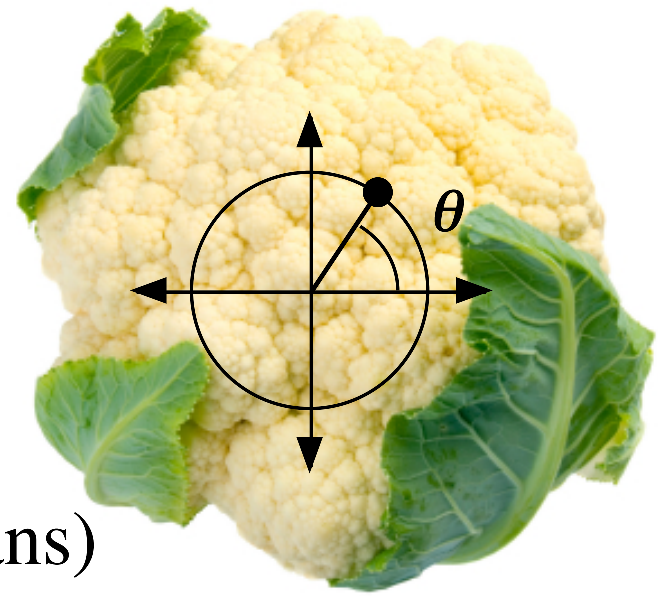
Angular Displacement

$$\pi = \frac{circumference}{diameter}$$

$$\pi = \frac{circumference}{2 \times radius}$$

$$circumference = 2\pi r$$

Arc length $s = r\theta$, (where θ in radians)



$$1rev = 360° = 2\pi \text{ rads}$$

$$\Delta\theta = \theta_f - \theta_i = \text{"angular displacement"}$$

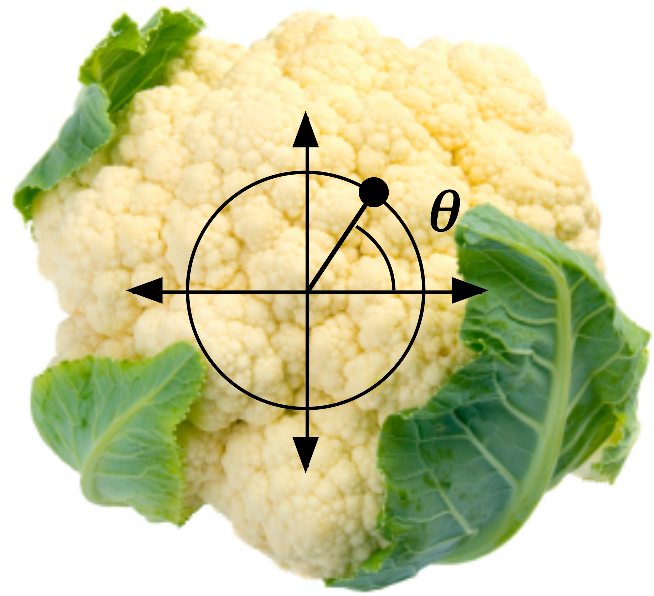
Angular Kinematics

$$\omega_{avg} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

$$\omega = \frac{d\theta}{dt}$$

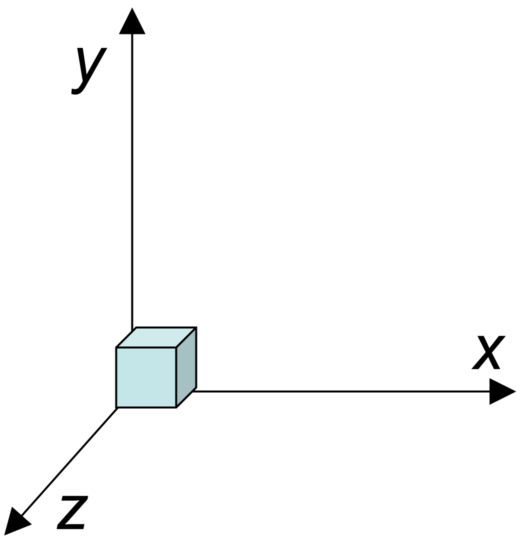
$$\alpha_{avg} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{t}$$

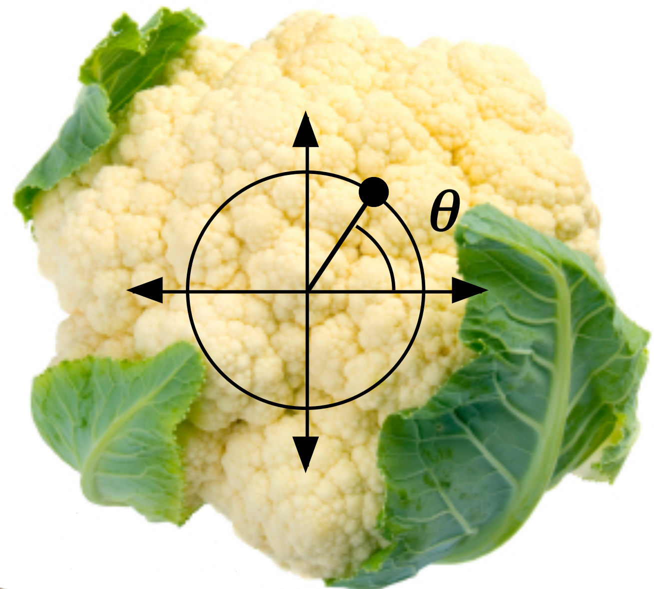
$$\alpha = \frac{d\omega}{dt}$$



Angular Direction?

Based on *right-handed* coordinate system





Angular Kinematics

$$v_{avg} = \frac{\Delta x}{\Delta t}$$

$$a_{avg} = \frac{\Delta v}{\Delta t}$$

$$s = r\theta$$

$$v = r\omega$$

$$a = r\alpha$$

$$\omega_{avg} = \frac{\Delta \theta}{\Delta t}$$

$$\alpha_{avg} = \frac{\Delta \omega}{\Delta t}$$

$$v_f = v_i + at$$

$$x_f = x_i + v_i t + \frac{1}{2}at^2$$

$$v_f^{\,2} = v_i^{\,2} + 2ax$$

$$\omega_f = \omega_i + \alpha t$$

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

$$\omega_f^{\,2} = \omega_i^{\,2} + 2\alpha\theta$$

Example 1

The turntable shown starts from rest, and begins rotating in the direction shown. 10 seconds later, it is rotating at 33.3 revolutions per minute

a)What is the turntable's final angular velocity (mag and dir)?

b)What is the turntable's average acceleration (mag and dir)?

c)What distance in meters does a bug located 10cm from the axis of rotation, travel during the time the turntable is accelerating?

