Apply a torque, get a rotation . . .

The Island Series:

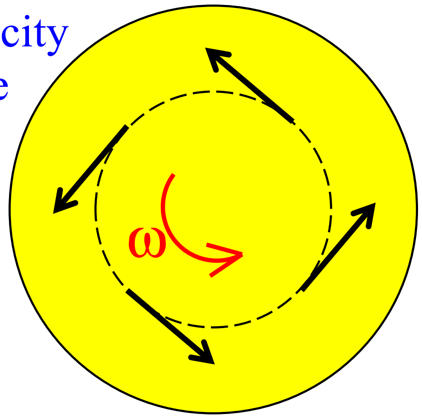
You have been kidnapped by a crazed physics nerd and left on an island with twenty-four hours to solve the following problem. Solve the problem and you get to leave. Don't solve the problem and you don't.

The problem: A large disk is set spinning with constant speed. The *translational velocity* of two points on the disk are identified along with each point's position relative to the axis of rotation. It is pointed out that at a minimum, two bits of information are required to characterize the motion of each point. Your captor is a minimalist, so the question is, "How can the motion of *each point* be characterized with only one bit of information?" Put a little differently, if someone with another disk somewhere else on the island wanted to duplicate your disk's motion, what single bit of information might you give that individual that would allow them to accomplish the task?

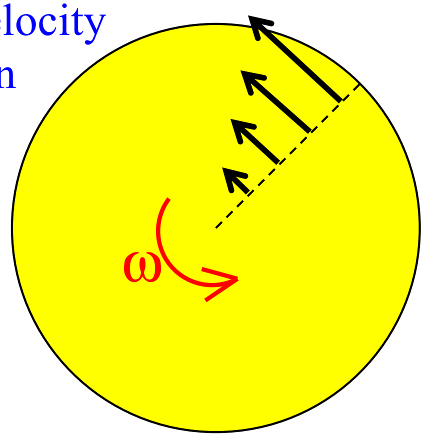
Solution to Island Problem

If you want to characterize the velocity of the particles of a spinning disk, alluding to their translational velocities is *REALLY inefficient*. Bits of mass that have a common radius from the *center of the rotation* will all have the same *velocity MAGNITUDE*, but each bit's velocity *vector* will have a different *direction.* And following a radial line out from the center will produce bits that all have the *same DIRECTION*, but each bit will have a different velocity *magnitude* (as you get out farther, the velocities will go up).

same velocity magnitude



same velocity direction



What IS common to all of the bits is their *angular velocity*. Each bit will sweep out the *same number of radians per second* as the disk rotates. That's why using rotational parameters for rotating systems is so useful. It is suffused with economy.

CHAPTER 10:
Rotational Motion

For every parameter that exists with the world of translational motion, there exists a rotational counterpart. To get into that world, we need to begin with some definitions.

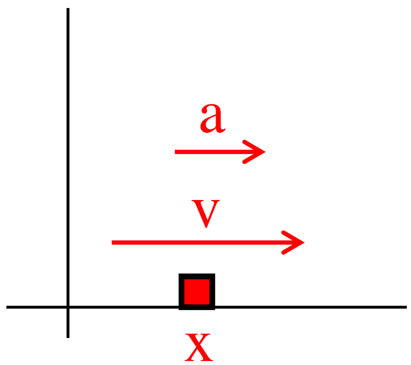
That is:

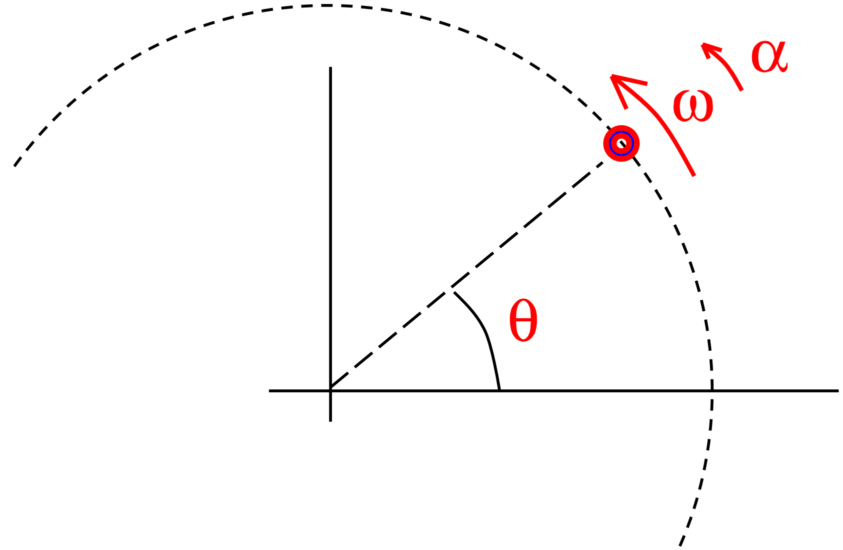
--*whereas* a body's coordinate position is defined with an "x" or "y" (units meters), a body's *angular* position is defined by a θ (measured in radians);

--*whereas* a body's translational velocity (the number of meters it traverse per unit time) is defined with a "v" (units m/s), a body's *angular* velocity (the number of radians it sweeps through per unit time) is defined using an ω (measured in radians/sec), and;

--*whereas* a body's translational acceleration (its velocity change per unit time) is defined with an "a" (units m/s/s), a body's *angular* acceleration (its angular velocity change per unit time) is defined using an α (measured in rad/sec/sec).

Summary:

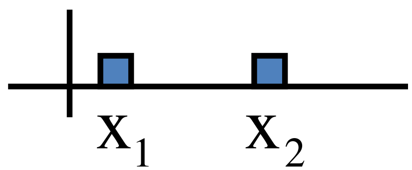




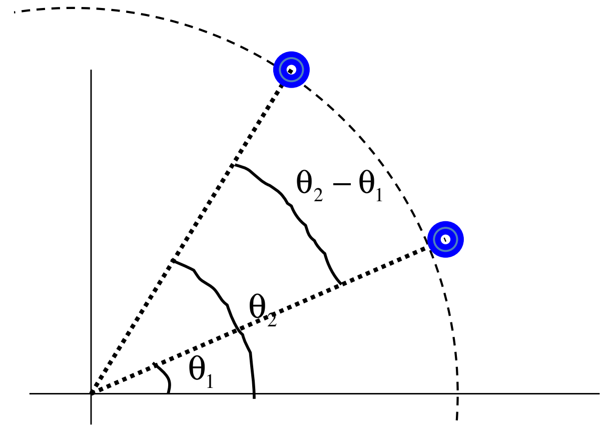
position	rate of change of position	rate of change of velocity
coordinate position	translational velocity	translational acceleration
x (meters)	v (m/s)	a (m/s^2)
angular position	angular velocity	angular acceleration
θ (radians)	ω (rad/s)	α (rad/s^2)

Average Parameters

body translates $x_2 - x_1$
in time "t"



body rotates $\theta_2 - \theta_1$
in time "t"



average velocity

translational $\quad v_{avg} = \dfrac{x_2 - x_1}{\Delta t} \qquad v_{avg} = \dfrac{v_1 + v_2}{2}$

rotational $\quad \omega_{avg} = \dfrac{\theta_2 - \theta_1}{\Delta t} \qquad \omega_{avg} = \dfrac{\omega_1 + \omega_2}{2}$

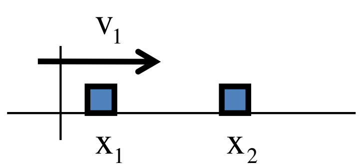
average acceleration

$$a_{avg} = \frac{v_2 - v_1}{\Delta t}$$

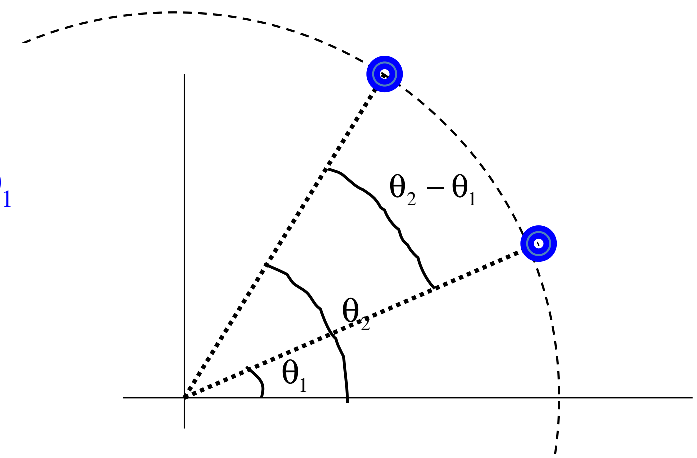
$$\alpha_{avg} = \frac{\omega_2 - \omega_1}{\Delta t}$$

Kinematics

body with initial velocity v_1
accelerates between $x_2 - x_1$
in time t



body with initial angular velocity ω_1
angularly accelerates between $\theta_2 - \theta_1$
in time t



translation motion

$$x_2 = x_1 + v_1(\Delta t) + \frac{1}{2}a(\Delta t)^2 \quad \text{or} \quad \Delta x = v_1(\Delta t) + \frac{1}{2}a(\Delta t)^2$$

$$v_2 = v_1 + a\Delta t \quad \text{or} \quad a = \frac{v_2 - v_1}{\Delta t}$$

$$v_2^{\,2} = v_1^{\,2} + 2a(\Delta x) \quad \text{or} \quad v_2^{\,2} = v_1^{\,2} + 2a(x_2 - x_1)$$

angular motion

$$\theta_2 = \theta_1 + \omega_1(\Delta t) + \frac{1}{2}\alpha(\Delta t)^2 \quad \text{or} \quad \Delta\theta = \omega_1(\Delta t) + \frac{1}{2}\alpha(\Delta t)^2$$

$$\omega_2 = \omega_1 + \alpha\Delta t \quad \text{or} \quad \alpha = \frac{\omega_2 - \omega_1}{\Delta t}$$

$$\omega_2^{\,2} = \omega_1^{\,2} + 2\alpha(\Delta\theta) \quad \text{or} \quad \omega_2^{\,2} = \omega_1^{\,2} + 2\alpha(\theta_2 - \theta_1)$$

Rotational Vectors

The relationship $\vec{v} = -(3m/s)\hat{i}$ is really code. It is telling you three things:

a.) the magnitude of the velocity (in this case, it's 3 m/s);

b.) the *line* of the velocity (the $\hat{i}$ tells you the vector is along the x-axis, versus being along the y-axis or z-axis or some combination thereof); and

c.) the + or – tells you the *actual* direction along the line (in this case, it's in the NEGATIVE x-direction, versus the POSITIVE x-direction);

You know how to decode the above expression. The following is also a code.

$$\vec{\omega} = -(3 \text{ rad/sec})\hat{i}$$

The question is, "What three things does *this* coding tell you?"

The relationship $\omega = -(3 \text{ rad/sec})\hat{i}$ tells you:

a.) the magnitude of the *angular* velocity (in this case, it's 3 rad/s);

b.) the *DIRECTION OF THE AXIS* about which the angular velocity proceeds (this will be *perpendicular* to the plane of the motion, so an "$\hat{i}$" tells you the motion is in the *y-z plane*); and

c.) the + or – tells you the whether the rotation is clockwise or counterclockwise, as viewed from the positive side of the axis (in this case, it's NEGATIVE, so the rotation will be *clockwise*—more about this later).

Clarification concerning *parts c* above. Both physics and standard mathematics use what is called a right-handed coordinate system. That means that if you place your right hand along the +x direction and curl your fingers in the +y direction, your thumb will point in the +z direction. The reason this is significant is that in doing so, you will be curling your fingers counterclockwise. So if you want to characterize a body moving counterclockwise in the x-y plane, giving the direction as +k makes sense as that is the direction your thumb would point if you made the fingers of your right hand curl along the direction of motion.

In short, though, if you know how to do the decoding, the notation is as simple as

.
$$\vec{v} = -(3m/s)\hat{i}$$

Summary Example 1: A turntable (record player) is rotating as shown in the sketch (courtesy of Mr. White). The *magnitude* of its angular speed is .3 rad/sec. After rotating half a turn, its angular speed *as a vector* is .1 rad/sec.

a.) Write out its initial angular velocity as a vector.

Vector direction in a rotational setting defines the *axis about which the rotation takes place*. It is *perpendicular* to the plane of the rotation. If you place your right hand on the record so your finger's rotate with the record, your thumb will point *into* the page in the $-\hat{k}$ *direction*. THAT is the "direction" of the angular velocity direction. In whole, the vector can be written as:

$$\vec{\omega}_o = (.3 \text{ rad/sec})(-\hat{k})$$
$$= -(.3 \text{ rad/sec})\hat{k}$$

Kindly note: The record is rotating clockwise. *Clockwise rotation*, as viewed from the positive side of the axis, *always* denotes a negative vector.

Summary Example 1: A turntable (record player) is rotating as shown in the sketch (courtesy of Mr. White). The magnitude of its angular speed is .3 rad/sec. After rotating half a turn, its angular speed *as a vector* is .1 rad/sec.

b.) What is the record's *angular acceleration*?

This is a rotational kinematic problem. Tracking a point on the table with an initial angular velocity of $\omega_1 = -.3 \text{ rad/sec}$, and assuming its initial position is $\theta_1 = 0$ and final position $\theta_2 = -\pi \text{ radians}$ (if its *angular velocity* is *negative*, its *angular displacement* will be *negative*, also), we can write:

$$(\omega_2)^2 = (\omega_1)^2 + 2\alpha(\theta_2 - \theta_1)$$

$$\Rightarrow \quad \alpha = \frac{(\omega_2)^2 - (\omega_1)^2}{2(\theta_2 - \theta_1)}$$

$$= \frac{(.1 \text{ r/s})^2 - (-.3 \text{ r/s})^2}{2(-\pi - 0)}$$

$$= .0123 \text{ r/s}^2$$

Summary Example 1: A turntable (record player) is rotating as shown in the sketch (courtesy of Mr. White). The magnitude of its angular speed is .3 rad/sec. After rotating half a turn, its angular speed *as a vector* is .1 rad/sec.

c.) Through how many radians will it have moved during the first 2 seconds of its rotation?

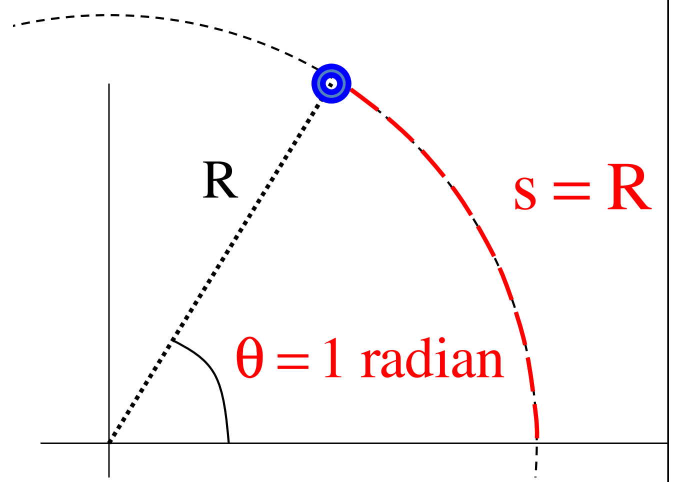
$$\omega_1 = -.3 \text{ rad/sec}$$
$$\alpha = .0123 \text{ rad/sec}^2$$

$$\theta_2 = \theta_1 + \omega_1 \Delta t + \tfrac{1}{2}\alpha(\Delta t)^2$$
$$\Rightarrow \quad \Delta\theta = \omega_1 \Delta t + \tfrac{1}{2}\alpha(\Delta t)^2$$
$$= (-.3 \text{ rad/sec})(2 \text{ sec}) + \tfrac{1}{2}(.0123 \text{ rad/sec}^2)(2 \text{ sec})^2$$
$$= -.5754 \text{ rad}$$

Rotational versus Translational Parameters

Definition of a radian?

If you lay out a *one radius* arc-length, the angle subtended is defined as one radian (see sketch).



So what arc-length is associated with a 2 radian angle?

$$s_2 = 2R$$

And what arc-length is associated with a 1/2 radian angle?

$$s_{1/2} = \left(\frac{1}{2}\right)R$$

And what is the arc length associated with a $\Delta\theta$ radian angle?

$$s = R\theta$$

where R's units are *meters per radian* and θ's units are in radians.

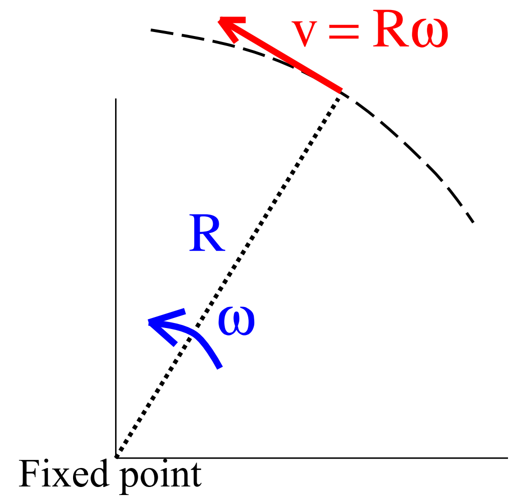
Taking the derivative of both sides yields:

$$\frac{ds}{dt} = R \frac{d\theta}{dt}$$

or $v\ (\text{m/s}) = R\ (\text{m/rad})\ \omega\ (\text{rad/sec})$

more commonly written as: $v = R\omega$

where v is the *velocity of a point moving* with *angular velocity* ω upon an arc *R units from the fixed center*.



Taking the derivative of both sides again yields:

$$\frac{dv}{dt} = R \frac{d\omega}{dt}$$

$$a\ (\text{m/s}^2) = R\,(\text{m/rad})\ \alpha\ (\text{rad/sec}^2)$$

more commonly written as:

$$a = R\alpha$$

These are NOT kinematic relationships! They work whether the acceleration is a constant or not.

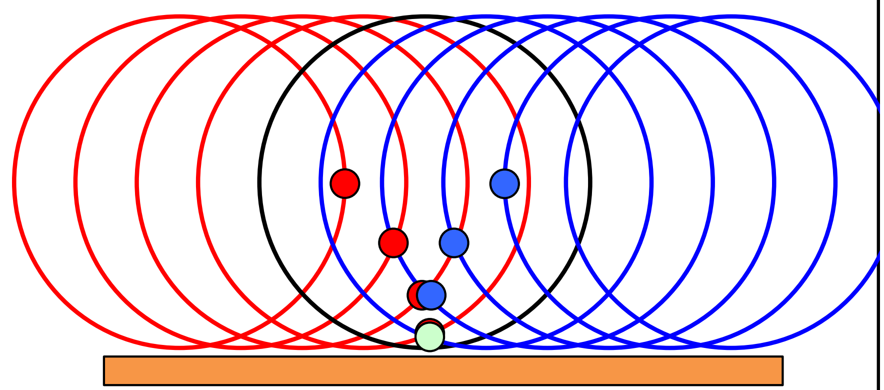
Rolling Point-of-Contact Action

Note that tonight's homework is on *moment of inertia*, which this and the next several sections do not address. These topics are important, but they will be fitted in at the end of daily lectures as time allows.

Question: An object is rolling across at tabletop. What the *instantaneous velocity* of object's contact point with the table?

Looking at the sketch makes it clear that the *velocity* of the point of contact of a rolling object is ZERO!

Additional justification: The wheel's velocity is clearly zero in the y-direction as the point of contact, as that point is executing a *turn-around* at that point. And in the x-direction, the table top is not moving (i.e., it's velocity is zero) and the wheel is not *sliding* over the table top, so the contact point must also be zero velocity.