Example 2

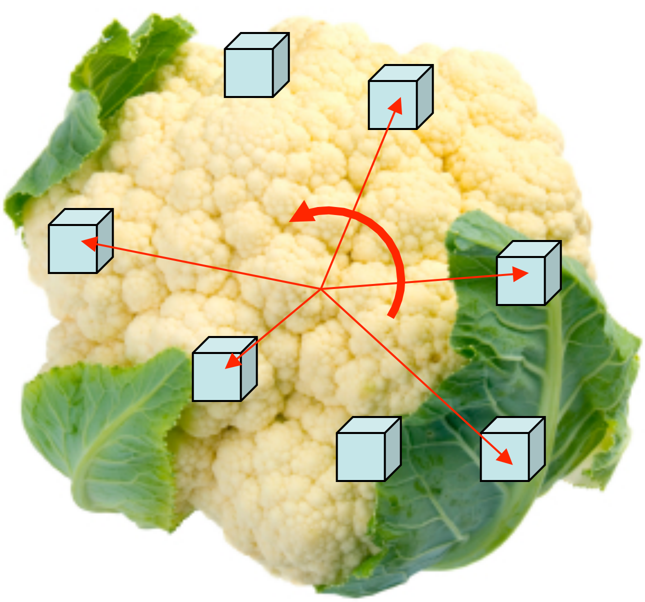
A wheel rotates with a constant angular acceleration of 3.50 rad/s^2. At time t=0, its angular velocity is 2.00 rad/s.

a) What angle does the wheel rotate through in 2.00 s?

b) What is the angular speed at that moment?

Rotational *K*

How can we determine the kinetic energy associated with the distributed mass of a rotating object?



$$K_{rotational} = \sum K_i$$

$$K_{rotational} = \sum \frac{1}{2} m_i v_i^2$$

$$K_{rotational} = \sum \frac{1}{2} m_i (r_i \omega)^2$$

$$K_{rotational} = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

$$K_{rotational} = \frac{1}{2} I \omega^2$$

I = "moment of inertia"

Moment of Inertia

$$I = \sum m_i r_i^2$$

The rotational quantity *I* is analogous to the linear quantity *m*, and represents an object's resistance to change in its rotation.

Example 3

2 children (m=20 kg) and 2 adults (m=70 kg) sit in seats on the teacup ride at Disneyland. The center of mass for the adults is located 70 cm from the axis of rotation, while the cm for the children is 80 cm from the axis of rotation.

a) Calculate the cup's moment of inertia.

b) Calculate the cup's rotational energy if it spins at 1.3 revolutions/second.

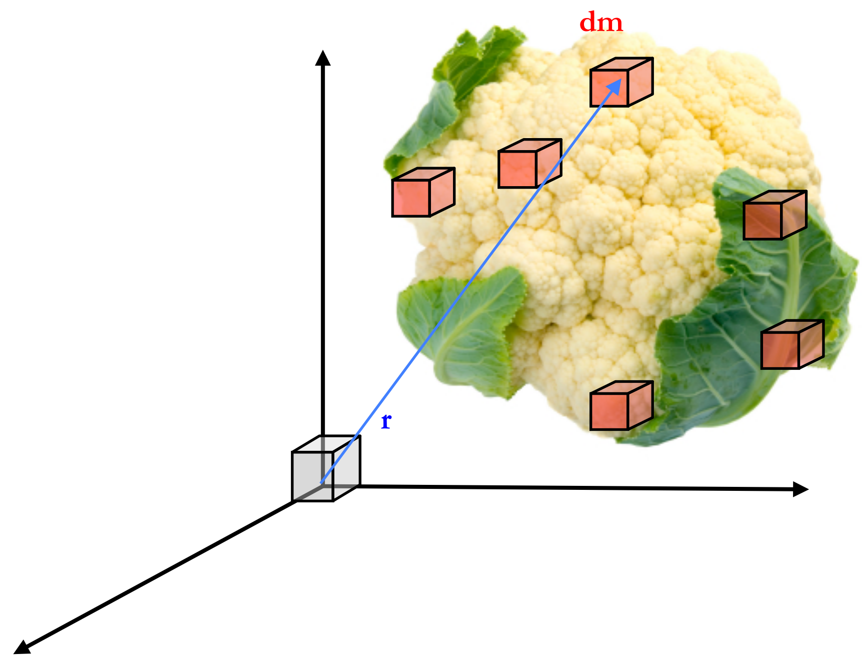
$I=94.2\ kg{\bullet}m^2$, $K=3140\ J$

I for Continuous Dist.

$$I = \sum m_i r_i^2$$

$$I = \lim_{\Delta m_i \to 0} \sum \Delta m_i r_i^2$$

$$I = \int r^2 dm$$



$$dm = \lambda \; dr$$

$$dm = \sigma \; dA$$

$$dm = \rho \; dV$$

Example 4

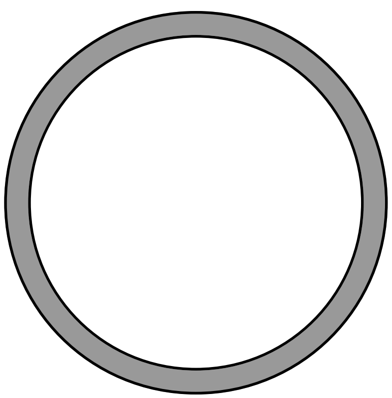
Calculate the moment of inertia for a uniform hoop of mass *M* and radius *R*.

$$I = \int r^2 dm$$

$$I = \int R^2 dm$$

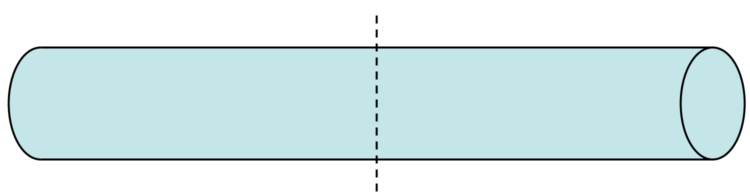
$$I = R^2 \int dm$$

$$I = MR^2$$



Example 5

Calculate the moment of inertia for a uniform rod, of mass *M*, with length *L*, rotating about its center of mass.



$$I = \int r^2 dm$$

$$dm = \lambda dr$$

$$I = \int r^2 \lambda dr$$

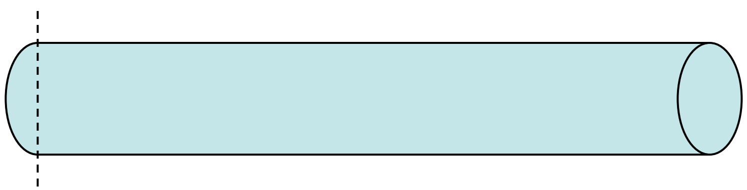
$$I = \lambda \ \tfrac{1}{3} r^3 \Big|_{-L/2}^{+L/2}$$

$$I = \frac{M}{L}\left(\tfrac{1}{3}\left(\tfrac{L}{2}\right)^3 - \tfrac{1}{3}\left(\tfrac{-L}{2}\right)^3 \right)$$

$$I = \tfrac{1}{12} ML^2$$

Example 6

Calculate the moment of inertia for a uniform rod, of mass *M*, with length *L*, rotating about an axis at one end of the rod.



$$I = \int r^2 dm$$

$$dm = \lambda dr$$

$$I = \int r^2 \lambda dr$$

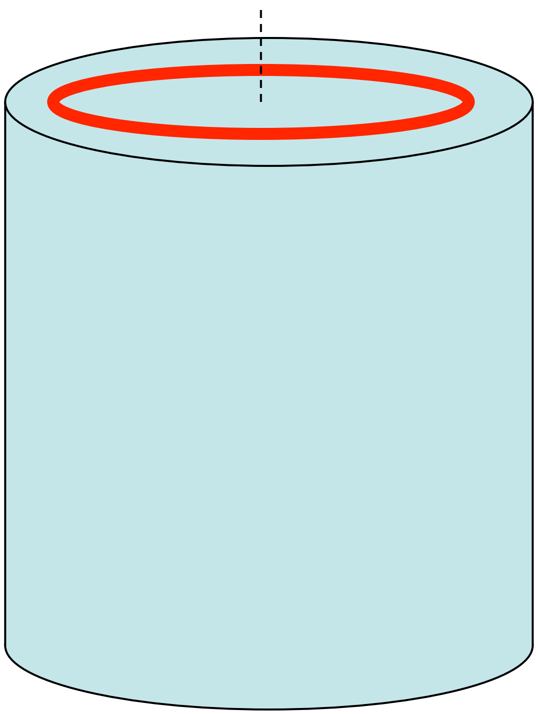
$$I = \lambda \left. \tfrac{1}{3} r^3 \right|_0^L$$

$$I = \frac{M}{L}\left(\frac{L^3}{3}\right)$$

$$I = \frac{1}{3} ML^2$$

Example 7

Calculate the moment of inertia around the central axis of a uniform, solid cylinder, with mass *M*, radius *R*, and length *L*.



$$I = \int r^2 dm$$

$$dm = \rho dV$$

$$I = \int r^2 \rho \; dV$$

$$dV = 2\pi r L \; dr$$

$$I = \int r^2 \rho 2\pi r L \; dr$$

$$I = 2L\pi\rho \int r^3 \; dr$$

$$I = 2L\pi\rho \; \tfrac{1}{4} r^4 \Big|_0^R = \frac{L\pi\rho R^4}{2}$$

$$\rho = \frac{M}{V} = \frac{M}{\pi R^2 L}$$

$$I = \frac{L\pi R^4}{2}\left(\frac{M}{\pi R^2 L}\right) = \frac{1}{2} MR^2$$

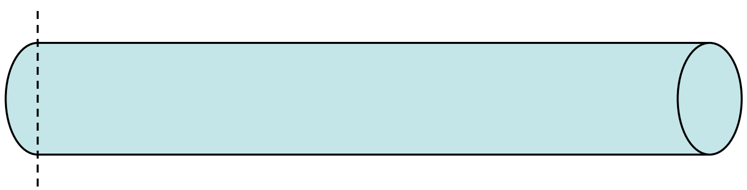
Parallel-Axis Theorem

If one knows the moment of inertia about the center of mass of an object, one can determine the moment of inertia about any other axis parallel to the original one.

$$\boxed{I = I_{cm} + MD^2}$$

Example 8

Use the parallel axis theorem to calculate the moment of inertia for a uniform rod, of mass *M*, with length *L*, rotating about an axis at one end of the rod.