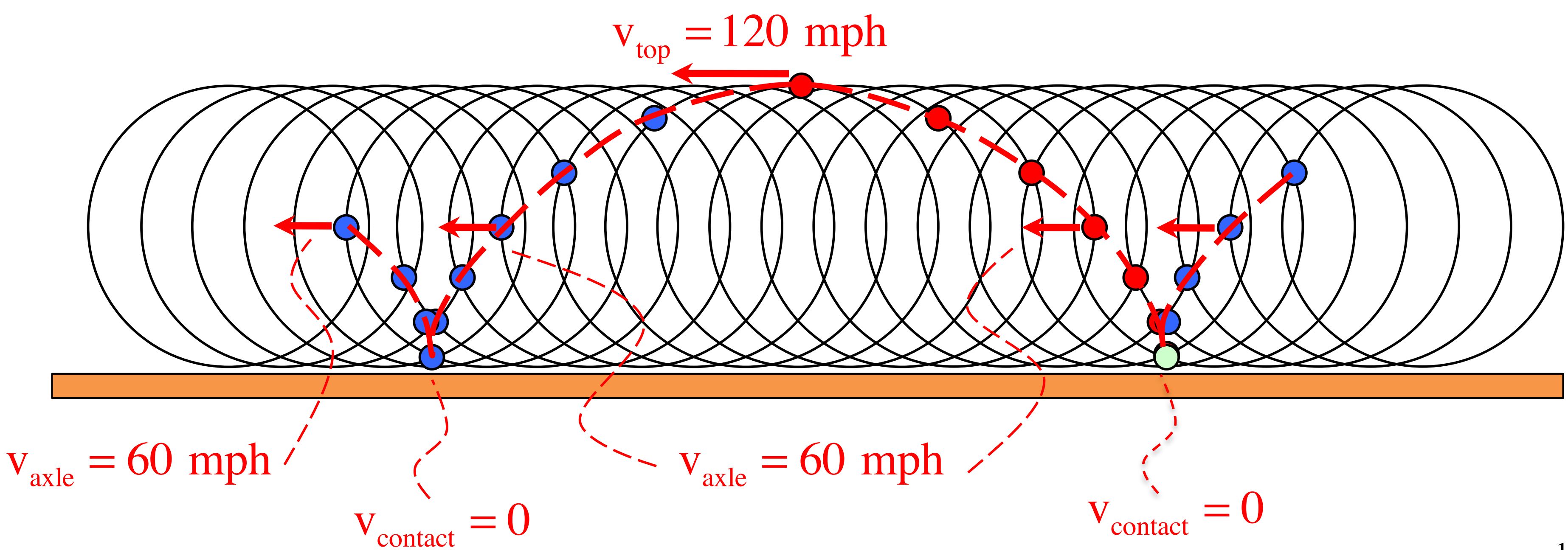
As a wild sidepoint: Think about a car traveling 60 mph. What happens to the section-of-tire that goes from *contact with the ground* to the *top of the tire* and back down to contact with the road?

--*At the contact point,* the section won't be moving at all.

--*But when at the level of the axle,* the section will move at the speed of the car.

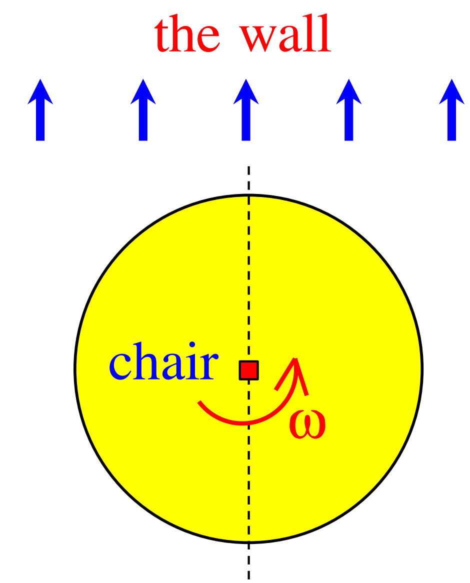
--*And when at the top of its motion,* the section will be moving at twice the speed of the car, or 120 mph (as $v_{top} = (2R)\omega$ suggests).

In other words, the wheel's circumference will accelerate from zero to 120 mph and back to zero over a few meters distance, over and over and over again as the car moves down the freeway. CRAZY, eh!



As an additional bit of craziness, if you know the *angular velocity* about one point on a rotating object, that will be the the angular velocity *about ALL points* on the object. How so?

Consider a rotating platform with a chair at its center that is rigged to ALWAYS face toward the wall:



You sit in the seat. It takes 10 seconds for the platform to rotate through one complete rotation.

a.) What does the motion look like from your perspective, assuming a constant *angular velocity*?

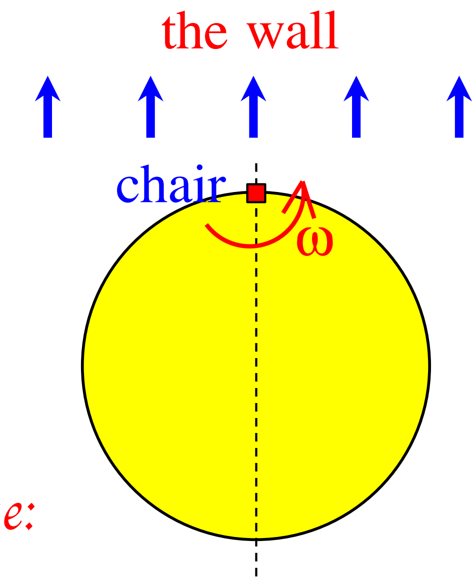
(It will move around you.)

b.) Relative to the axis you are sitting on, what will be the platform's *angular velocity*?

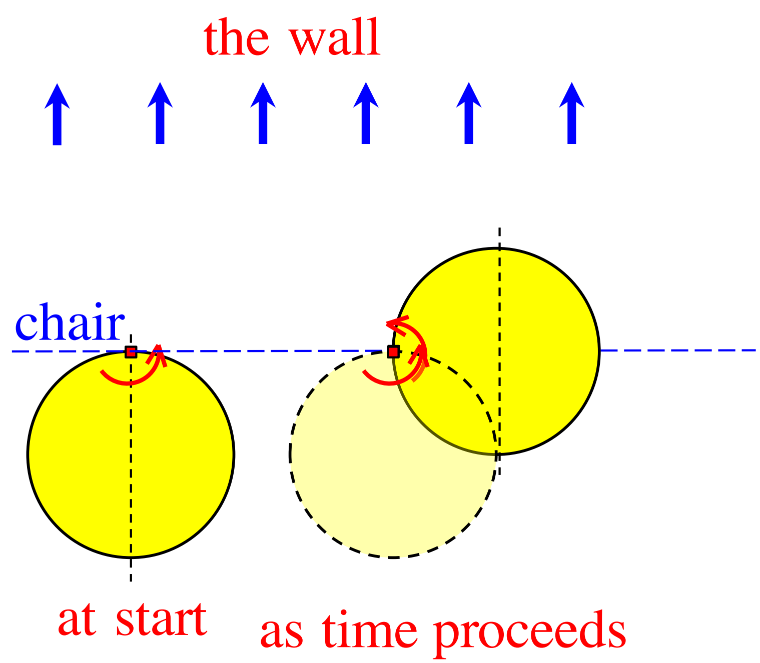
$$\omega = \frac{2\pi \text{ rad}}{10 \text{ sec}}$$

$$= .2\pi \text{ rad/sec}$$

The chair is now placed at the edge of the platform. It is still rigged to always face toward the wall. Just as was the previous case, it takes 10 seconds for the disk to move through one rotation. From your perspective, what does the motion look like, and what is the angular velocity of the disk about your position?



Following the motion as seen by you in the chair at the edge:

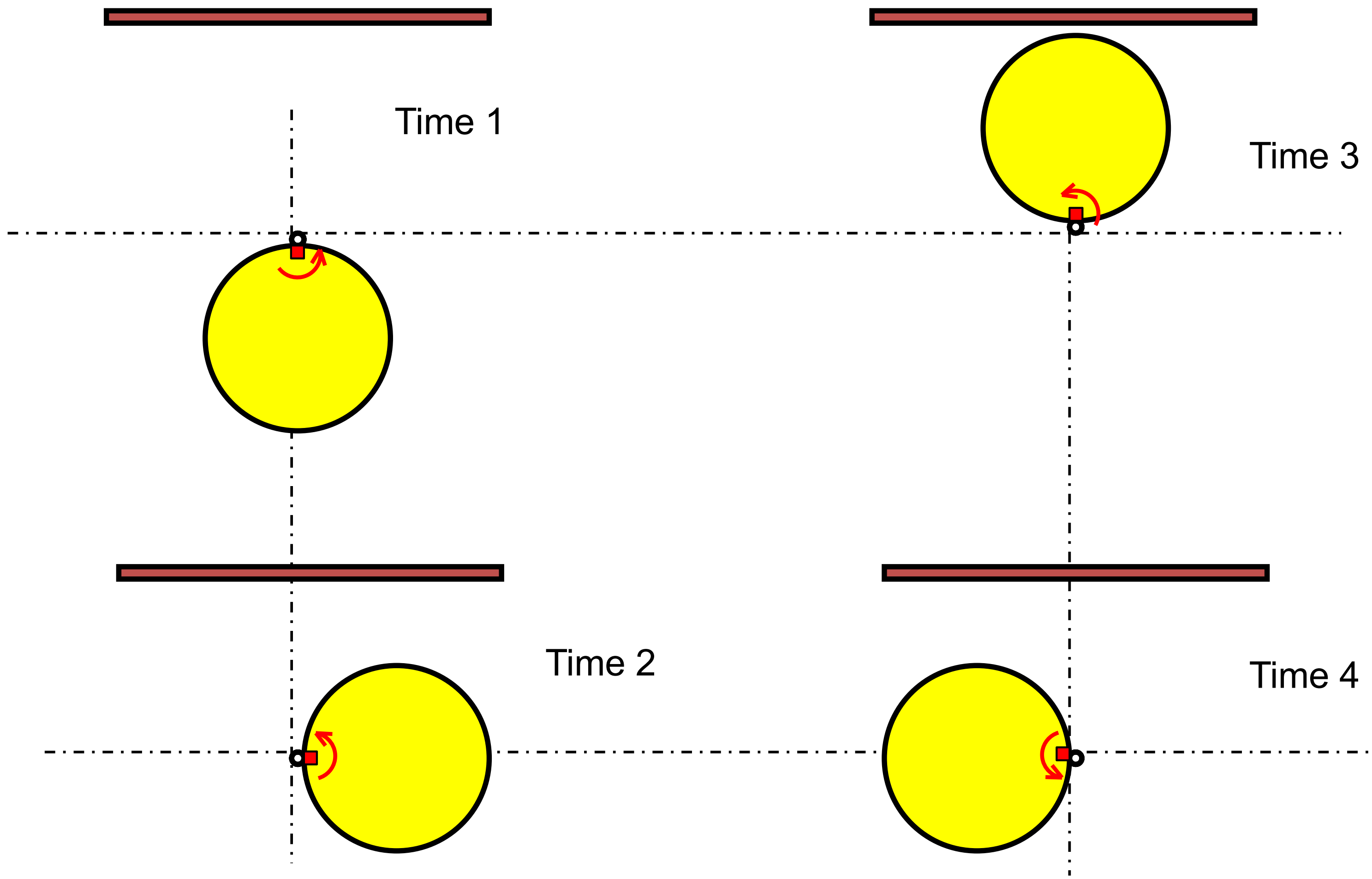


You start facing away from the disk, seeing none of it (looking at the wall).

As the disk rotates, you continue to face the wall and the disk begins to come into view on your right. In other words, the disk appears to be *rotating around the axis upon which you sit*.

Progression of motion

from watcher's perspective (remember, the watch is ALWAYS facing the wall!).



And what is the angular velocity of the disk about your vantage point?

You will sweep out 2π radians in 10 seconds, so you'll get:

$$\omega = \frac{2\pi \text{ rad}}{10 \text{ sec}}$$

$$= .2\pi \text{ rad/sec}$$

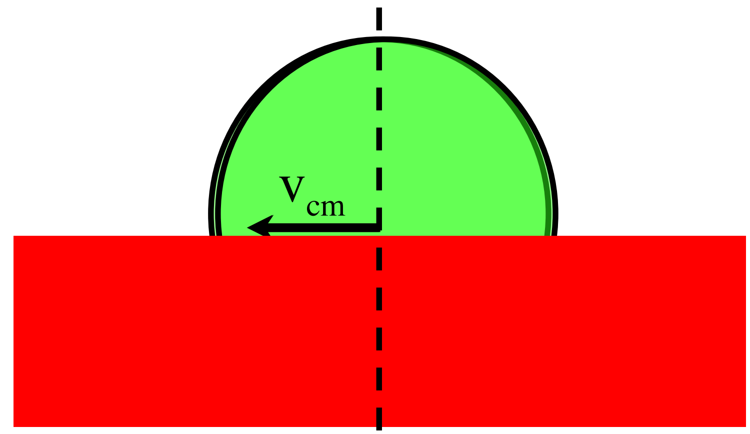
The same as about the central axis!!!!!

The point: The amount of time it takes the for the platform to rotate around you is the same in both the "center seat" situation and the "edge seat" situation. Additionally, the angular displacement in both cases during one revolution's worth of time is 2π radians.

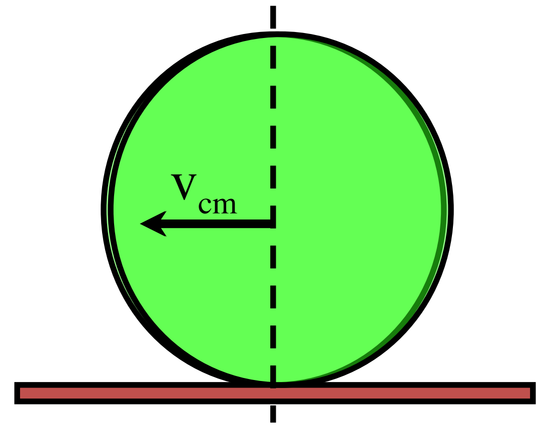
Sooooo (in other words), if the object appears to be rotating around you, the *angular velocity* you observed *will be the same* no matter where on the platform you are standing.

Translation: If you know the *angular velocity* of an object about any point on the object, you know the *angular velocity* about any other point on the object.

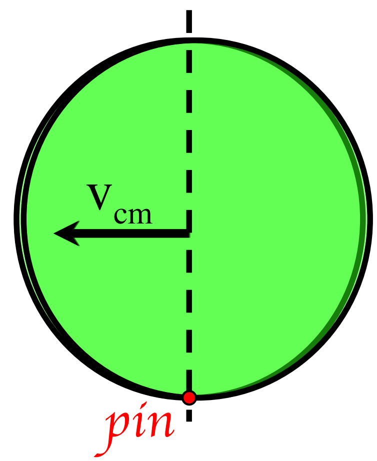
The consequence of this is the acknowledgement that if you were to see a moving disk covered as shown to the right, you wouldn't know if:



The disk was *rolling on a floor*; or

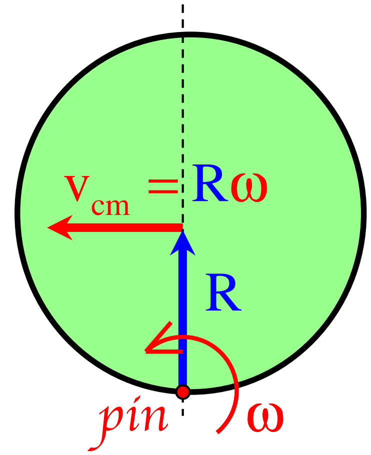


The disk was *pivoting about a fixed point*.



What is hugely important about this is that it means we can relate the *velocity of* a rolling object's *center of mass* to the *angular velocity* of the rolling object *about* it's center of mass. How so?

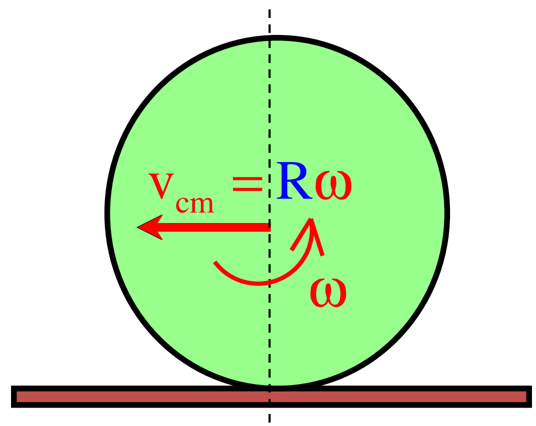
Consider a disk rotating with *constant angular velocity* ω around a pin at its edge. What is the *velocity* of a point at it's *center of mass*?



The velocity of a point R units from a fixed point on an object (like, at the body's center of mass) moving with *angular velocity* ω about a pin point is:

$$v_{cm} = R\omega$$

BUT BECAUSE you can't tell the difference instantaneously between a pinned, rotating disk and a disk that is *rolling on a tabletop*, that means the relationship between the angular velocity ω of a rolling object and its *center of mass* velocity must be:

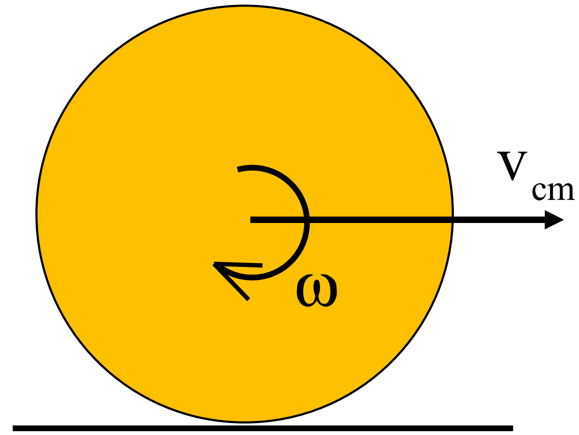


$$v_{cm} = R\omega \quad \Rightarrow \quad a_{cm} = R\alpha$$

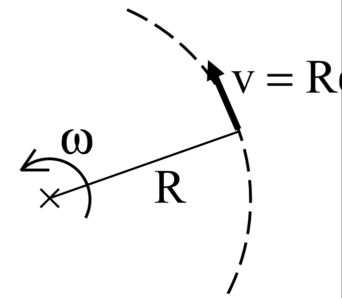
This is mucho importante!

But why is this important, *really*?

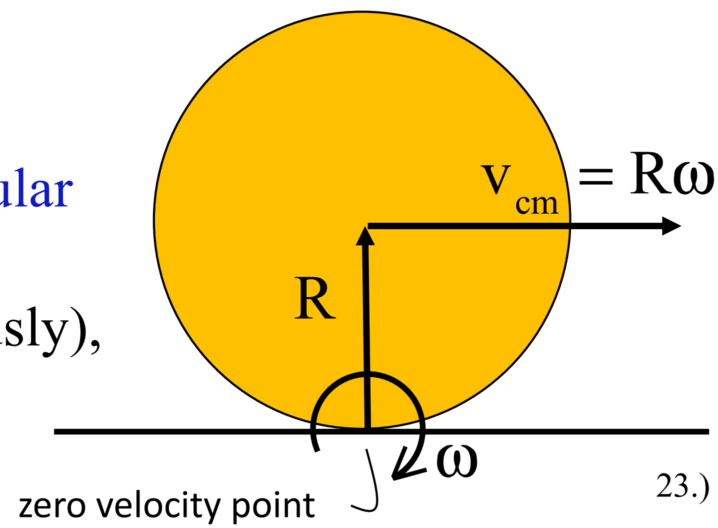
Consider a ball rolling across a table. It's center of mass has some velocity v_{cm} and all of the body's mass is rotating about the center of mass with some angular velocity ω. So how do we relate those two parameters (and how do we justify that relationship)?



We only have one relationship between the angular velocity of a mass moving in a circular path and its instantaneous velocity in that motion, and that is $v = R\omega$, but that requires *rotation around a fixed point*.



But if the contact point of the rolling ball is instantaneously fixed (zero velocity), and if the angular velocity about the center of mass is the same as the angular velocity about that fixed point (instantaneously), then, as the sketch shows, it follows that $v_{cm} = R\omega$.