$$I = \int r^2 dm$$

$$I_{cm} = \frac{1}{12} ML^2$$

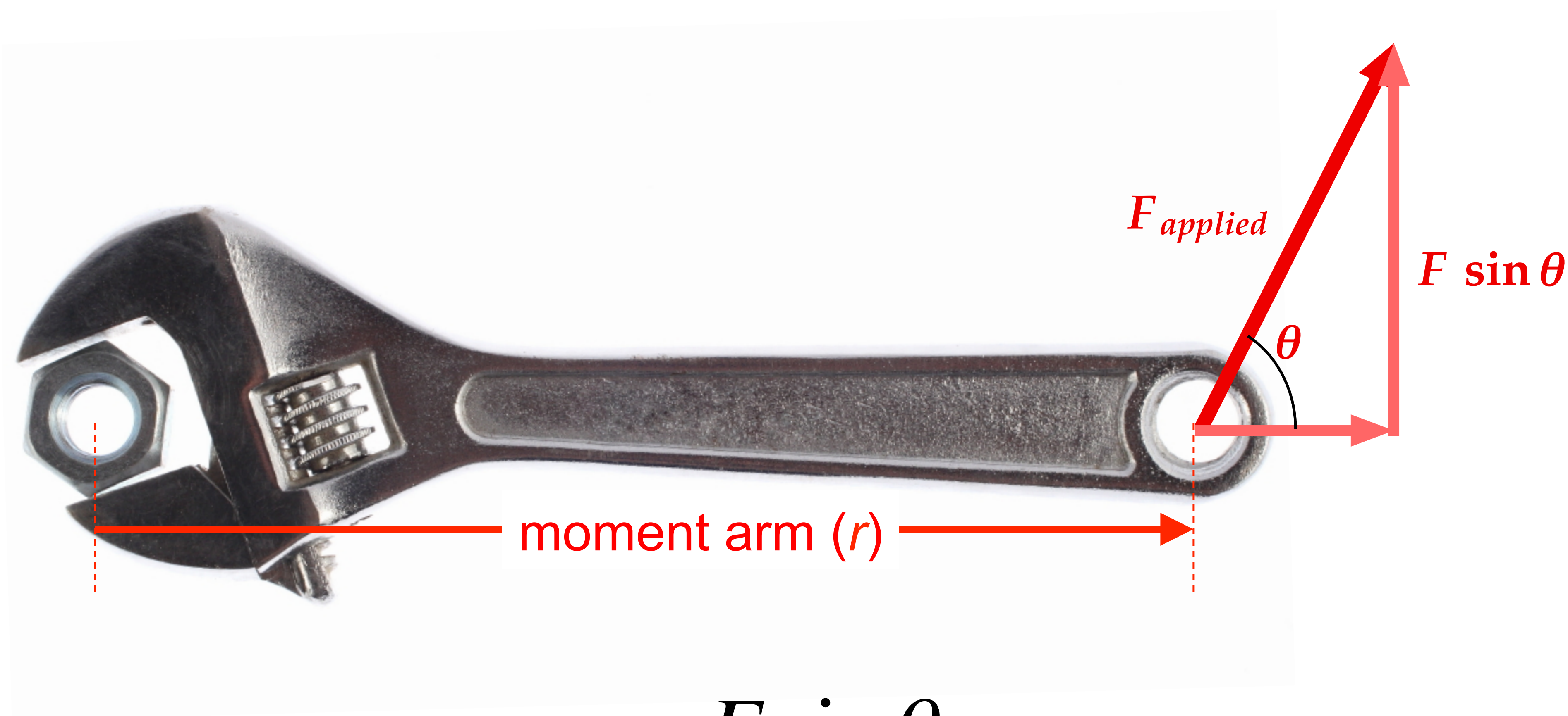
$$I_{end} = I_{cm} + MD^2$$

$$I_{end} = \frac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2$$

$$I_{end} = \frac{1}{12} ML^2 + \frac{3}{12} ML^2$$

$$I = \frac{1}{3} ML^2$$

Torque

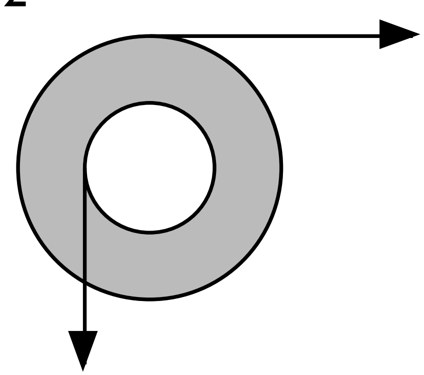


$$\tau = rF\sin\theta$$

$$\tau = Fd$$

Example 9

A one-piece cylinder, shown here, has a core-section that protrudes from a larger drum. A rope wrapped around the large drum of radius R_1 exerts a force F_1 to the right, while a rope wrapped around the core, radius R_2, exerts a force F_2 downward.



a) Calculate the net torque acting on the cylinder in terms of the variables given.

b) If F_1=5.0 N, R_1=1.0m, F_2=6.0N, and R_2=0.50m, which way does the cylinder rotate?

Torque → Cross Product

$$W = \vec{\mathbf{F}} \bullet \vec{\mathbf{s}}$$

$$W = Fs\cos\theta$$

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

$$\tau = rF\sin\theta$$

Determinants

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\begin{aligned} \vec{\mathbf{A}} \times \vec{\mathbf{B}} = & (A_y B_z - A_z B_y)\vec{\mathbf{i}} + \\ & (A_z B_x - A_x B_z)\vec{\mathbf{j}} + \\ & (A_x B_y - A_y B_x)\vec{\mathbf{k}} \end{aligned}$$

Cross Products in 3-d

$$ixi = jxj = kxk = 0$$

$$ixj = k \qquad -jxi = k$$

$$jxi = -k \qquad -ixj = -k$$

$$jxk = i \qquad -kxj = i$$

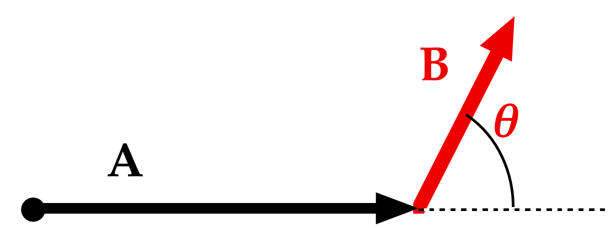
$$kxj = -i \qquad -jxk = -i$$

$$kxi = j \qquad -kxi = -i$$

$$ixk = -j \qquad -ixk = i$$

$$\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$$

$$C = AB\sin\theta$$



Example 10

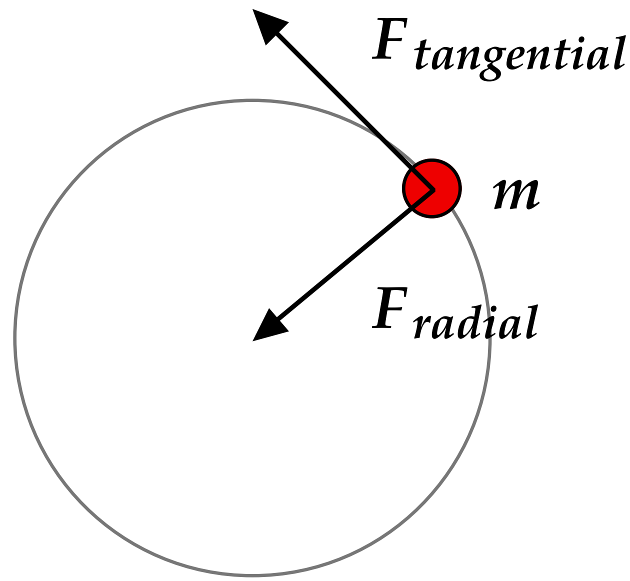
a) Find the cross product of 5i+6j and -3i + -2j

b) What is the angle between these two vectors?