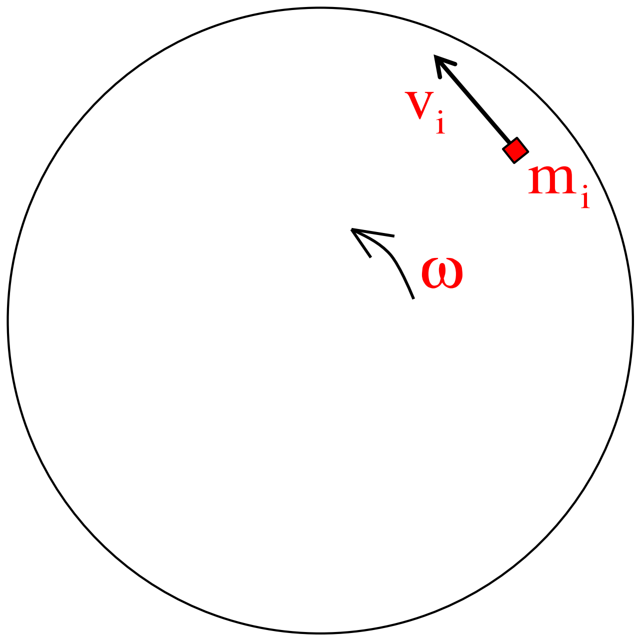
This is important!!!

More Minutia: Rotational Inertia and the Moment of Inertia

There is a rotational counterpart for every *translational concept* and *parameter* out there. So what can we say about the *energy content* of a rotating disk?

Moved a distance r_i units from the center of a disk rotating with *angular velocity* ω and you will find the i^{th} mass m_i moving with *translational velocity* v_i. It's *kinetic energy* calculates as:

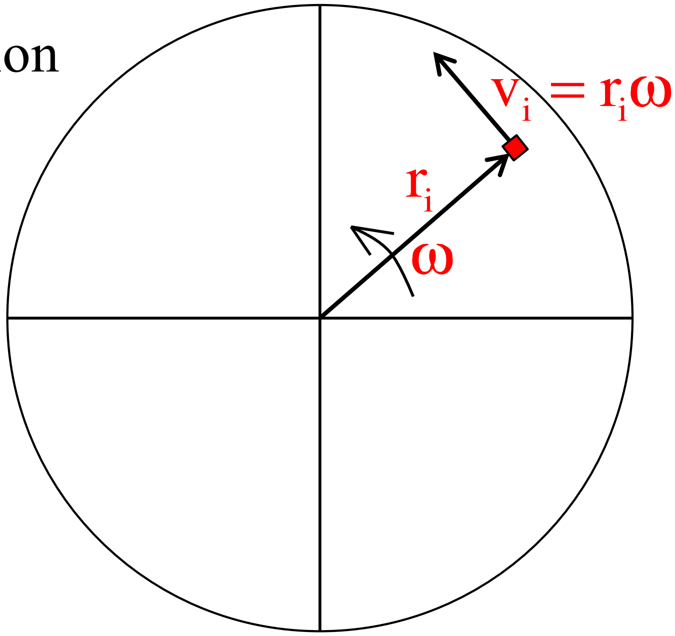
$$KE_i = \frac{1}{2} m_i \left(v_i\right)^2$$



To get the total kinetic energy for the entire mass, this process has to be done for all the masses with the results summed, or.

$$KE = \sum KE_i = \sum \frac{1}{2} m_i \left(v_i\right)^2$$

Noting that $v_i = r_i\omega$, we can rewrite that summation yielding:



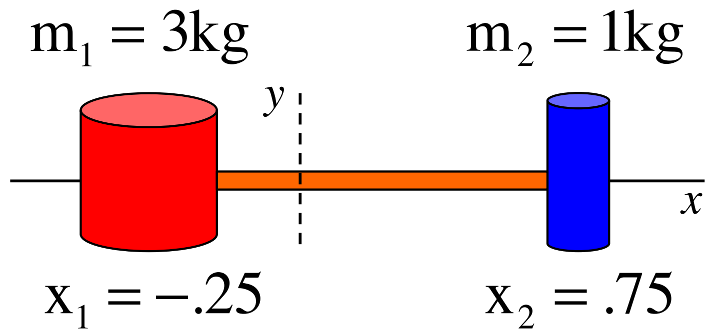
$$
\begin{aligned}
KE &= \sum_i KE_i \\
&= \sum_i \frac{1}{2} m_i (v_i)^2 \\
&= \sum_i \frac{1}{2} m_i (r_i\omega)^2 \\
&= \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2
\end{aligned}
$$

Comparing this to $KE = \frac{1}{2}(m)v^2$ leaves us with ½ and a velocity term squared, and a *mass related term* in parentheses.

This mass-related term is $I = \sum m_i r_i^2$. It is called *moment of inertia*. It is *always* defined relative to an axis and it is the rotational counterpart to mass . . . which is to say, it is a relative measure of a body's resistance to changing its rotational motion, or its *rotational inertia*.

Example 2: Consider two masses m and 3m located a distance 1.0 meter apart. Relative to the coordinate axes shown:

a.) Determine the *moment of inertia* about the y-axis through the x-axis *center of mass*:



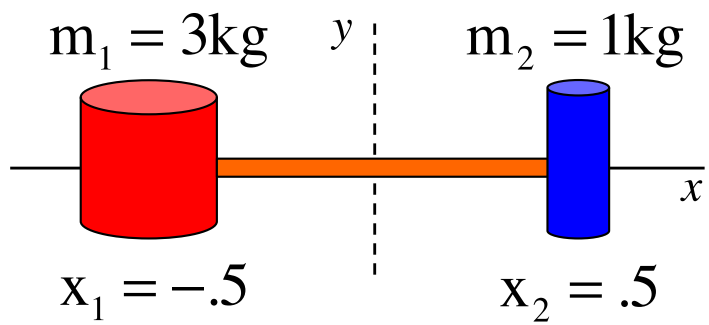
$$I_y = \sum m_i (x_i)^2$$

$$= m_1 (x_1)^2 + m_2 (x_2)^2$$

$$= (3 \text{ kg})(-.25 \text{ m})^2 + (1 \text{ kg})(.75 \text{ m})^2$$

$$= .75 \text{ kg} \bullet \text{m}^2$$

b.) Determine the *moment of inertia* about the y-axis as shown:



$$I_y = \sum m_i (x_i)^2$$

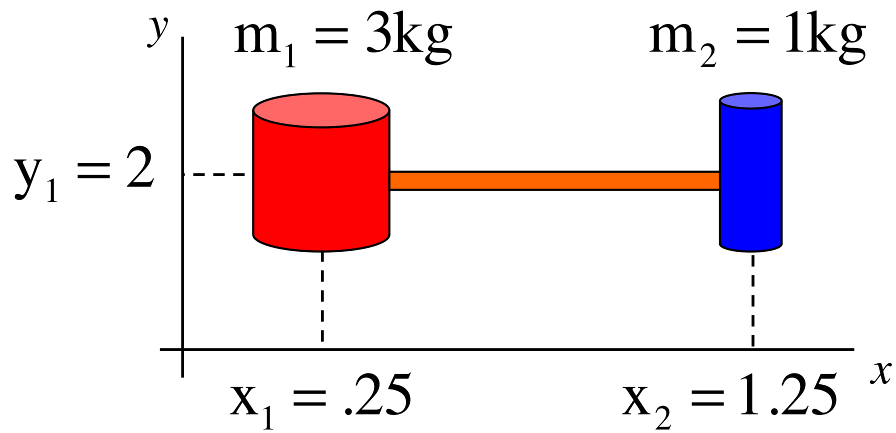
$$= m_1 (x_1)^2 + m_2 (x_2)^2$$

$$= (3 \text{ kg})(-.5 \text{ m})^2 + (1 \text{ kg})(.5 \text{ m})^2$$

$$= 1.0 \text{ kg} \bullet \text{m}^2$$

Notice the *moment of inertia* about the *center of mass* is smaller than about the other axes denoted. This is always true. I_{cm} is always a minimum.

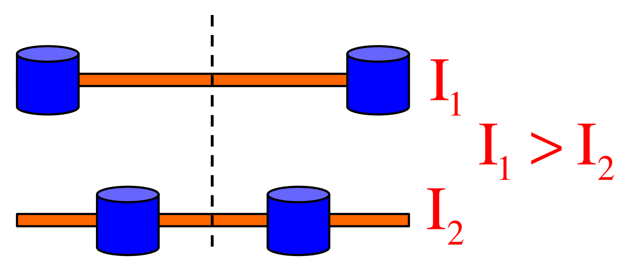
c.) Determine the *moment of inertia* about the x-axis:



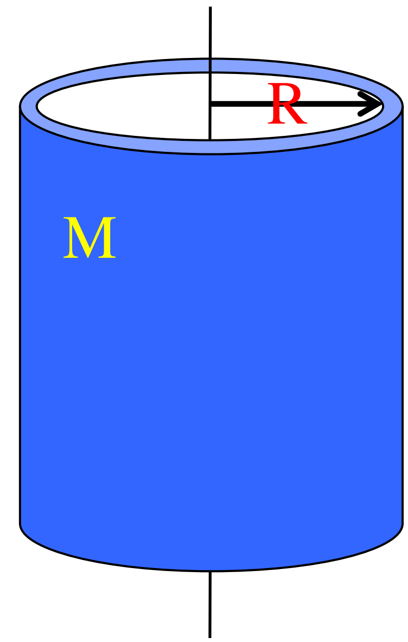
So what is this approach asking you to do? It is asking that you begin at the axis of interest, proceed outward until you run into some mass, multiply the mass by the distance-out-quantity-squared, and sum all those quantities up. For this problem, that will look like:

$$\begin{aligned} I_y &= \sum m_i (y_i)^2 \\ &= (m_1 + m_2)(y_1)^2 \\ &= (3\text{ kg} + 1\text{ kg})(2\text{ m})^2 \\ &= 16\text{ kg} \bullet \text{m}^2 \end{aligned}$$

Note: The closer a body is to an axis of rotation, the smaller its *moment of inertia* is about that axis.



Example 3: Keeping in mind what this approach is asking you to do, what is the moment of inertia of a thin cylindrical shell of mass M and radius R *about its central axis*?



This doesn't really take any math. If you move out from the axis until you find some mass, you find all the mass a distance R units out (assuming the cylinder really is THIN). Multiplying that mass by that distance-squared yields:

$$I_{thincylinder} = MR^2$$

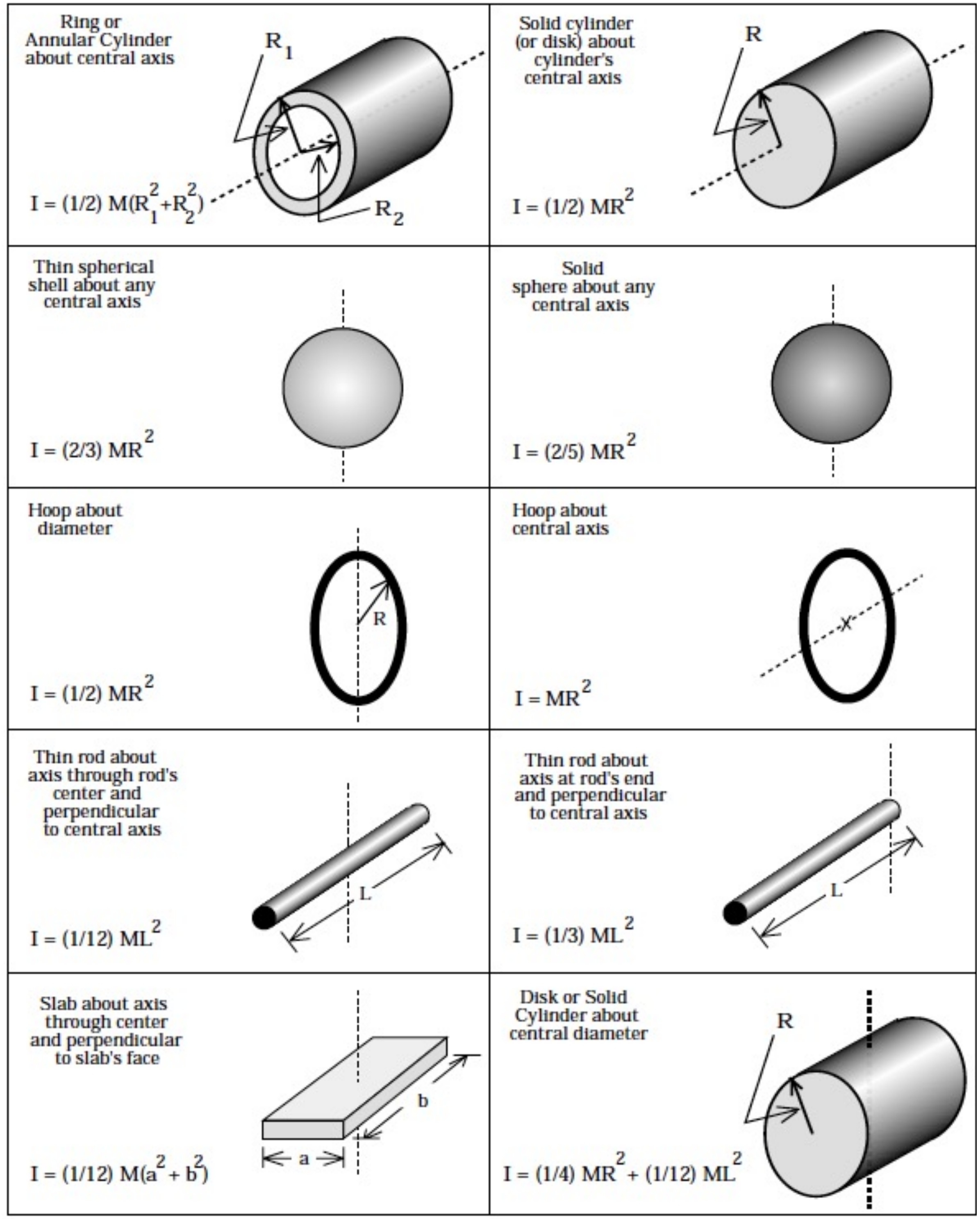
In other words, although this won't be the case in general, this was more of a conceptual exercise than anything else.

In general, when you need a *moment of inertia* expression for an object, you can either derive it or find it on page 287 of your text (or page 257 of Fletch's text).

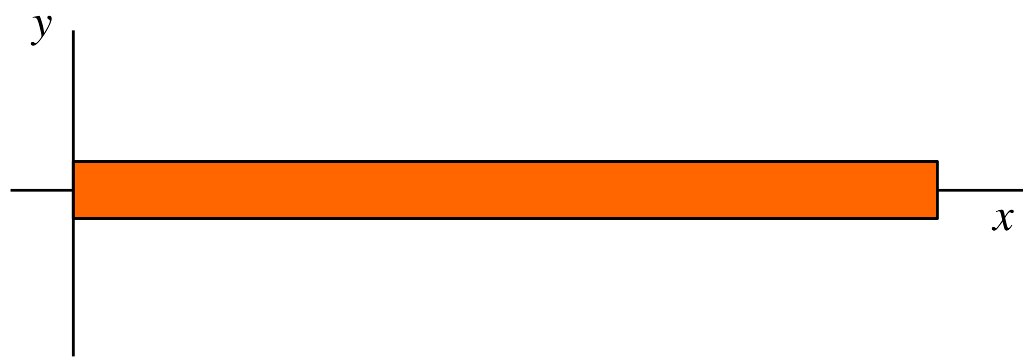
The table from Fletch's book is shown to the right. Notice that the *moment of inertia* of a hoop (which is just a very short, thin cylinder) about its central axis is quote as:

$$I_{thincylinder} = MR^2$$

just as we surmised. Still, you need to know generally how to do the derivations . . .

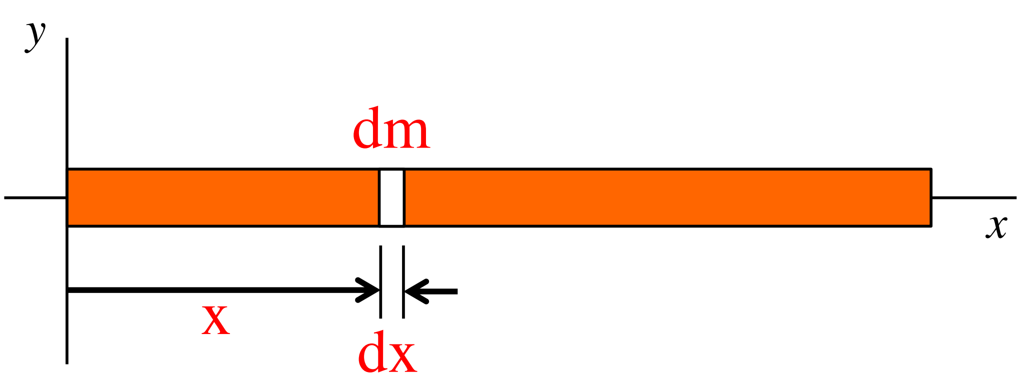


Example 4: Remembering what the approach is asking you to do, derive an expression for the moment of inertia of a homogeneous rod of length L about one end.



Although this is a continuous mass, the principle is the same. Move out along the x-axis until you find some mass, multiply by the distance-squared, then sum that quantity for all the masses found.

The problem? The system is not made up of discrete pieces of mass. That means that after moving an arbitrary distance "x" units down the axis, you need to create a differentially thin section of the rod of width "dx" to define a differentially small piece of mass "dm," do the required multiplication, then sum all such pieces using integration. Doing this yields:



$$I = \int x^2 \, dm$$

As we did with *center of mass* problems, we need to relate the position x of the bit of mass to the amount of mass *dm* that is there. To do that, we need to invoke a *density function*.

Due to the geometry, we will use a *linear density function*. With that in mind, we can write:

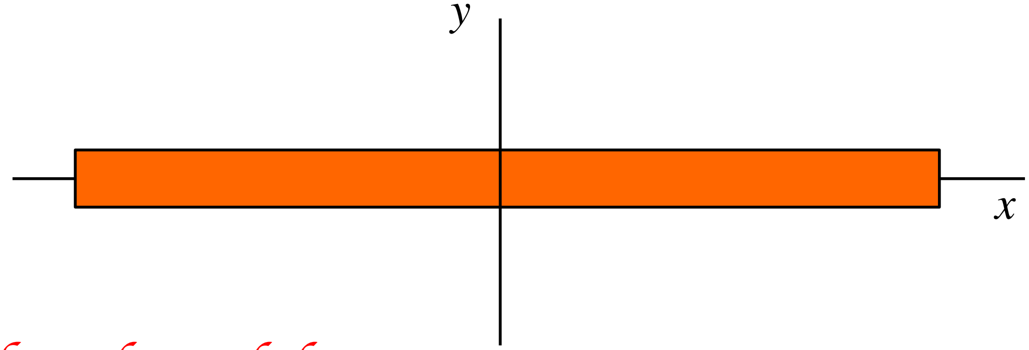
$$\lambda = \frac{M}{L} \quad \text{and} \quad \lambda = \frac{dm}{dx}$$

$$\Rightarrow \quad dm = \lambda dx$$

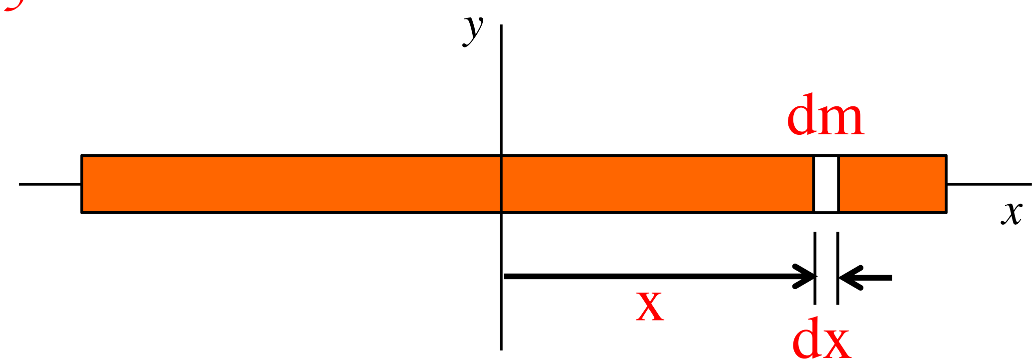
And with all that, we have:

$$\begin{aligned}
I_{end} &= \int x^2\, dm \\
&= \int_{x=0}^{L} x^2 (\lambda dx) \\
&= \lambda \int_{x=0}^{L} x^2\, dx \\
&= \left(\frac{M}{L}\right)\left(\frac{x^3}{3}\right)\Bigg|_{x=0}^{L} \\
&= \frac{1}{3} mL^2
\end{aligned}$$

Example 5: Derive an expression for the *moment of inertia* of a homogeneous rod of length L about its central bisecting axis.



To define dx and dm:



$$
\begin{aligned}
I_{end} &= \int x^2 \, dm \\
&= \int_{x=-L/2}^{L/2} x^2 (\lambda dx) \\
&= \lambda \int_{x=-L/2}^{L/2} x^2 \, dx \\
&= \left(\frac{M}{L}\right)\left(\frac{x^3}{3}\right)\Bigg|_{x=-L/2}^{L/2} = \left(\frac{M}{3L}\right)\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] \\
&= \frac{1}{12} mL^2
\end{aligned}
$$

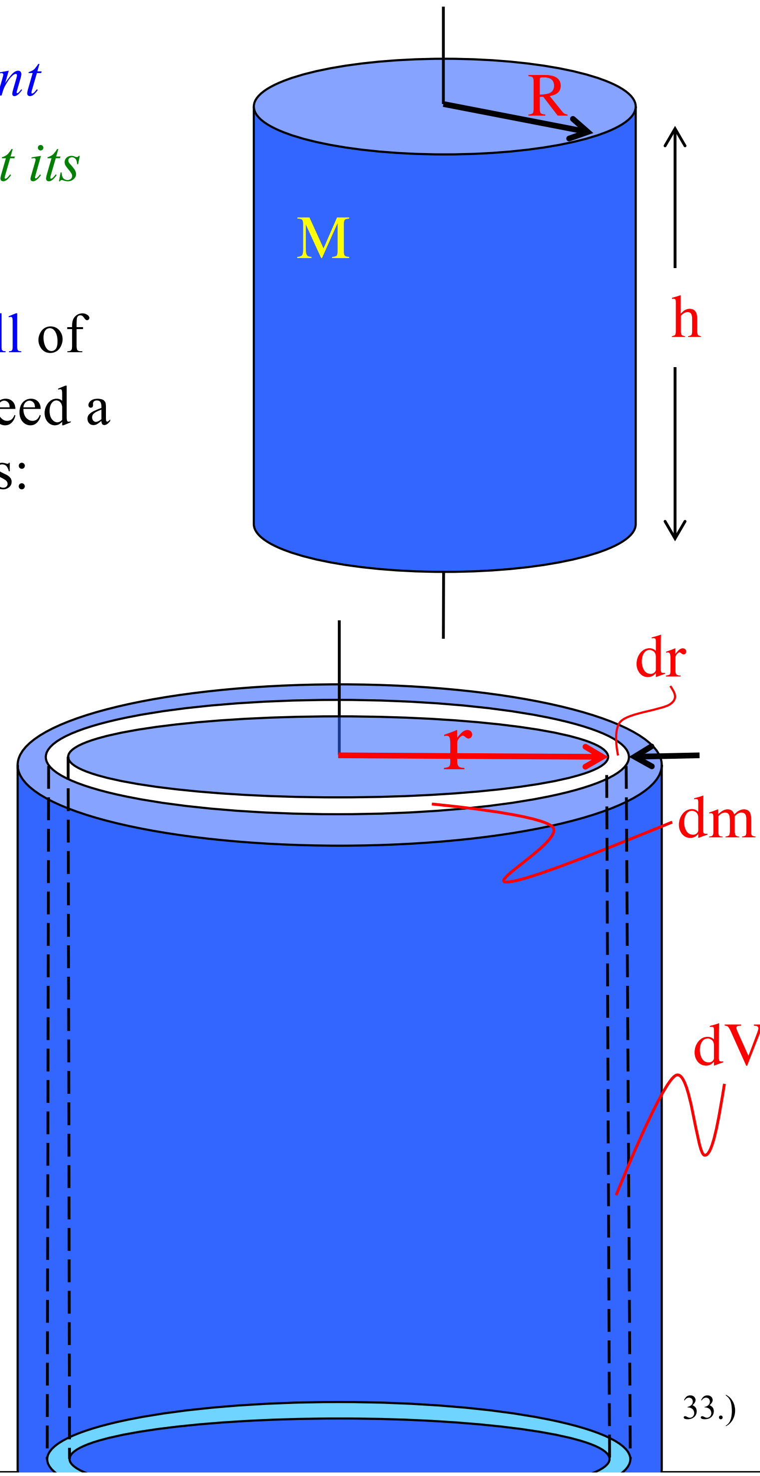
Example 6: Derive an expression for the *moment of inertia* of a cylinder of mass M and radius R *about its central axis*?

This requires a differentially thin cylindrical shell of radius *r* and thickness *dr*, which means we will need a volume density function. One way to write this is:

$$\rho = \frac{M}{Ah} = \frac{M}{\left(\pi R^2\right)h}$$

Noting that the *differential volume* of the differentially thin cylindrical shell will equal the shell's differential area (it's circum-ference times its thickness dr) times the cylinder's height, we can also write:

$$\rho = \frac{dm}{dV} \quad \Rightarrow \quad dm = \rho dV = \rho\left(dA\right)h$$
$$= \rho\left(2\pi r\ dr\right)h$$



With that information, we can do our integral.

$$I = \int r^2\, dm$$

$$= \int_{r=0}^{R} r^2 (\rho dV) = \int_{r=0}^{R} r^2 \left[\rho (2\pi r\ dr) h\right]$$

$$= 2\pi\rho h \int_{r=0}^{R} r^3\, dr$$

$$= 2\pi \left(\frac{M}{\pi R^2 h}\right) h \int_{r=0}^{R} r^3\, dr$$

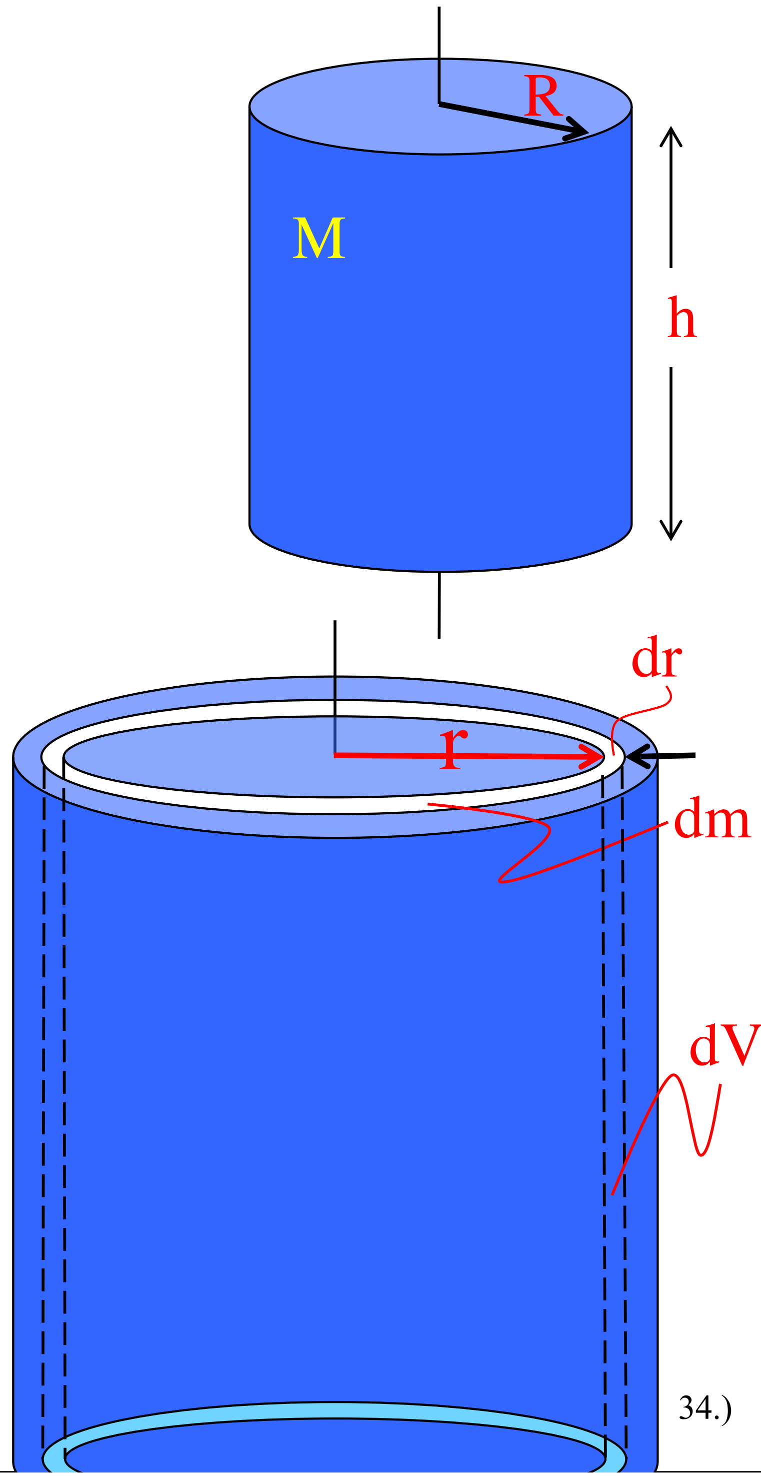
$$= 2\pi \left(\frac{M}{\pi R^2 h}\right) h \left(\frac{r^4}{4}\Big|_{r=0}^{R}\right)$$

$$= \cancel{2}\cancel{\pi} \left(\frac{M}{\cancel{\pi}\cancel{R^2}\cancel{h}}\right) \cancel{h} \left(\frac{R^{\cancel{4}\,2}}{\cancel{4}\,2}\right)$$

$$= \frac{1}{2} MR^2$$

Note: We could as easily have used a surface density function σ, with $\sigma = M/\pi R^2 = dm/dA$ and

$$I = \int_{r=0}^{R} r^2\, dm = \int_{r=0}^{R} r^2 (\sigma dA).$$

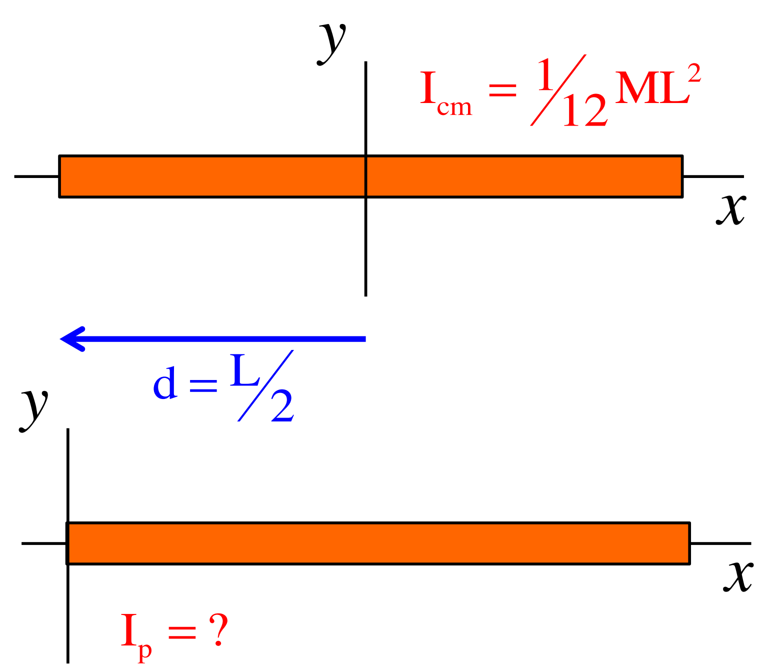


Parallel Axis Theorem

If you know the *moment of inertia* about an axis through the *center of mass* (I_{cm}), and want the *moment of inertia* about an axis parallel to that axis and a distance d units away (I_p), the *parallel axis theorem* will allow you to determine that value. That expression is:

$$I_p = I_{cm} + Md^2$$

Example 7: Given that the *moment of inertia* of a beam about a central axis is $\frac{1}{12}ML^2$, use the parallel axis theorem to determine the *moment of inertia* of the beam about one end.



$$\begin{aligned} I_p &= I_{cm} + Md^2 \\ &= \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 \\ &= \frac{1}{3}ML^2 \end{aligned}$$

. . . as calculated previously!

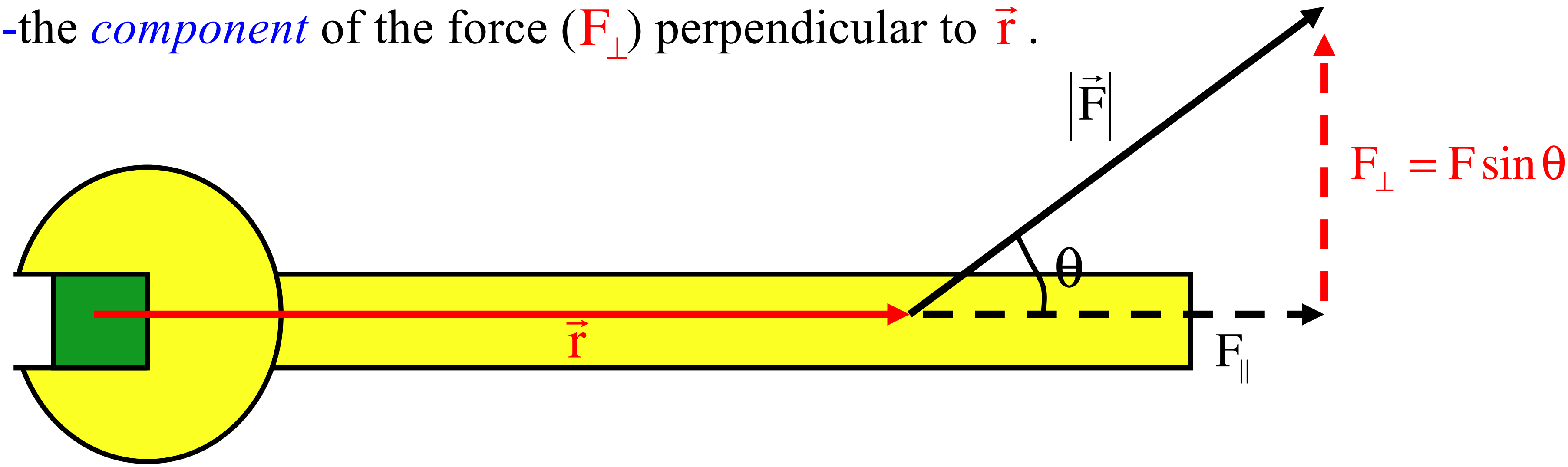
Torque

Consider the wrench shown below. Notice that the amount of rotational umph (this is a technical term) provided by the wrench on the bolt depends upon:

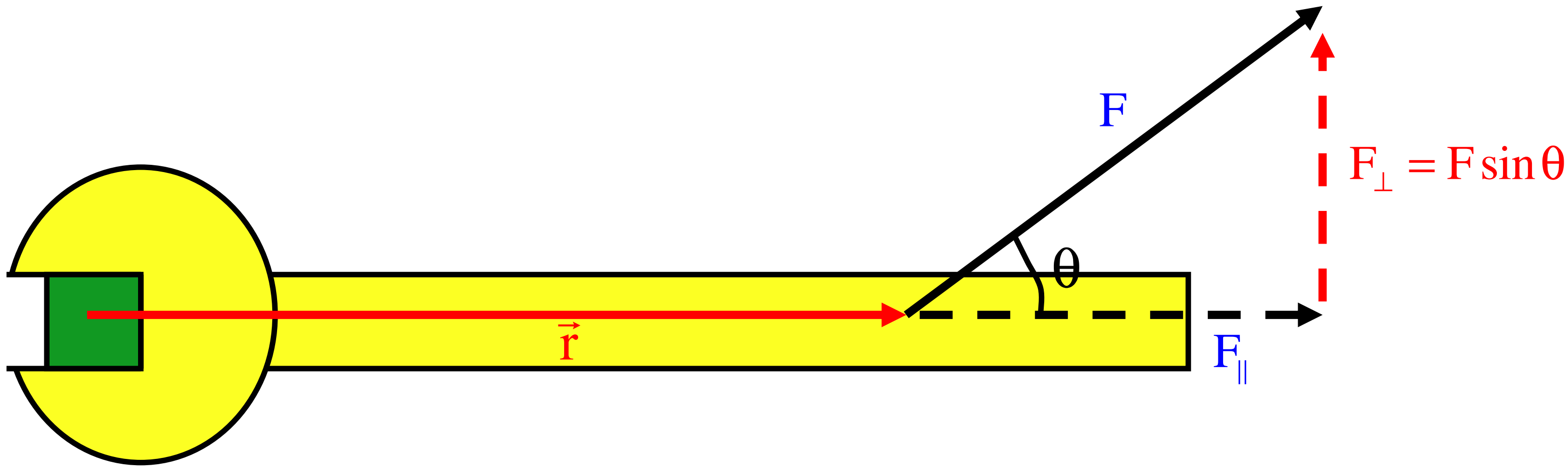
--how far out the force is applied ($\vec{r}$), and

--how big the force is ($|\vec{F}|$), and

--the *component* of the force ($F_{\perp}$) perpendicular to $\vec{r}$.



The product of $F_{\perp}$ and $|\vec{r}|$ generates the rotational counterpart to force, a vector called *torque*. When a net *torque* is applied to a stationary object that is free to rotate, the object will *angularly accelerate*.



So formally defined, the MAGNITUDE of the torque $|\vec{\tau}|$ generated by the force on the wrench is mathematically equal to: $|\vec{\tau}| = |\vec{r}|\ F_\perp$

As can be seen in the graphic, the perpendicular component of the force is equal to:

$$F_\perp = F\sin\theta$$

where θ is defined as the angle between *the line of the force* and *the line of the position vector* $\vec{r}$. (This definition is going to be important later.)