8k

164°=(180-16.4)

Torque & α



$$\tau = F_{\text{tangential}} r$$

$$\tau = (m a_{\text{tangential}}) r$$

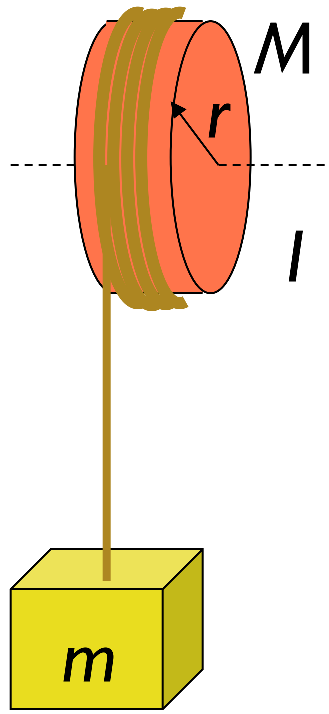
$$\tau = (m(\alpha r)) r$$

$$\tau = (m r^2) \alpha$$

$$\tau = I \alpha$$

Example 11

Calculate the linear acceleration of the object *m*, the angular acceleration of the wheel, and the tension in the cord (using *m*, *r*, and *I*).



$$Mass: F_{net} = ma$$

$$F_g - F_T = ma$$

$$Wheel: \tau = I\alpha$$

$$r \times F_T = I\alpha$$

$$F_T = mg - ma$$

$$r(mg - ma) = I\tfrac{a}{r}$$

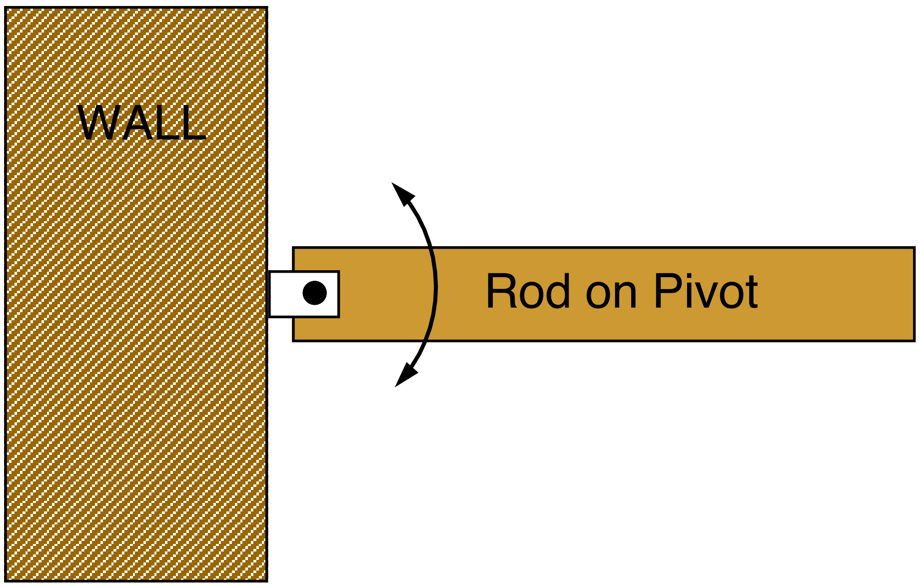
$$a = \frac{r^2 mg}{r^2 m + I}$$

$$r(mg - ma) = r(mg - mr\alpha) = I\alpha$$

$$\alpha = \frac{rmg}{I + r^2 m}$$

Example 12

A uniform rod has a mass M and length L, and can pivot as shown.



a) If the rod is released from rest in the position shown what is the initial angular acceleration?

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}} = I\alpha$$

$$\frac{L}{2}(mg) = \frac{1}{3}ML^2\alpha$$

$$\alpha = \frac{3g}{2L}$$

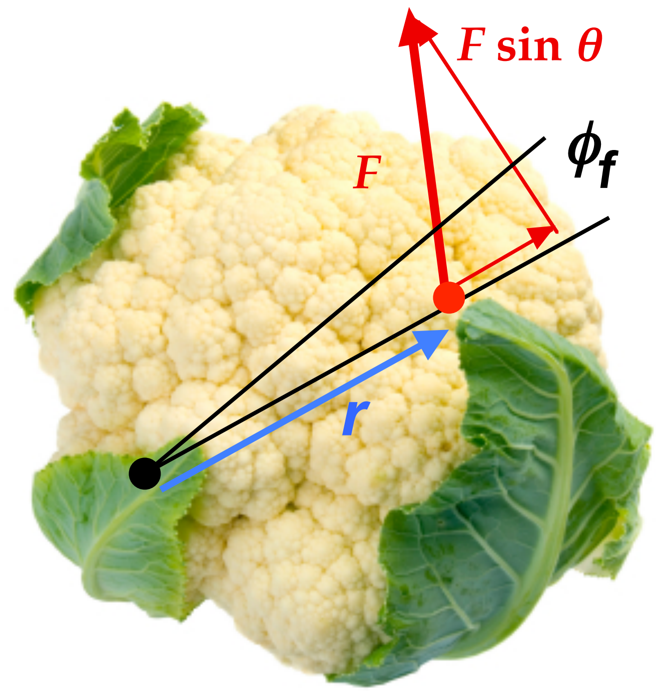
b) What is the initial linear acceleration of the right end of the rod?

$$\alpha = \frac{3g}{2L}, \ so \ \frac{a}{r} = \frac{3g}{2L}, \ or \ \frac{a}{L} = \frac{3g}{2L}$$

$$a = \frac{3}{2}g$$

Rotational Work & Power

How much Work is done when a Force is applied to a point, causing the object to rotate a distance *ds*?



$$dW = \vec{\mathbf{F}} \bullet d\vec{\mathbf{s}}$$

$$dW = F \sin\theta \bullet ds$$

$$dW = F \sin\theta \bullet r \bullet d\phi$$

$$\tau = rF \sin\theta$$

$$dW = \tau \bullet d\phi$$

$$W = \int \tau \bullet d\phi$$

How much Power is required to do that work?