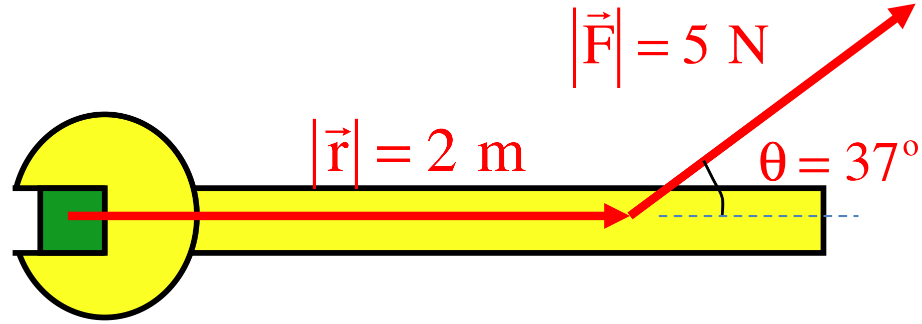
This means the torque can also be written as:

$$\begin{aligned} |\vec{\tau}| &= |\vec{r}|\ F_\perp \\ &= |\vec{r}|\,|\vec{F}|\sin\theta \end{aligned}$$

In fact, there are three ways to calculate a torque using polar information:

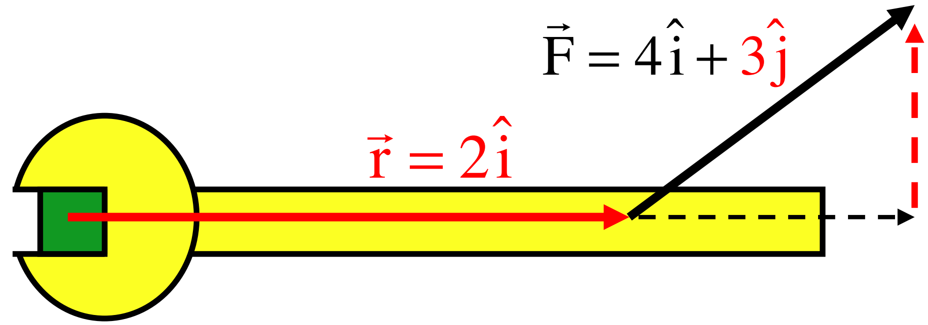
Definition approach:

$$\left|\vec{\tau}\right| = \left|\vec{r}\right|\left|\vec{F}\right|\sin\theta$$
$$= (2\text{ m})(5\text{ N})\sin 37^\circ$$
$$= 6\text{ N}\bullet\text{m}$$



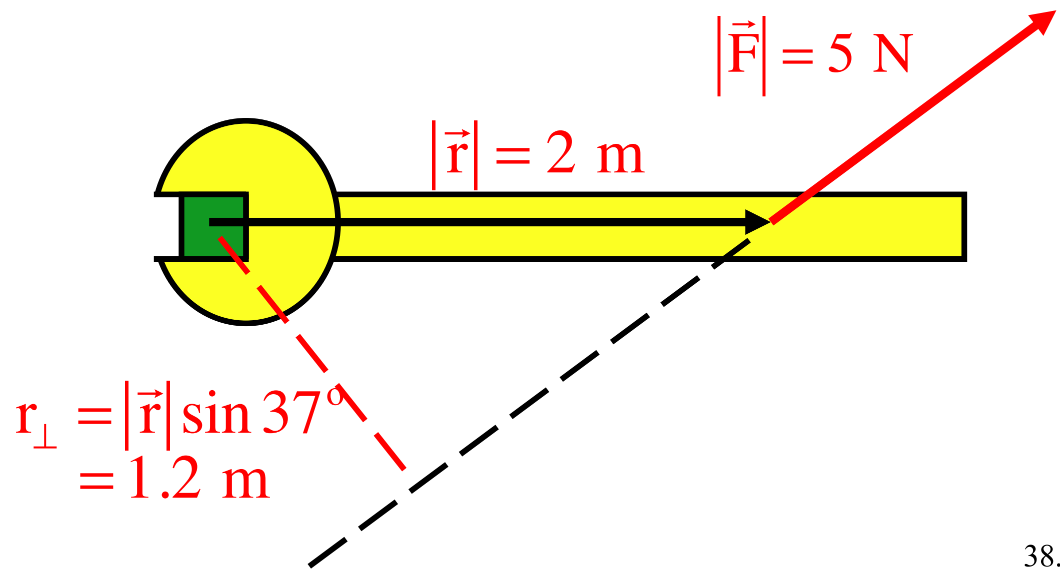
F-perpendicular approach:

$$\left|\vec{\tau}\right| = F_{\perp}\left|\vec{r}\right|$$
$$= (3\text{ N})(2\text{ m})$$
$$= 6\text{ N}\bullet\text{m}$$



r-perpendicular approach:

$$\left|\vec{\tau}\right| = r_{\perp}\left|\vec{F}\right|$$
$$= (1.2\text{ m})(5\text{ N})$$
$$= 6\text{ N}\bullet\text{m}$$



Why is the r-perpendicular approach so powerful (and mostly preferred)?

Can you find the *shortest distance between a point and a line*? If so, you can find $r_\perp$ for any situation. Called the moment arm, that is what $r_\perp$ is, the shortest distance between the *point about which you are taking the torque* and *the line of the force*. With it, $|\vec{\tau}| = r_\perp |\vec{F}|$.

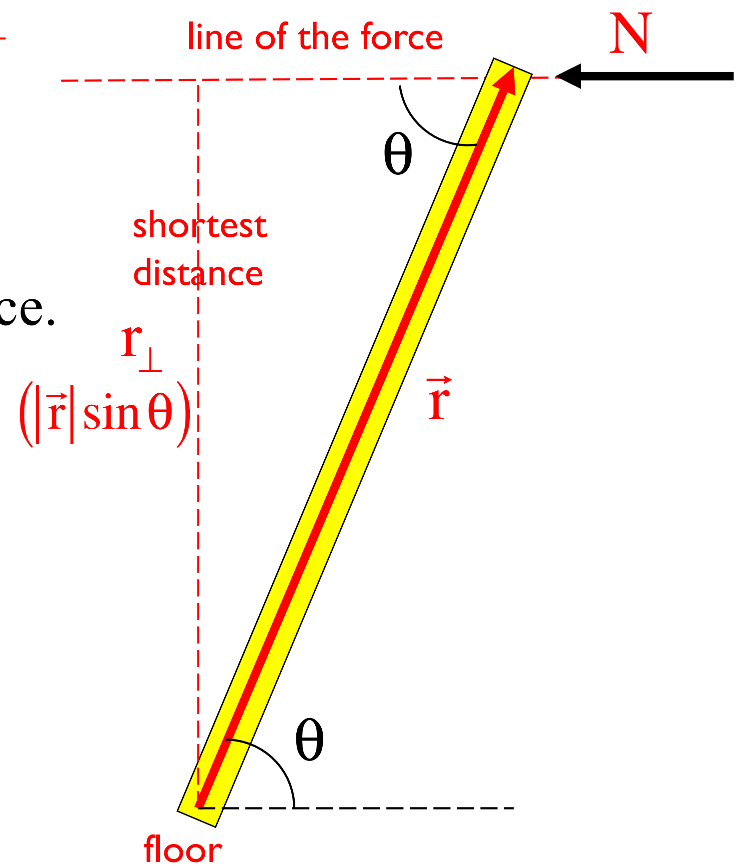
Example 6: A ladder sits against a wall. Using the $r_\perp$ approach, determine the torque generated by the wall's normal force *N* about the ladder's contact with the floor.

1. Identify the position vector $\vec{r}$ for the normal force.

2. Identify the *line of N*.

3. Identify shortest distance between *floor contact* and *line of N*, then express in terms of $|\vec{r}|$.

4. Determine the torque with $|\vec{\tau}| = r_\perp |\vec{F}|$.



Side point: *Although we started* by looking at a wrench, there are all sorts of instances in physics when we want to product of the magnitude of one vector and the perpendicular component of the second vector.

Because it pops up so often, this process is called *a cross product*. For two vectors $\vec{C}$ and $\vec{D}$, the magnitude of $\vec{D}x\vec{C}$ is defined such that:

$$\left|\vec{D}x\vec{C}\right| = \left|\vec{D}\right|\left|\vec{C}\right|\sin\phi,$$

where ϕ is the angle between *the line of* $\vec{C}$ and *the line of* $\vec{D}$.

The direction of a cross product will be *perpendicular to the plane* determined by $\vec{C}$ and $\vec{D}$, and can be determined using the *right hand rule*.

Note that if $\vec{r}$ and $\vec{F}$ are in the x-y plane, the direction of the *cross product* will be in the + or – $\hat{k}$-direction.

This is all fine and swell if you are dealing with polar notation, but what about vectors in *unit vector notation*? Specifically, if:

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

and

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

Then $\vec{B}x\vec{A} = \left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)x\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$

$$= \left[\left(B_x\hat{i}\right)x\left(A_x\hat{i}\right)\right] + \left[\left(B_y\hat{j}\right)x\left(A_x\hat{i}\right)\right] + \ldots$$

But $\left(B_x\hat{i}\right)x\left(A_x\hat{i}\right) = B_xA_x \sin 0^0 = 0$, so all the like-terms go to zero, and $\left(B_y\hat{j}\right)x\left(A_x\hat{i}\right) = B_yA_x \sin 90^0 = B_yA_x$ in the -k direction, so we end up with 6 non-zero parts.

What's interesting is that those six parts fall out with the evaluation of the matrix:

$$\vec{B}x\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix}$$

Question is, how do you evaluate a matrix like this?

The operation is fairly simple (something you do over and over again).

$$\vec{B} x \vec{A} = \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ B_x & B_y & B_z & B_x & B_y \\ A_x & A_y & A_z & A_x & A_y \end{array} \right.$$

$$= \hat{i}\left[(B_y)(A_z) - (B_z)(A_y) \right]$$

Blank out the column and row in which exists the unit vector $\hat{i}$.

Evaluate the two-by-two matrix that is to the immediate right, and multiply it by $\hat{i}$.

Adding in the $\hat{j}$ term looks like:

$$\vec{B} x \vec{A} = \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ B_x & B_y & B_z & B_x & B_y \\ A_x & A_y & A_z & A_x & A_y \end{array} \right.$$

$$= \hat{i}\left[(B_y)(A_z) - (B_z)(A_y)\right] + \hat{j}\left[(B_z)(A_x) - (B_x)(A_z)\right]$$

Finishing off with the $\hat{k}$ term gives us:

$$\vec{B} x \vec{A} = \left| \begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ B_x & B_y & B_z & B_x & B_y \\ A_x & A_y & A_z & A_x & A_y \end{array} \right.$$

$$= \hat{i}\left[(B_y)(A_z) - (B_z)(A_y)\right] + \hat{j}\left[(B_z)(A_x) - (B_x)(A_z)\right] + \hat{k}\left[(B_x)(A_y) - (B_y)(A_x)\right]$$

DON'T MEMORIZE the end result. *Know the approach*!

Example 8: Determine $\vec{D}x\vec{C}$ if:

$$\vec{C} = (-1)\hat{i} + (2)\hat{j}$$

$$\vec{D} = (4)\hat{i} + (5)\hat{j} + (6)\hat{k}$$

$$\vec{D}x\vec{C} = \left|\begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & 6 \\ -1 & 2 & 0 \end{array}\right| \begin{array}{cc} \hat{i} & \hat{j} \\ 4 & 5 \\ -1 & 2 \end{array}$$

Example 8: (con't) Determine $\vec{D}x\vec{C}$ if:

$$\vec{C} = (-1)\hat{i} + (2)\hat{j}$$

$$\vec{D} = (4)\hat{i} + (5)\hat{j} + (6)\hat{k}$$

Solution:

$$\vec{D}x\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & 6 \\ -1 & 2 & 0 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ 4 & 5 \\ -1 & 2 \end{matrix} + \vec{D}x\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & 6 \\ -1 & 2 & 0 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ 4 & 5 \\ -1 & 2 \end{matrix} + \vec{D}x\vec{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & 6 \\ -1 & 2 & 0 \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ 4 & 5 \\ -1 & 2 \end{matrix}$$

$$\vec{D}x\vec{C} = \hat{i}\left[(5)(0) - (6)(2)\right] + \hat{j}\left[(6)(-1) - (4)(0)\right] + \hat{k}\left[(4)(2) - (5)(-1)\right]$$

$$= \quad -12\hat{i} \quad -6\hat{j} \quad +13\hat{k}$$

Newton's Second Law Problems

Just as a *net force* acting on a mass is proportional to the acceleration of the body, with a proportionality constant being the mass of the body (i.e., a relative measure of the body's resistance to change its motion), the *net torque* a body experiences about a point will be proportional to the *angular* acceleration of the body about that point, with the proportionality constant being the *moment of inertia* about that point (i.e., a relative measure of the body's resistance to changing its *rotational* motion about that point). In other words, just as:

$$\vec{F}_{net} = m\vec{a}$$

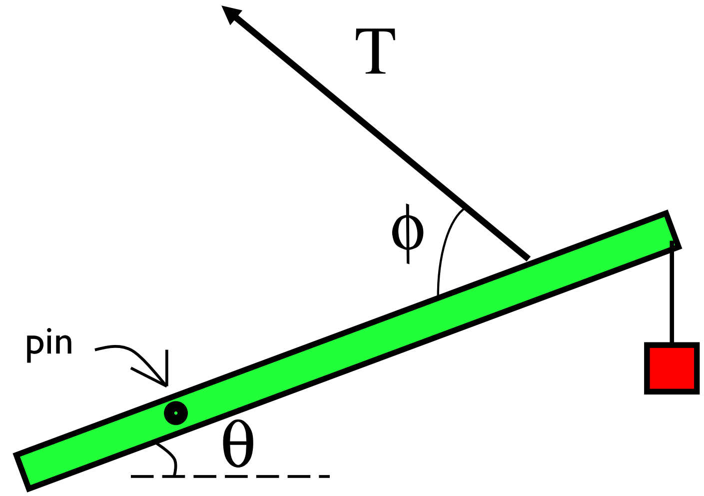
so also is:

$$\vec{\tau}_{net} = I\vec{\alpha}$$

Because all of our problems are going to have rotation in the plane of the page, they will all have rotational vector directions of $\pm\hat{k}$. We will need to keep track of signs, but we won't need the unit vector. As a consequence, the rotational version of N.S.L. can be written as:

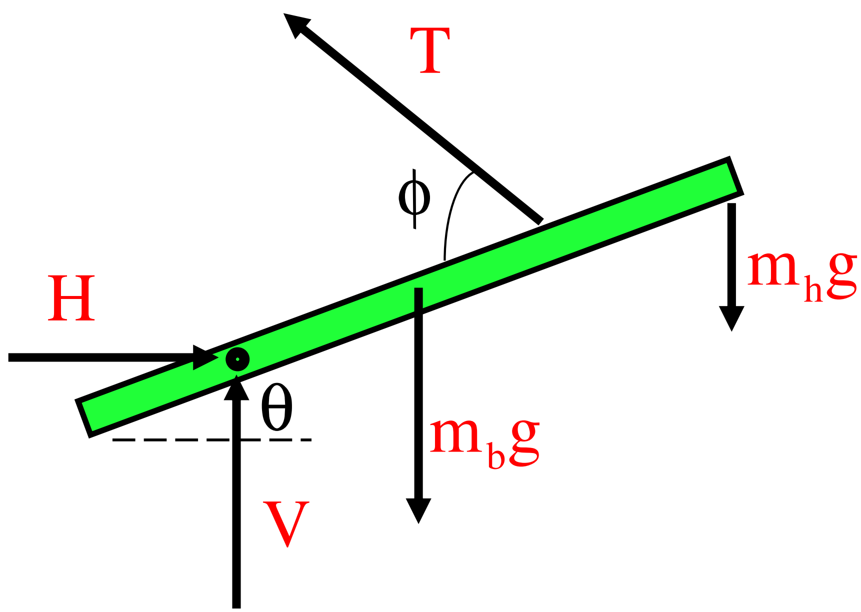
$$\tau_{net} = I\alpha$$

Example 9: A beam of mass m_b and length L is pinned at an angle θ a quarter of the way up the beam (i.e., at $L/4$). A hanging mass m_h is attached at the end. Tension in a rope three-quarters of the way from the end keeps it stationary. What is known is:



$$m_b, m_h, L, g, \theta, \phi \text{ and } I_{cm,beam} = \frac{1}{12} m_b L^2$$

a.) Draw a f.b.d. identifying all the forces acting on the beam.

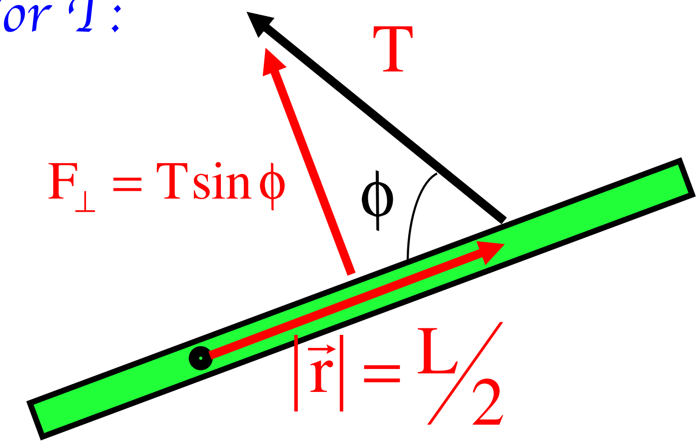


$$m_b, m_h, L, g, \theta, \phi \text{ and } I_{cm,beam} = \frac{1}{12} m_b L^2$$

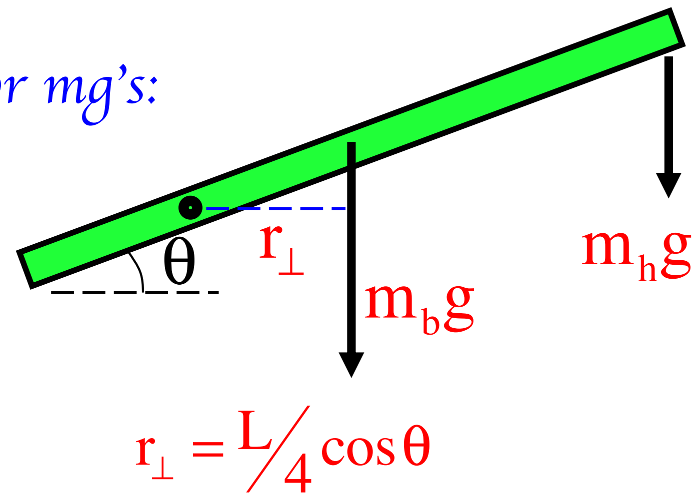
b.) Derive an expression for the tension in the line.

The clever thing to do here is to sum the torques about the pin. That will eliminate both *H* and *V* leaving you with only one unknown, *T*. What's even more clever is to use the *F-perpendicular approach* on *T* as that component is REALLY easy to determine (given ϕ).

For T:



For mg's:



$\sum \tau_{pin}$:

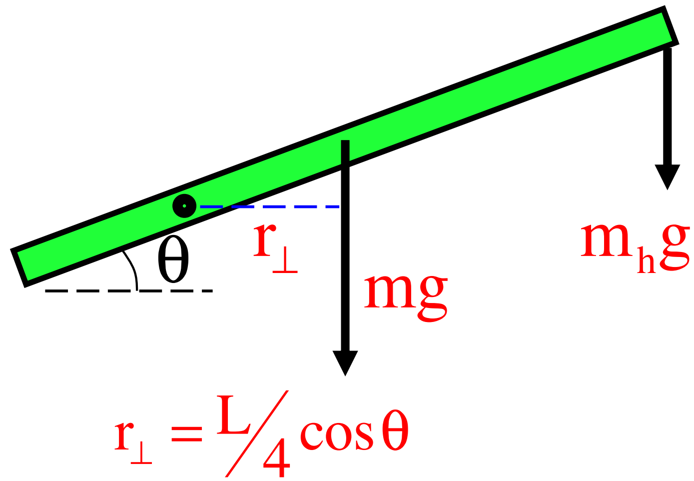
$$T\sin\phi\left(\frac{\cancel{L}}{2}\right) - m_b g\left(\frac{\cancel{L}}{4}\cos\theta\right) - m_h g\left(\frac{3\cancel{L}}{4}\cos\theta\right) = I_{pin}\cancelto{0}{\alpha}$$

$$\Rightarrow \quad T = \frac{(m_b + 3m_h)g}{2}\left(\frac{\cos\theta}{\sin\phi}\right)$$

$$m_b, m_h, L, g, \theta, \phi \text{ and } I_{cm,beam} = \frac{1}{12} m_b L^2$$

c.) Derive an expression for the angular acceleration of the beam *about the pin* just after the cable is cut.

This is the same problem you just did with one major difference. You don't have to do a torque calculation for the tension because T is no more, and the $I\alpha$ term (where I is the moment of inertia *about the pin*) is no longer zero. With the non-zero torques about the pin being gravity and the hanging mass, and we can write:



$$\sum \tau_{pin}:$$

$$-m_b g\left(\frac{L}{4}\cos\theta\right) - m_h g\left(\frac{3L}{4}\cos\theta\right) = -I_{pin}\alpha$$

Problem: We weren't given I_{pin}, we were given I_{cm}. Enter *the parallel axis theorem*!