$$\frac{dW}{dt} = \tau \bullet \frac{d\theta}{dt}$$

$$P = \tau \bullet \omega$$

Work $\longrightarrow$ K?

$$W = \int \vec{\mathbf{F}} \bullet d\vec{\mathbf{s}}$$

$$W = \int ma \bullet ds$$

$$W = m\int_{s_i}^{s_f} \frac{dv}{dt} \bullet ds = m\int_{s_i}^{s_f} \frac{dv}{dx}\frac{dx}{dt} \bullet dx$$

$$W = m\int_{v_i}^{v_f} v \bullet dv = \frac{1}{2}mv_f^{\,2} - \frac{1}{2}mv_i^{\,2}$$

Example 13

Show that the Work done by Torque produces a change in an object's rotational K.

$$\tau = \mathbf{I}\alpha$$

$$\tau = \mathbf{I}\frac{d\omega}{dt} = \mathbf{I}\frac{d\omega}{d\theta}\frac{d\theta}{dt} \text{ (by chain rule)}$$

$$\tau = \mathbf{I}\frac{d\omega}{d\theta}\omega$$

$$dW = \tau \ d\theta, \text{ so } \boldsymbol{\tau} = \frac{dW}{d\theta}, \text{ and}$$

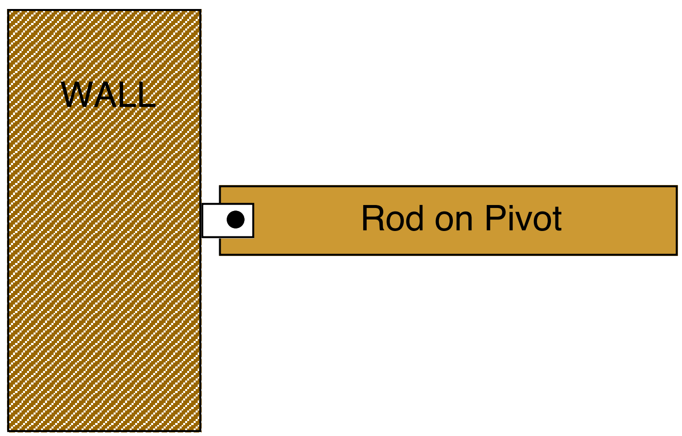
$$\frac{dW}{d\theta} = \mathbf{I}\frac{d\omega}{d\theta}\omega$$

$$dW = I\omega \ d\omega$$

$$W = I\int_{\omega_i}^{\omega_f}\omega \ d\omega = \frac{1}{2}I\omega_f^{\ 2} - \frac{1}{2}I\omega_i^{\ 2}$$

Example 14

A rod of length L is free to rotate on a frictionless pin as shown, the rod is released from rest in the horizontal position.



a) What is the angular speed of the rod in its lowest position (right before it hits the wall)?

omega = $\sqrt{(3g/L)}$

b) What is the linear speed of the center of mass at this lowest position?

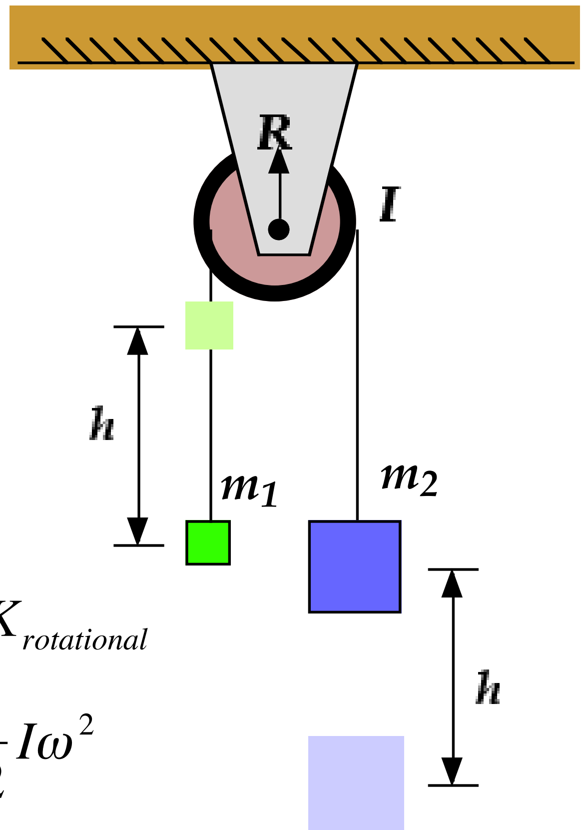
v(cm)=1/2 $\sqrt{(3gL)}$

c) What is the linear speed of the end of the rod at this lowest position?

v(end)=$\sqrt{(3gL)}$

Example 15

Two masses are connected to a pulley as shown and the system is released from rest. Find the linear speeds of the masses after m_2 has descended a distance h, and the angular speed of the pulley at that moment.



$$U_{1i} + U_{2i} + K_{1i} + K_{2i} + K_{rotational} = U_{1f} + U_{2f} + K_{1f} + K_{2f} + K_{rotational}$$

$$0 + 0 + 0 + 0 + 0 = m_1 gh + m_2 g(-h) + \frac{1}{2} m_1 v_1^{\ 2} + \frac{1}{2} m_2 v_2^{\ 2} + \frac{1}{2} I\omega^2$$

$$v = r\omega, \ and \ \ speed \ \ v_1 = v_2, \ \ so$$

$$gh(m_2 - m_1) = \frac{1}{2} m_1 (r\omega)^2 + \frac{1}{2} m_2 (r\omega)^2 + \frac{1}{2} I\omega^2$$

$$2gh(m_2 - m_1) = \omega^2 (m_1 r^2 + m_2 r^2 + I)$$

$$\omega = \sqrt{\frac{2gh(m_2 - m_1)}{m_1 r^2 + m_2 r^2 + I}}$$

Translational ⟶ Rotational

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

$$\sum \mathbf{F} = m\mathbf{a}$$

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t$$

$$\mathbf{x}_f - \mathbf{x}_i = \mathbf{v}_i t + \tfrac{1}{2}\mathbf{a}t^2$$

$$\mathbf{v}_f^{\,2} = \mathbf{v}_i^{\,2} + 2\mathbf{a}(\mathbf{x}_f - \mathbf{x}_i)$$