$m_b, m_h, L, g, \theta, \phi$ and $I_{cm,beam} = \frac{1}{12} m_b L^2$

The parallel axis theorem states:

$$I_p = I_{cm} + Md^2$$

where *d* is the distance between the two parallel axes. In this case, that distance is *L/4*, so we can write:

$$\begin{aligned} I_p &= I_{cm} + Md^2 \\ &= \frac{1}{12} mL^2 + m\left(\frac{L}{4}\right)^2 \\ &= \frac{7}{48} mL^2 \end{aligned}$$

and

$$\sum \tau_{pin}:$$

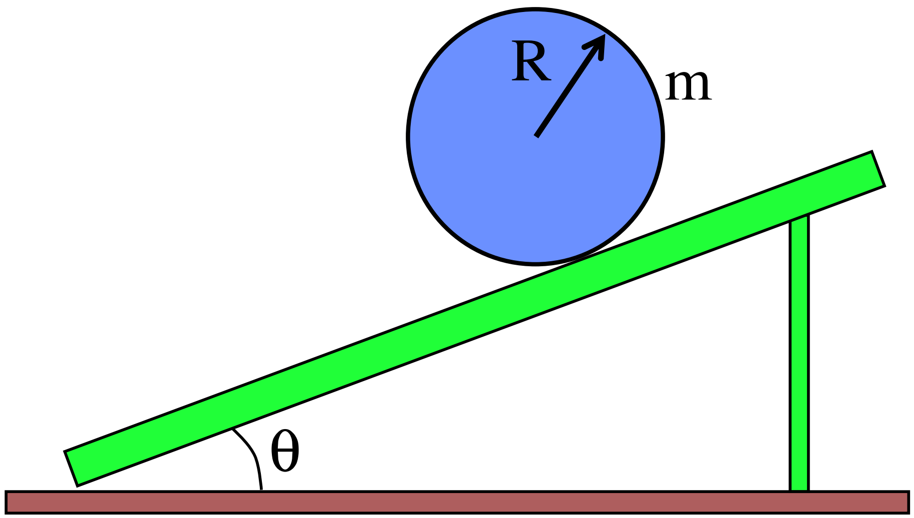
$$\begin{aligned} -m_b g\left(\frac{\cancel{L}}{\cancel{4}}\cos\theta\right) - m_h g\left(\frac{3\cancel{L}}{\cancel{4}}\cos\theta\right) &= -I_{pin}\alpha \\ &= -\left(\frac{7}{\underset{12}{\cancel{48}}} m_b L^{\cancel{2}}\right)\alpha \end{aligned}$$

$$\Rightarrow \quad \alpha = \frac{12\left[m_b + 3m_h\right] g\cos\theta}{7 m_b L}$$

Example 10: A ball of radius R rolls without slipping down an incline of angle θ. a.) What is the acceleration of the *center of mass*, and b.) what is its speed after it has dropped a distance *h*?

Assume you know:

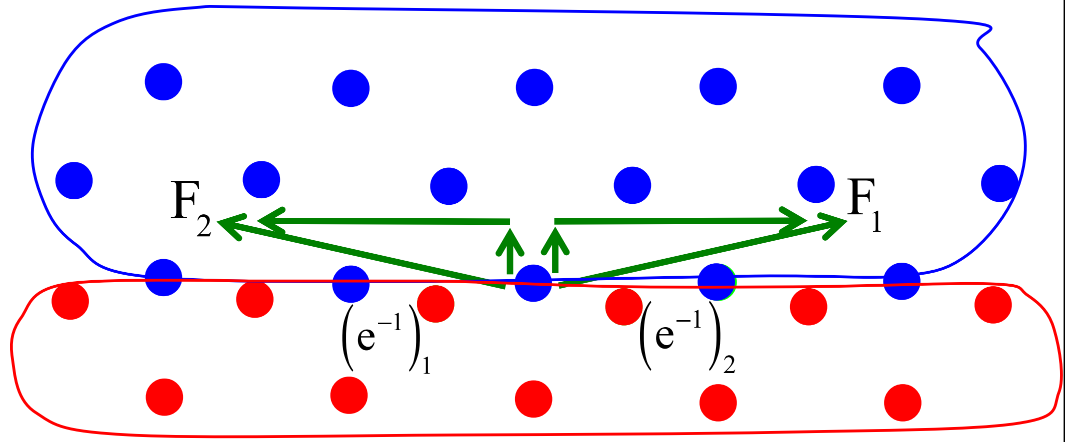
$$m, R, g, \theta, \text{ and } I_{cm,ball} = \frac{2}{5} m R^2$$



Before we can do this problem, there is something that needs to be noted about objects rolling up or down an incline. We need to talk about rolling friction and, more specifically, the direction of rolling friction.

Thinking back to what we said about static friction and its cause. The molecular structure of the bodies in contact meld into one another, and the electron-repulsion of their elements create a kind of bonding that needs to be released before motion can occur (the slide I used to animate this idea is shown on the next page). With sliding, that release takes the form of shearing and translates into what we call kinetic friction. With rolling, that release takes the form of pealing and is called *rolling friction*. What is important to note is that both have the underlying melding mechanism associated with static friction.

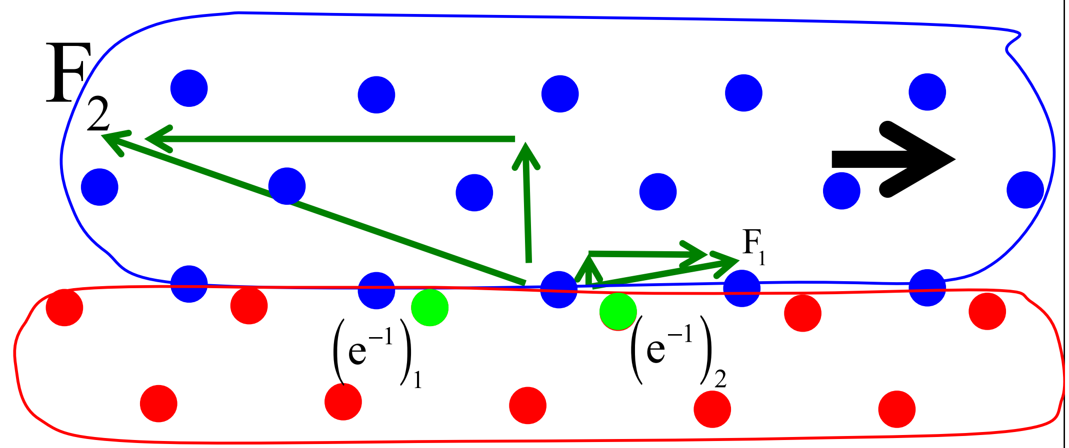
A little closer look is instructive.



As the electrons of the upper object (in blue) nestle into the electron configuration of the lower object (in red), they apply a repulsive force to one another (see sketch).

The horizontal components add to zero. The vertical components producing the normal force that supports the upper object.

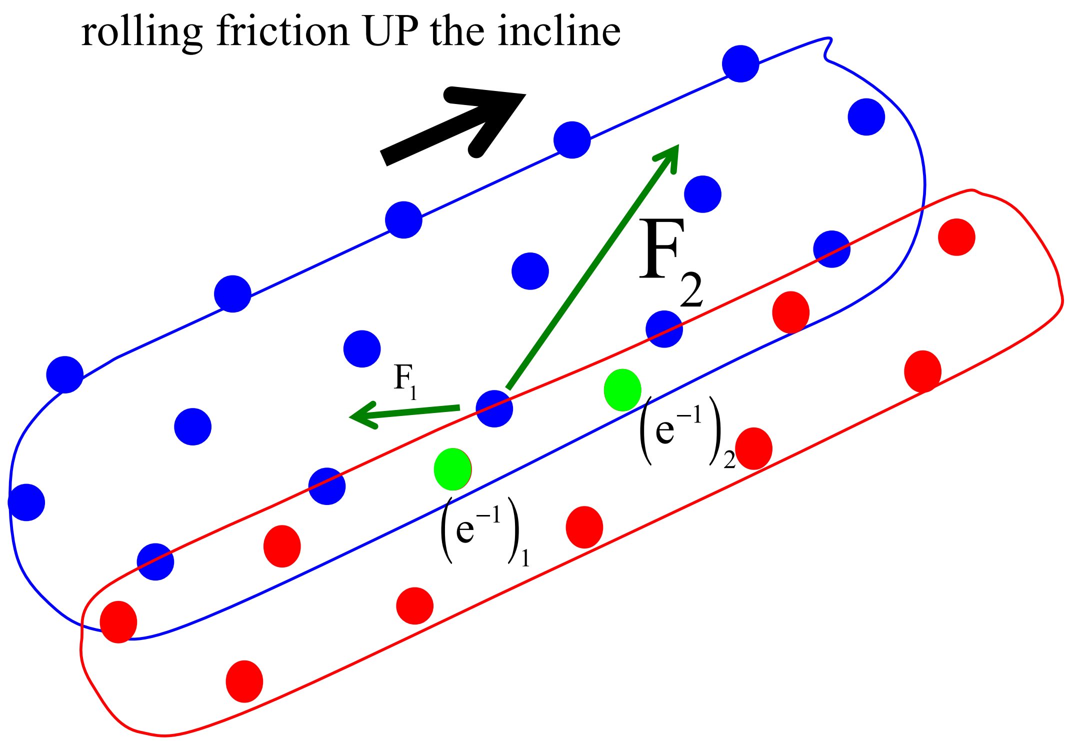
But try to move the upper body to the right and the horizontal components will no longer cancel.



This net horizontal force is known as the *static frictional force* between the two bodies. It is the force that has to be overcome before the upper body can actually accelerate to the right. Put a little differently, for the top body to accelerate, an external force to the right that is large enough to effectively shearing the repulsive bonds that exist between electrons has to be applied.

What changes if the one of the bodies is round and the second object, the underneath object, is an incline. The round body's weight will motivate it to slough down against the repulsing (green) electrons of the incline (see sketch). That tendency to ease down the incline is what produces the static frictional force UP the incline (again, notice offset of electrons in sketch).

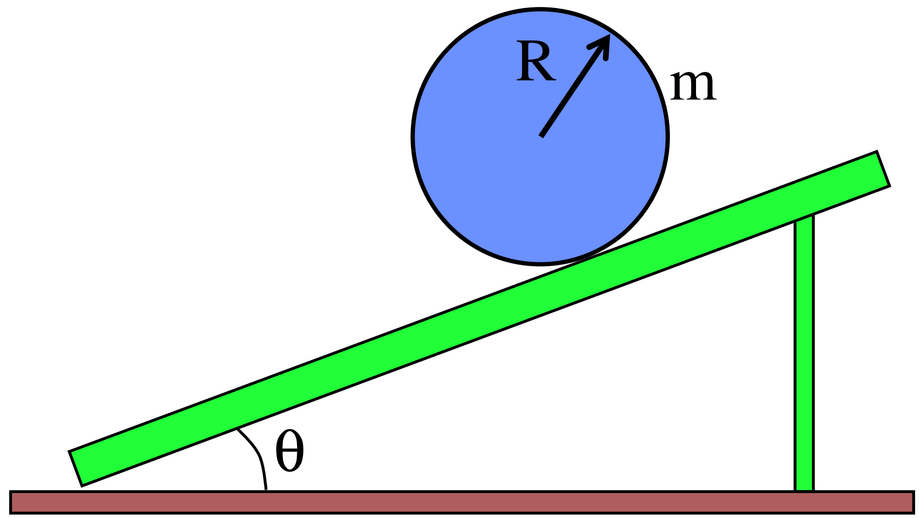
In fact, it doesn't matter whether the round body is rolling up the incline or down it, that electron interaction will always produce a static friction force that is UP the incline.



Example 10: A ball of radius R rolls without slipping down an incline of angle θ. a.) What is the acceleration of the *center of mass*, and b.) what is its speed after it has dropped a distance *h*?

Assume you know:

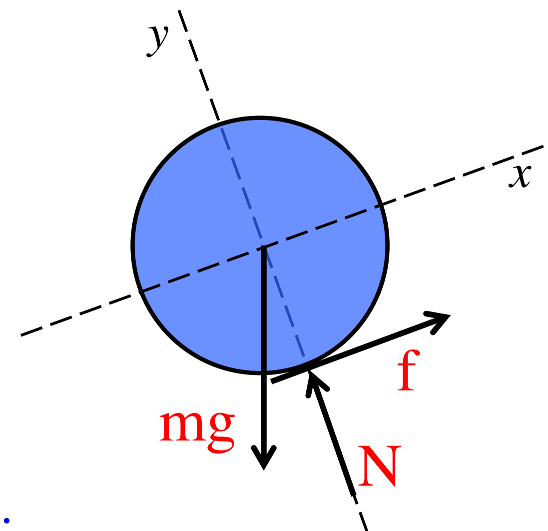
$$m, R, g, \theta, \text{ and } I_{cm,ball} = \frac{2}{5}mR^2$$



a.) What is the acceleration of the center of mass of the ball?

Because we are going to be taking torques, we need to place the forces on our f.b.d. *where they actually act* on the body.

There are two ways to do a problem like this. We'll execute both.



The first approach looks at the motion from the perspective of the *center of mass*.

$$m, R, g, \theta, \text{ and } I_{cm,ball} = \frac{2}{5} mR^2$$

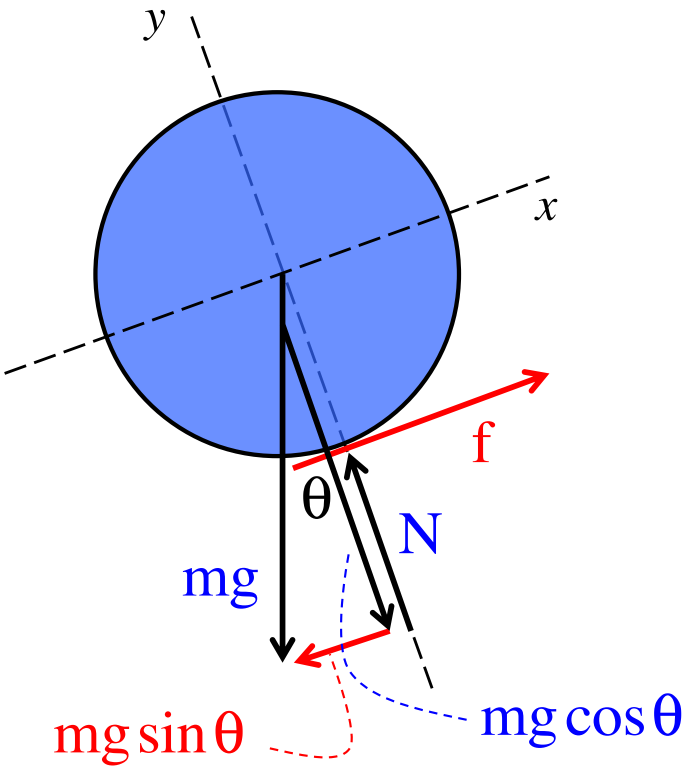
a.) con't.

The center of mass both accelerates and has mass *angularly accelerate* about itself.

Dealing with the translational motion first:

According to Newton's Second, summing the forces along the *line of the acceleration* will equal the mass times the acceleration *of the center of mass* along that line.

Our f.b.d. shows the forces and force components along the line (and perpendicular to the line) of acceleration. Doing the summation yields:



$$\sum F_x :$$

$$f - mg\sin\theta = -ma_{cm} \qquad \text{(Equ. A)}$$

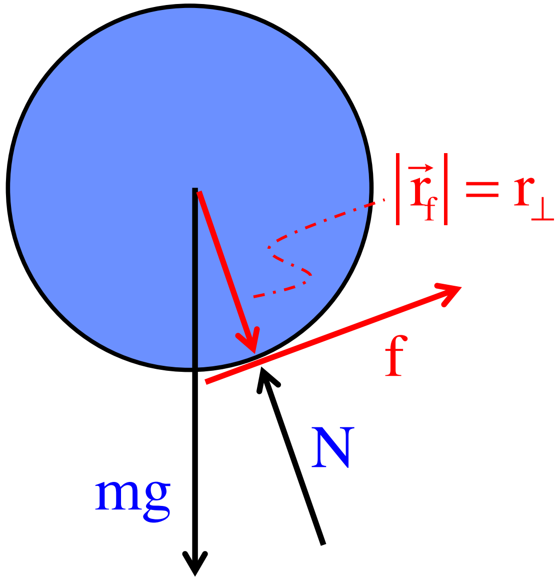
$$m, R, g, \theta, \text{ and } I_{cm,ball} = \frac{2}{5}mR^2$$

a.) con't.

Dealing with the rotational motion:

According to the *rotational version* of Newton's Second, the sum of the torques about the *center of mass* will equal the *moment of inertia* about the center of mass times the angular acceleration of the body about the center of mass.

A dressed down f.b.d. showing the forces without axes or components allowing us to identify any $\vec{r}$ vectors and $r_\perp$ quantities. With those, the torque summation yields:



$$\sum \tau_{cm}:$$

$$f\cancel{R} = I_{cm}\alpha$$

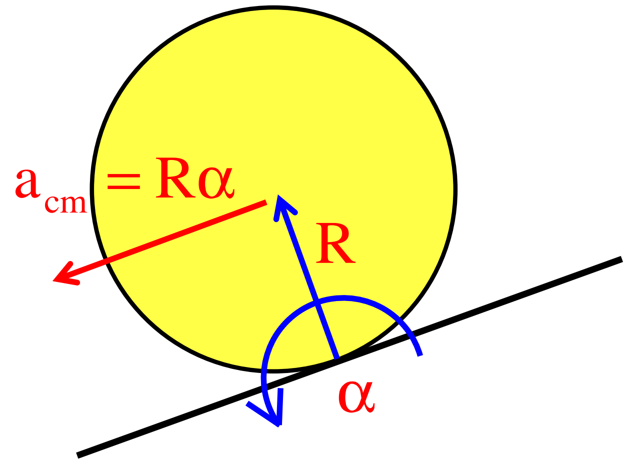
$$= \left(\tfrac{2}{5}mR^{\cancel{2}}\right)\alpha$$

$$\Rightarrow \quad f = \left(\tfrac{2}{5}mR\right)\alpha \quad \text{(Equ. B)}$$

$$m,\ R,\ g,\ \theta,\ \text{and}\ I_{cm,ball} = \frac{2}{5} mR^2$$

a.) con't.

Remembering that $a_{cm} = R\alpha$ for a rolling object, *Equ. B* becomes:



$$f = \left(2/5\, mR\right)\alpha$$

$$\Rightarrow \quad f = \left(2/5\, m\cancel{R}\right)\left(\frac{a_{cm}}{\cancel{R}}\right)$$

$$\Rightarrow \quad f = 2/5\, m a_{cm}$$

From *Equ. A,* then:

$$f - mg\sin\theta = -ma_{cm}$$

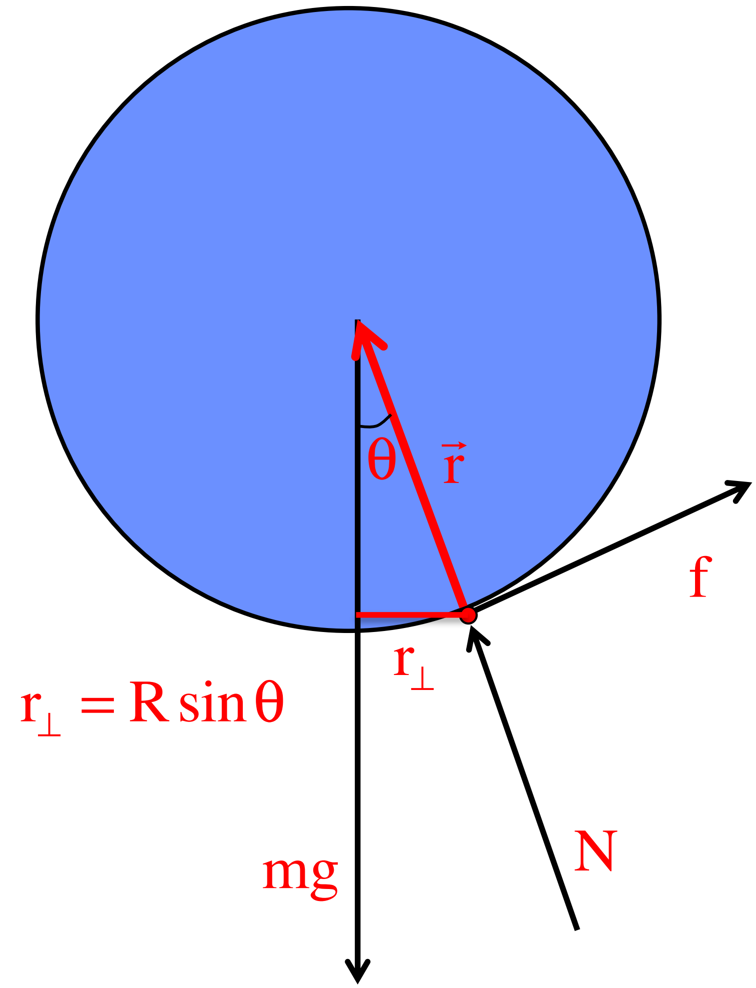
$$\Rightarrow \quad \left(2/5\, \cancel{m} a_{cm}\right) - \cancel{m} g\sin\theta = -\cancel{m} a_{cm}$$

$$\Rightarrow \quad a_{cm} = \frac{5}{7} g\sin\theta$$

$$m, R, g, \theta, \text{ and } I_{cm,ball} = \frac{2}{5} mR^2$$

a.) con't.

The second approach looks at the motion from an *instantaneous fixed point at the point of contact* perspective (remember, you can't tell the difference between the motion of the two situations, instantaneously). From that perspective, summing the torques about the contact point eliminates the *normal* and *friction* leaving only a torque due to *gravity*, and we can write:



$$\sum \tau_p :$$

$$r_\perp \left|\vec{F}\right| = I_p \alpha$$

$$(R\sin\theta) mg = I_p \alpha$$

The torque is about the axis at P, so the *moment of inertia* must be about P. Enter (again) the *parallel axis theorem*.

The parallel axis theorem maintains:

$$I_p = I_{cm} + Md^2$$
$$= \frac{2}{5}mR^2 + mR^2$$
$$= \frac{7}{5}mR^2$$

That means:

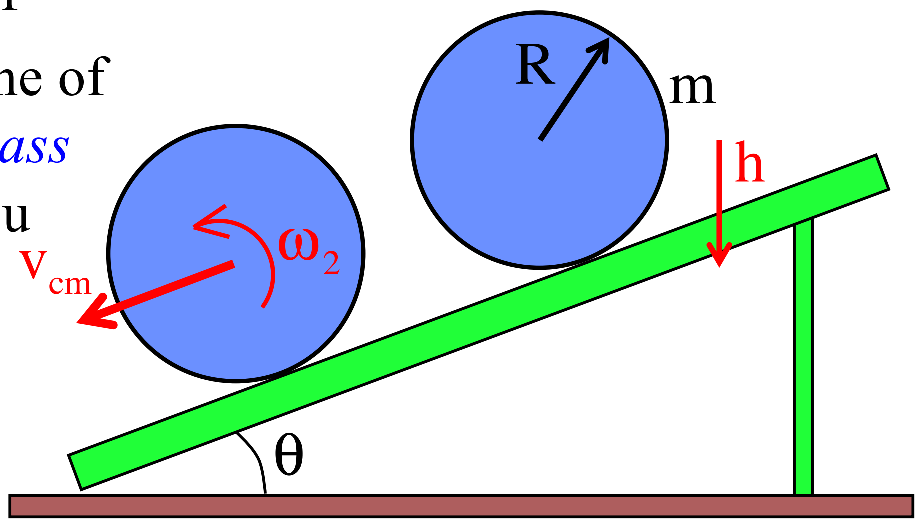
$$(R\sin\theta)mg = I_p\alpha$$
$$\Rightarrow \quad \cancel{m}g\cancel{R}\sin\theta = \left(\frac{7}{5}\cancel{m}\cancel{R}^{\cancel{2}}\right)\left(\frac{a_{cm}}{\cancel{R}}\right)$$
$$\Rightarrow \quad a_{cm} = \frac{5}{7}g\sin\theta$$

This is the same solution we got using the previous approach.

So which approach should you use in a given problem? Use the one that most naturally reflects the situation. In this case, the body's *center of mass* was both accelerating and had mass angularly accelerating around it, so the c.of m. approach was the most reasonable. In the case of a pinned beam, the mass is angularly accelerating around a fixed pin, so the fixed point approach is best. Use the approach that best reflects what's actually happening in the system.

Example 10 continued: A ball of radius R rolls without slipping down an incline of angle θ. What is the speed of its *center of mass* after it has dropped a distance *h*? Assume you know:

$$m, R, g, \theta, \text{ and } I_{cm,ball} = \frac{2}{5}mR^2$$

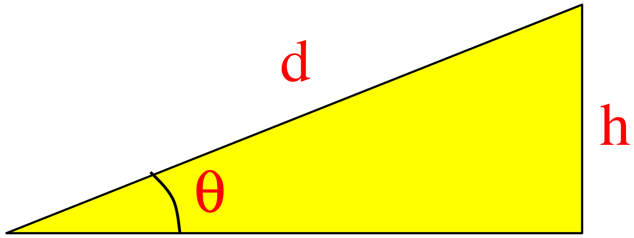


b.) What is the *velocity* of the *center of mass* after the ball drops a distance *h*?

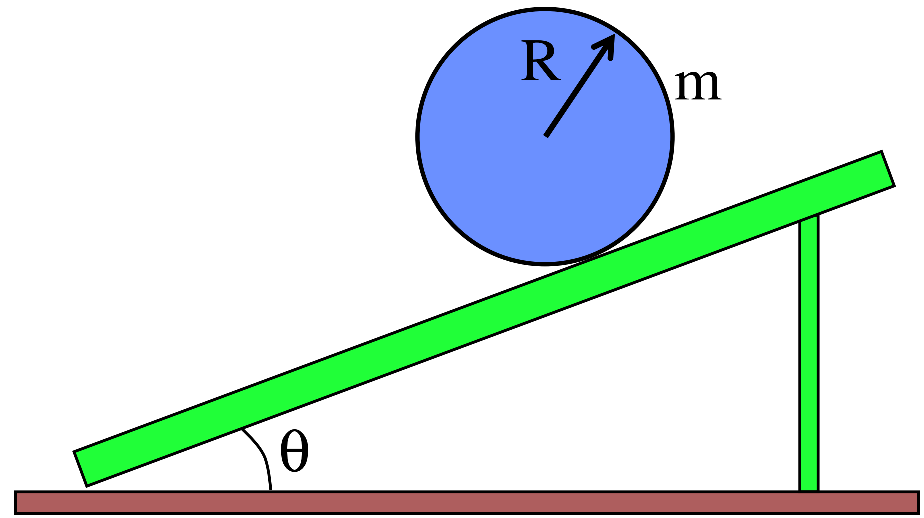
There are two ways to do *this* problem, one using *kinematics* and one using *energy considerations*. We'll do the kinematics approach, first.

We've already determined the *acceleration* of the *center of mass* down the incline to be $a_{cm} = \frac{5}{7} g\sin\theta$. We know the *initial velocity* is ZERO, and we know the body travels *d*, or:

$$\sin\theta = \frac{h}{d} \quad \Rightarrow \quad d = \frac{h}{\sin\theta}$$



$$m, R, g, \theta, \text{ and } I_{cm,ball} = \frac{2}{5} mR^2$$



With that information, we can write:

$$(v_2)^2 = (v_1)^2 + 2ad$$

$$\Rightarrow \quad v_2 = \sqrt{(\cancelto{0}{v_1})^2 + 2ad}$$

$$= \sqrt{2\left(\frac{5}{7} g \sin\theta\right)\left(\frac{h}{\sin\theta}\right)}$$

$$= \sqrt{\frac{10}{7} gh}$$

This is all fine and swell if you already know the ball's *acceleration* (or *angular acceleration*), but if you don't, doing the Newton's Second Law evaluation just so you can use kinematics is kinda dumb . . . especially when you have an approach that is MADE to deal with velocities . . . *conservation of energy*!!!!!

Energy Considerations

When we derived the general expression for a rotating body's moment of inertia, we introduced the idea of *rotational kinetic energy.* That expression was:

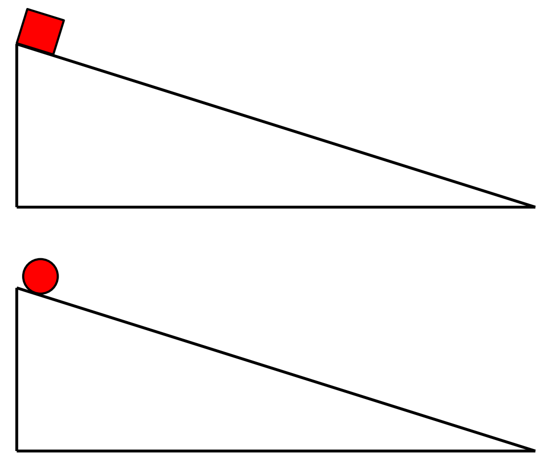
$$KE_{rot} = \frac{1}{2}I\omega^2$$

How might this play into a *conservation of energy* problem with *rotational motion*? In essence, we do NOT define *potential energy functions* for work being done by rotational agents, so the only terms that are generally affected in the *conservation of energy* relationship are the *kinetic energy* terms and, maybe, the *extraneous work* quantities.

But first . . .

A Preliminary Observation

Let's say you have a block sliding down a frictionless ramp, and a ball rolling down an identical frictional ramp (same angle). If both are released from rest at the same time and same height, which object will get to the bottom of the incline first, and which will have the greatest speed?



The answer: The block will get to the bottom first and be moving fastest, though possibly not for the reason you might think.

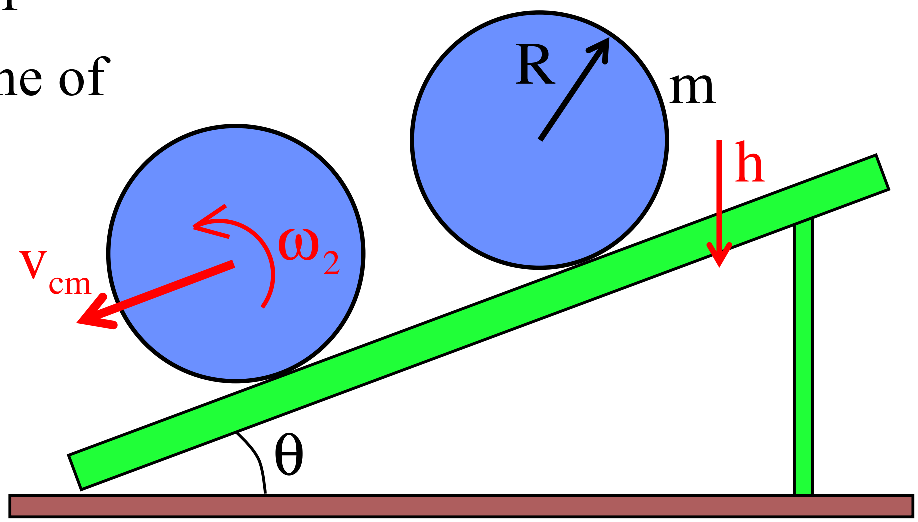
Specifically, both objects start with the same amount of potential energy, so it might be a temptation to assume that friction does extraneous work on the ball and, hence, it's velocity at the bottom will be less than that of the block moving over the frictionless surface. That isn't the key, and you need to understand why.

Think back to our discussion of friction. Objects in contact with one another nestles into one another at the molecular structure. What stops the incursion is repulsion between the electrons of the two structures. Try to move one of the objects and that electron repulsion becomes imbalanced, producing a retarding force that on the macroscopic level is called *static friction*. To move one of the objects, you have to exert enough force to overcome this bonding-like situation. Doing so produces motion. During relative motion, the melding continues to exist, just not as much. The need to continually overcome the electron repulsion set up by this lesser melding is what we associate with *kinetic friction*. When a body ROLLS over a surface, there is no shearing. Instead, the surfaces peal back from one another. This removes VERY LITTLE energy from the system.

Bottom line: It isn't friction that makes the ball move slower. It's the fact that in the case of the block, all of the gravitational potential energy goes into changing the body's translational kinetic energy. In the case of the ball, where not only the center of mass is accelerating but also there is angular acceleration *around* the center of mass, some of that initial gravitational potential energy goes into changing the body's translational KE (and, hence, it's translational velocity) but some also has to go into changing the body's *rotational KE*. Having less energy for translational KE, the ball moves slower than the block at the bottom of the ramp.

Example 10 continued: A ball of radius R rolls without slipping down an incline of angle θ. Assume you know:

$$m, R, g, \theta, \text{ and } I_{cm,ball} = \frac{2}{5}mR^2$$



b.) What is the *velocity* of the *center of mass* after the ball drops a distance *h*.

Starting from scratch, energy consideration is the way to go here. Remembering that the ball started from rest, we can write:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

$$0 + mgh + 0 = \left[\frac{1}{2}m(v_{cm})^2 + \frac{1}{2}I_{cm}\omega^2\right] + 0$$

$$\Rightarrow \quad \cancel{m}gh = \frac{1}{2}\cancel{m}(v_{cm})^2 + \frac{1}{2}\left(\frac{2}{5}\cancel{m}\cancel{R^2}\right)\left(\frac{v_{cm}}{\cancel{R}}\right)^2$$

$$\Rightarrow \quad v_{cm} = \sqrt{\frac{10}{7}gh}$$

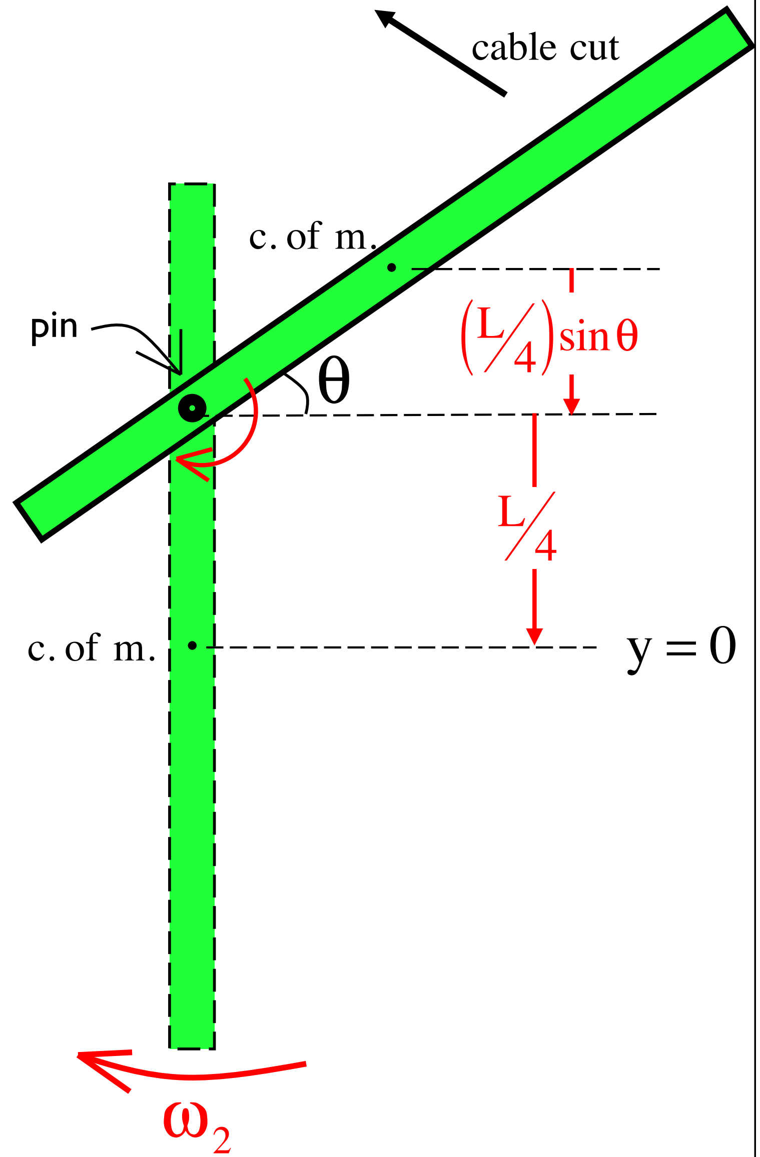
Example 11: So let's go back to the swinging beam of length L pinned at an angle θ a quarter of the way up the beam (i.e., at L/4). The cable is cut and the beam swings down. What is the *velocity* of it's *center of mass* as it passes through its lowest point. We know:

$$m, L, g, \theta, \phi \text{ and } I_{cm,beam} = \frac{1}{12} mL^2$$

and we calculated earlier the *moment of inertia* about the pin as:

$$I_{pin} = \frac{7}{48} m_{beam} L^2$$

In this case, the object is rotating about the pin, so it makes sense to evaluate its motion *relative to the fixed axis at the pin*. Tracking the *center of mass drop* for *potential energy positions* (see sketch), we can write:



$$m, L, g, \theta, \phi \text{ and } I_{cm,beam} = \frac{1}{12}mL^2,\ I_{pin} = \frac{7}{48}mL^2$$

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

$$0 + mg\left[\frac{L}{4} + \left(\frac{L}{4}\right)\sin\theta\right] + 0 = \frac{1}{2}I_{pin}(\omega_2)^2 + 0$$

$$\Rightarrow \quad mg\left[\frac{L}{4} + \left(\frac{L}{4}\right)\sin\theta\right] = \frac{1}{2}\left(\frac{7}{48}mL^2\right)\left(\frac{v_{cm}}{L/4}\right)^2$$

$$\Rightarrow \quad v_{cm} = \sqrt{\frac{3}{14}g\left[L + L\sin\theta\right]}$$

We could have looked at this from the perspective of the *center of mass*. That would look like:

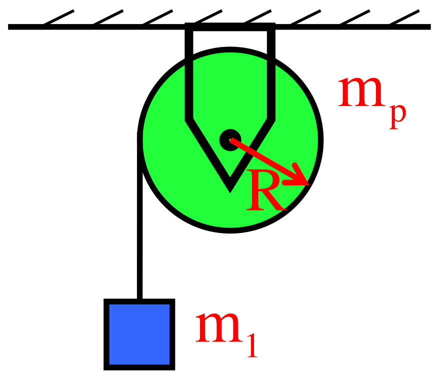
$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

$$0 + mg\left[\frac{L}{4} + \left(\frac{L}{4}\right)\sin\theta\right] + 0 = \left[\frac{1}{2}m(v_{cm})^2 + \frac{1}{2}I_{cm}(\omega_2)^2\right] + 0$$

Try the math. It will yields the same result.

Rotation, Translation and Pulleys

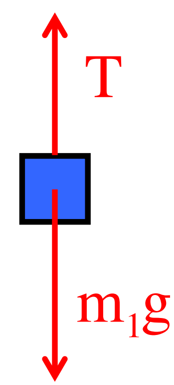
You have seen a problem in which a body has both *translated* and *rotated* (a ball rolling down an incline), but we haven't yet dealt with pulleys and situations with multiple masses. With that in mind, consider:



Starting simple—Example 11: Consider a hung pulley with a rope around it and a mass attached to its end. If the pulley is massive, and WITHOUT using *kinematics*:

a.) Derive an expression for the system's *acceleration*.

Starting with the translational version of N.S.L. on the hanging mass:



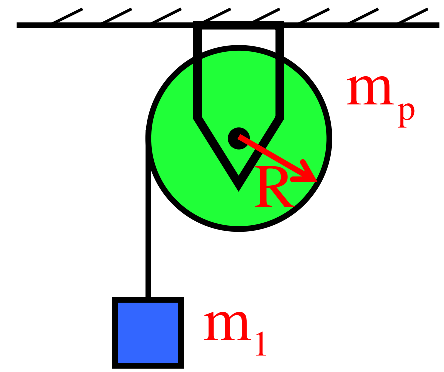
$$\sum F_y :$$

$$T - m_1 g = -m_1 a$$

Two unknowns (a and T)—require another equation.

a.) con't.