$$W = \int \mathbf{F} \bullet d\mathbf{x}$$

$$K = \tfrac{1}{2}mv^2$$

$$P = \frac{dW}{dt} = Fv$$

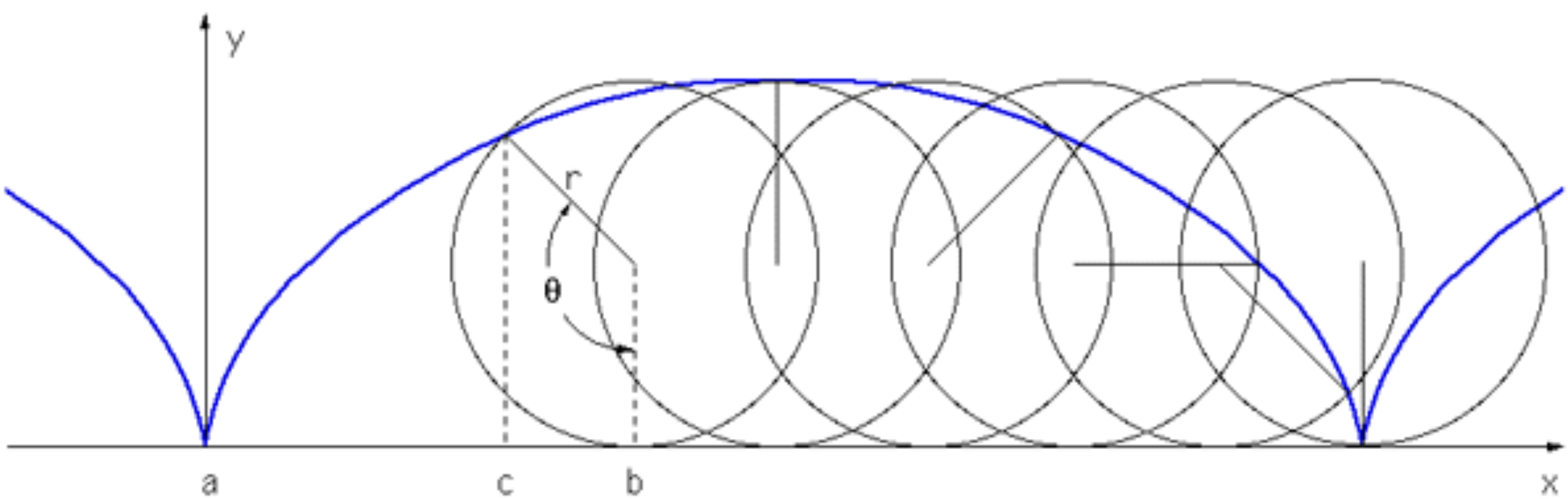
$$\mathbf{p} = m\mathbf{v}$$

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

Fixed Axle ⟶ Rolling!

$$x_{cm} = r\theta \qquad v_{cm} = \frac{ds}{dt} = \frac{d(r\theta)}{dt} = r\frac{d\theta}{dt} = r\omega$$

$$a_{cm} = \frac{dv_{cm}}{dt} = \frac{d(r\omega)}{dt} = r\frac{d\omega}{dt} = r\alpha$$



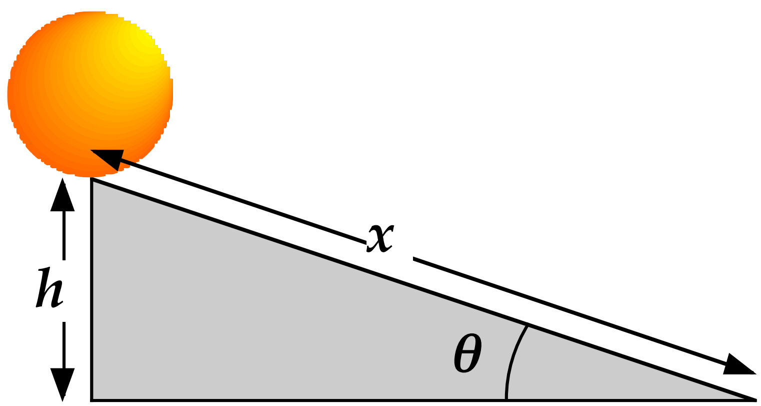
$K_{rolling\ body}$ =?

The total K energy of an object undergoing rolling motion is the sum of the rotational K energy about the center of mass, and the translational K energy of the center of mass.

$$K_{total} = K_{rotational} + K_{translational}$$

Example 16

Find the acceleration, and speed at the bottom, of a sphere rolling down an inclined plane as shown, using energy concepts.



$$I_{cm} = \frac{2}{5}MR^2$$

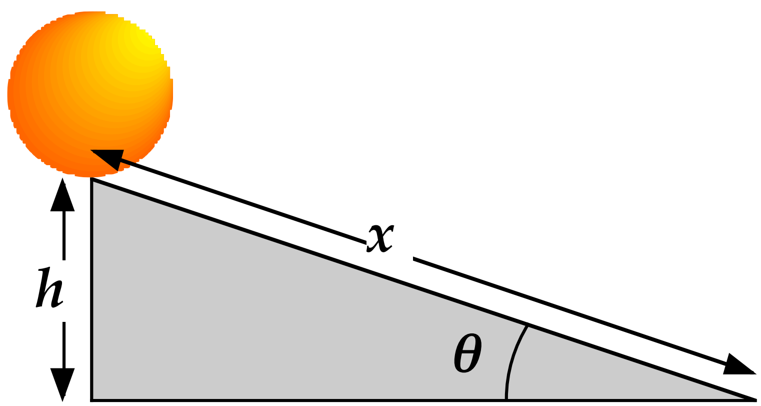
$$U_g = \frac{1}{2}I_{cm}\omega^2 + \frac{1}{2}mv_{cm}^{\ 2}$$

Solve to get $v = \sqrt{\frac{10}{7}gh}$, then use

$v^2 = v_i^{\ 2} + 2ax$, to get $a = \frac{5}{7}g\sin\theta$

Example 17

Find the acceleration, and speed at the bottom, of a sphere rolling down an inclined plane as shown, using a Force analysis.



$$F_{net} = ma_{cm}$$

$$F_{//} - F_{friction} = ma_{cm}$$

$$mg\sin\theta - F_{friction} = ma_{cm}$$

$$\tau = I_{cm}\alpha$$

$$rF_{friction} = \left(\frac{2}{5}mr^2\right)\left(\frac{a}{r}\right)$$

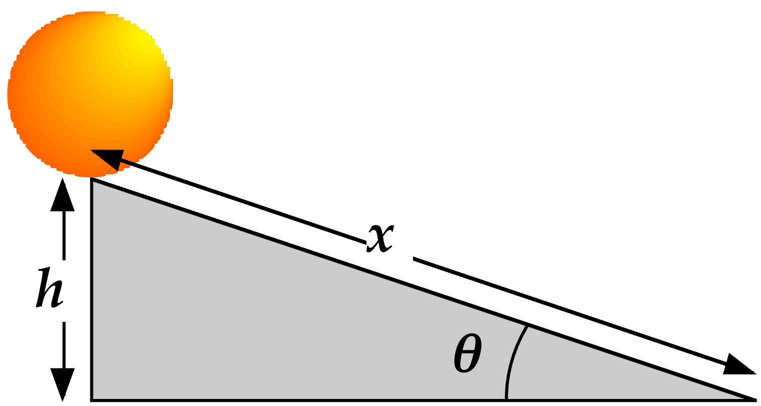
$$F_{friction} = \left(\frac{2}{5}ma\right)$$

$$mg\sin\theta - \left(\frac{2}{5}ma\right) = ma_{cm}$$

$$a_{cm} = \frac{5}{7}g\sin\theta$$

Example 18

Based on the previous analysis, what is the minimum μ necessary for the object to roll? What happens if the μ is less than that minimum value?



$$F_{friction} = \left(\frac{2}{5}ma\right) \;\&\; a_{cm} = \frac{5}{7}g\sin\theta,\; so$$

$$F_{friction} = \frac{2}{7}mg\sin\theta$$

$$\mu = \frac{F_{friction}}{F_{Normal}} = \frac{\frac{2}{7}mg\sin\theta}{mg\cos\theta} = \frac{2}{7}\tan\theta$$

A smaller μ implies that the sphere will be slipping as it rolls, losing some energy to frictional heat losses. Thus, it will have less total K by the time it reaches the bottom of the ramp. Its rotational speed will be less (why? Do a Force analysis) but its translational speed will be greater (why?)

Example 19

A hoop, a ball, and a solid cylinder are all released from the top of a ramp at the same time so that they roll down. Which one reaches the bottom first? Does it depend on the mass of the objects? Does it depend on their relative radii?

Tests – Comments

- Be careful what you're differentiating with respect to. (Work= $\int$ F•dx, Impulse= $\int$ F•dt)
- $\mathbf{F}_{Normal} \neq mg$ every time!
- Simplify when possible.
- When giving answers in variable form:
  - *don't* indicate units
  - *don't* write "9.8" for **g**
- Don't leave two paths to two answers on paper
- Don't cross out work if you're not replacing it with something else.

Photogates

Parallax is caused by the finite width of the photogate light beam, but the word parallax does not adequately describe what is happening. Refer to Figure-1. For the ideal case (an infinitely narrow light beam) the photo-detector signal would drop and rises like a square-wave pulse. If this really happened the measured time would be that shown as Δt(ideal) in Figure-1.

In the real case (a beam width of 0.5 to 2.0 mm) the edges of the flag take time to traverse the width of the beam, and the photo-detector signal gets rounded corners and slanted vertical lines. The measured time, Δt(obs), can be longer or shorter than Δt(ideal) depending upon how the logic circuitry responds to the photo-detector signal.

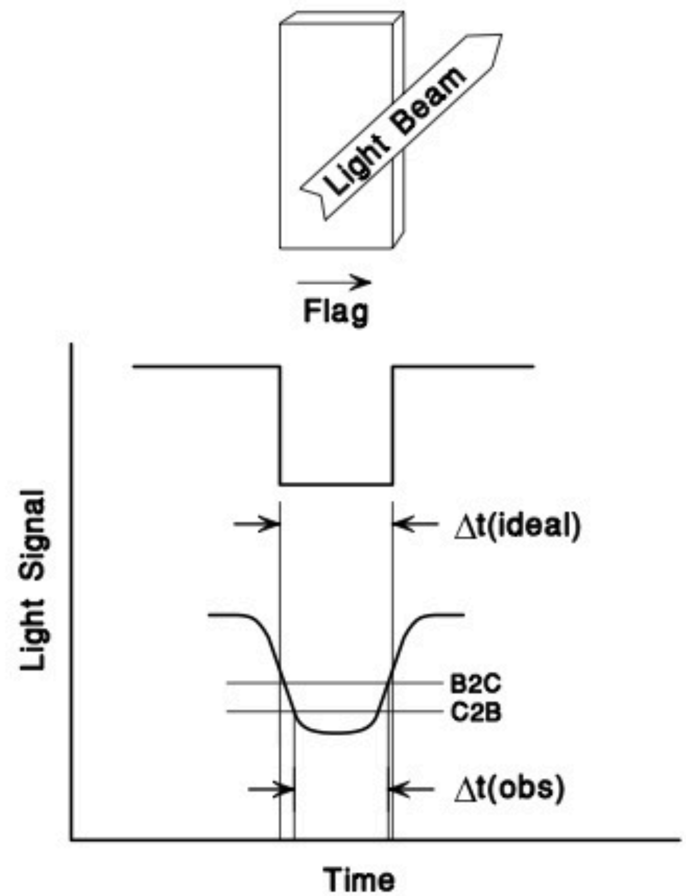


Figure 1

Usually Δt(real) is shorter than Δt(ideal) because the logic circuitry is designed with hysteresis and a substantial blockage of the beam is required. Hysteresis means the voltage level at which the clear-to-blocked transition is detected is different than the level at which the blocked-to-clear transition is detected. This is necessary to eliminate unwanted logic transitions caused by noise superimposed on the signal. In Figure-1 the hysteresis is set so the clear-to-blocked transition (C2B) occurs when the light is about 80-percent blocked, and the blocked-to-clear transition (B2C) occurs when the light is about 60-percent blocked. As can be seen, we the result that Δt(obs) is less than Δt(ideal).