Assuming the pulley is a disk, so it's *moment of inertia* can be approximated at:



$$I = \frac{1}{2} m_p R^2$$

we can write:

$$\sum \tau_{pully} :$$

$$TR = I\alpha$$

$$\Rightarrow \quad T\cancel{R} = \left(\frac{1}{2} m_p \cancel{R^2}\right)\left(\frac{a}{\cancel{R}}\right)$$

$$\Rightarrow \quad T = \frac{1}{2} m_p a$$

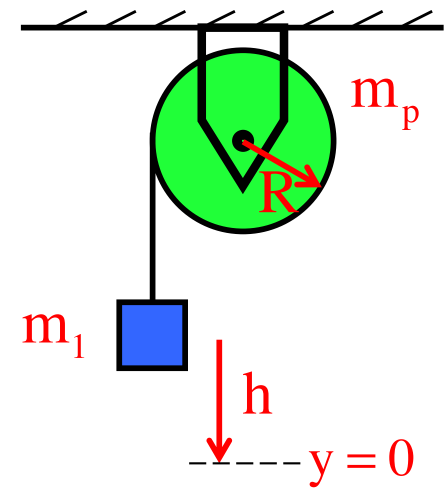
Combining our two relationships yields:

$$T - m_1 g = -m_1 a$$

$$\Rightarrow \quad \left(\frac{1}{2} m_p a\right) - m_1 g = -m_1 a$$

$$\Rightarrow \quad a = \frac{m_1}{m_1 + \frac{1}{2} m_P} g$$

b.) How fast is the hanging mass moving after it has fallen a distance *h*?



Without using kinematics, this is a *conservation of energy* problem. Remembering that the disk will have *rotational kinetic energy*, we can write:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

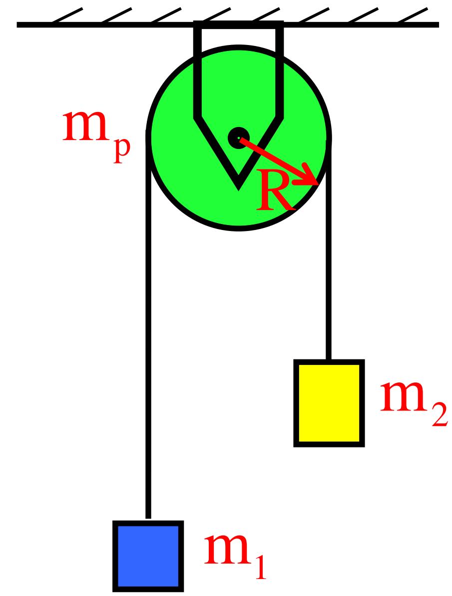
$$0 \quad + \ m_1gh \ + \quad 0 \quad = \left[\frac{1}{2}m_1\left(v_{cm}\right)^2 + \frac{1}{2}I_{pully}\omega^2\right] + \quad 0$$

$$\Rightarrow \quad m_1gh = \frac{1}{2}m_1\left(v_{cm}\right)^2 + \frac{1}{2}\left(\frac{1}{2}m_pR^2\right)\left(\frac{v}{R}\right)^2$$

$$\Rightarrow \quad v_{cm} = \sqrt{\frac{2m_1gh}{m_1 + {}^1\!/_2\, m_p}}$$

Interesting note: The calculated work *tension* did on the hanging mass as it fell was negative; the calculated work *tension* did on the pulley as it rotated was positive; and the two are equal summing to zero. That *internal force* did no net work on the system.

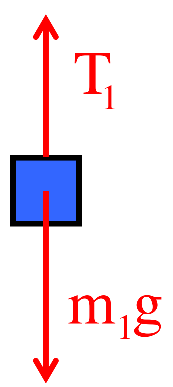
Example 12: something a little more complex: An Atwood Machine with a massive pulley of radius *R* (see sketch). Without using kinematics:



a.) Derive an expression for the system's acceleration.

Note that we've done this problem before! So what's different? The *tension in the lines* can no longer be equal. How so? If they were, the torques provided by the two tensions would be equal and opposite, so the net torque would be ZERO, hence no *angular acceleration*. But there *is* angular acceleration, so the tensions must be different. So . . .

f.b.d. on m_1

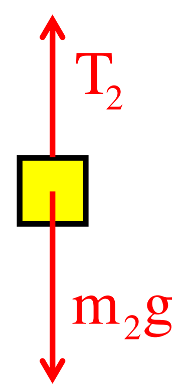


$$\sum F_y :$$

$$T_1 - m_1 g = m_1 a$$

$$\Rightarrow \quad T_1 = m_1 g + m_1 a$$

f.b.d. on m_2



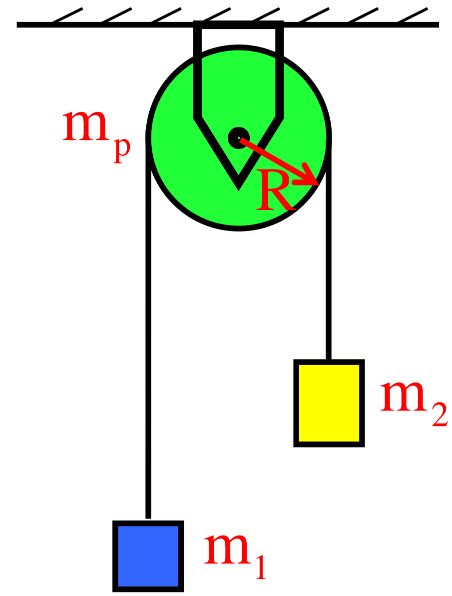
$$\sum F_y :$$

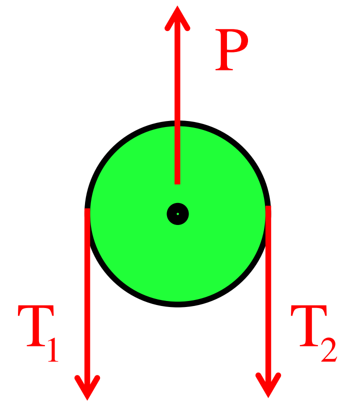
$$T_2 - m_2 g = -m_2 a$$

$$\Rightarrow \quad T_2 = m_2 g - m_2 a$$

a.) con't.

There are three unknowns, T_1, T_2 and a. Your third equation is coming from summing torques *about the pulley*. With m_1's acceleration defined *downward*, the pulley's angular acceleration must be clockwise and negative, so:





$$\sum \tau_{pin}:$$

$$T_1R - T_2R = -I\alpha$$

Substituting tensions

$T_1 = m_1g + m_1a$ and $T_2 = m_2g - m_2a$

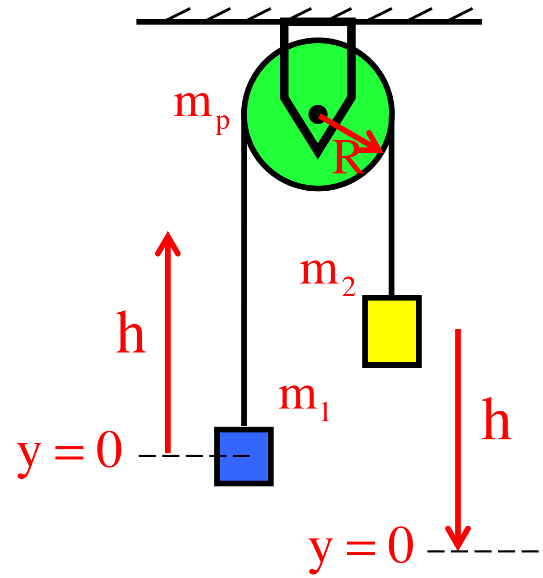
and we get:

$$T_1 R - T_2 R = -I_{pully}\alpha$$

$$\Rightarrow (m_1g + m_1a)\cancel{R} - (m_2g - m_2a)\cancel{R} = -\left(\frac{1}{2}m_p\cancel{R}^{\cancel{2}}\right)\left(\frac{a}{\cancel{R}}\right)$$

$$\Rightarrow a = \frac{-m_1g + m_2g}{\left(\frac{1}{2}m_p + m_1 + m_2\right)}$$

b.) How fast are the hanging masses moving after they have traversed a distance h?



Again, this is a conservation of energy problem. Defining the *zero-potential energy levels*, then writing out the governing equation for the system without solving yields:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

$$0 + m_2gh + 0 = \left[\frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 + \frac{1}{2}I_{pully}\omega^2\right] + m_1gh$$

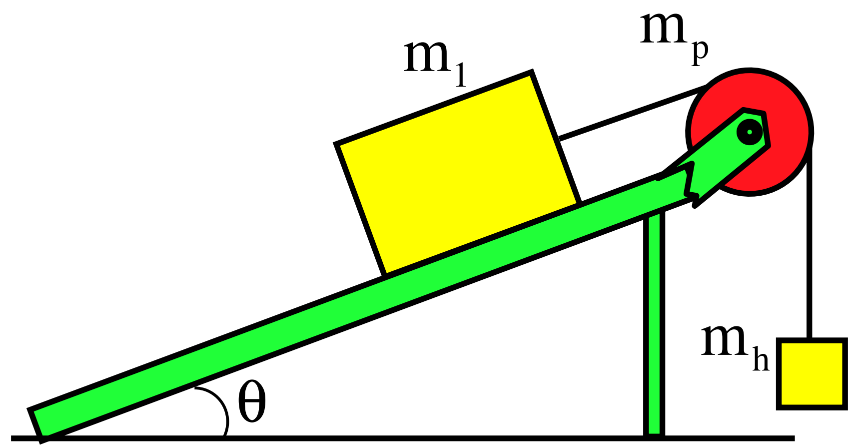
So two morals here:

1.) *The* tension on either side of a massive pulley is *different*; and

2.) *You can* assign each mass its own *zero potential energy level* for gravity (near the surface of the earth), *independent of any other mass in the system*.

3.) *Although it* may not be obvious at first glance, the extraneous work done by the two tensions added to the work done by the torque produced by those tensions on the pulley will add to zero (that's why W_{ext} is zero).

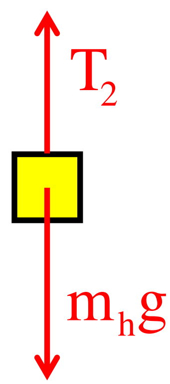
Example 13: Consider a string attached to a *hanging mass* at one end and to a block on an *incline* of angle θ at the other. The string is hung over a *massive pulley* of radius R and known I. What is given is:



$$m_1,\ m_h,\ m_p,\ R,\ g,\ \theta, \text{ and } I_{pulley} = \frac{1}{2} m_p R^2$$

a.) Derive an expression for the hanging mass's *acceleration* when the system is released.

For hanging mass (assuming acceleration *upward*):

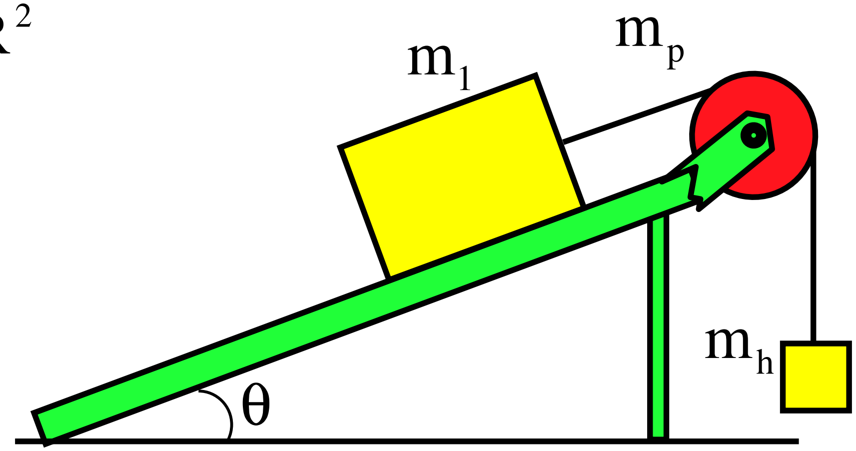


$$\sum F_y :$$

$$T_2 - m_h g = m_h a$$

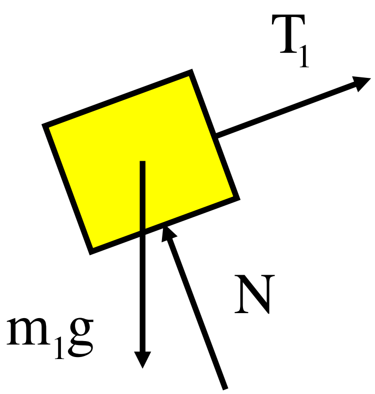
$$\Rightarrow \quad T_2 = m_h g + m_h a$$

m_1, m_h, m_p, R, g, θ, and $I_{pulley} = \frac{1}{2} m_p R^2$



a.) con't

For block (assuming acceleration *down the incline*):

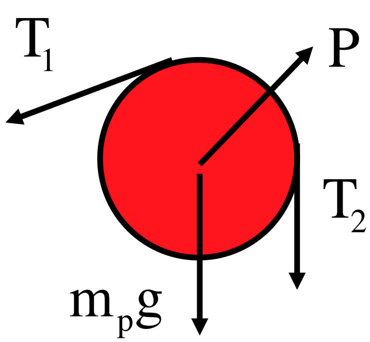


$$\sum F_{m_1,x}$$

$$T_1 - m_1 g \sin\theta = -m_1 a$$

$$\Rightarrow \quad T_1 = m_1 g \sin\theta - m_1 a$$

For pulley (assuming angular acceleration *clockwise*):

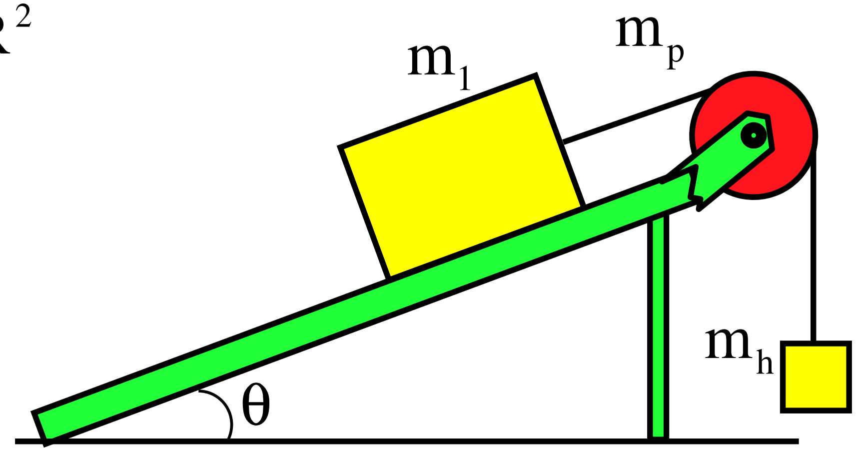


$$\sum \tau_{pin}$$

$$-T_2 R + T_1 R = I_{pulley}\alpha$$

$$\Rightarrow \quad -T_2 R + T_1 R = \left(\frac{1}{2} m_p R^2\right)\left(\frac{a}{R}\right)$$

m_1, m_h, m_p, R, g, θ, and $I_{pulley} = \frac{1}{2} m_p R^2$



a.) con't

Remembering that:

$$T_1 = m_1 g \sin\theta - m_1 a$$

and $T_2 = m_h g + m_h a$

we can write:

$$-T_2 \cancel{R} + T_1 \cancel{R} = \left(\frac{1}{2} m_p \cancel{R^2}\right)\left(\frac{a}{\cancel{R}}\right)$$

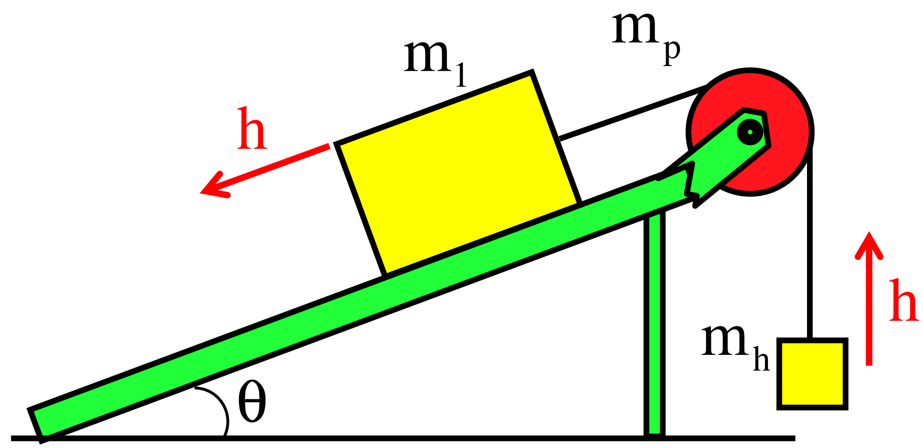
$$\Rightarrow \quad -T_2 + T_1 = \frac{1}{2} m_p a$$

$$\Rightarrow \quad -\left(m_h g + m_h a\right) + \left(m_1 g \sin\theta - m_1 a\right) = \frac{1}{2} m_p a$$

$$\Rightarrow \quad a = \frac{-m_h g + m_1 g \sin\theta}{\left(\frac{1}{2} m_p + m_1 + m_h\right)}$$

Example 13 (con't): Consider

a string attached to a *hanging mass* at one end and to a block on an *incline* of angle θ at the other. The string is hung over massive pulley of radius R and known I. What is given is:

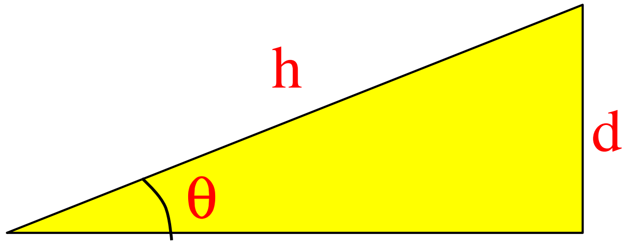


$$m_1, m_h, m_p, R, g, \theta, \text{and } I_{pulley} = \frac{1}{2} m_p R^2$$

b.) Derive an expression for the hanging mass's *velocity* after it has risen a distance *h*.

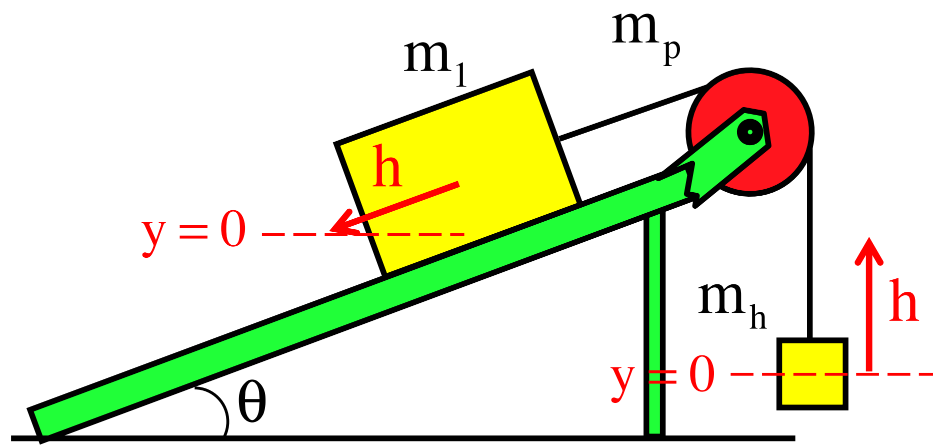
This is a conservation of energy problem:

Note that when the hanging mass rises a distance *h*, the block slides down the incline a distance *h* but drops a distance *d*:



$$\sin\theta = \frac{d}{h} \quad \Rightarrow \quad d = h\sin\theta$$

$$m_1,\ m_h,\ m_p, R, g, \theta, \text{ and } I_{pulley} = \frac{1}{2}m_pR^2$$



Where do we define the *zero potential energy* levels?

Assuming the zero level for the hanging mass is where it starts out, and the zero level for the block is where it ends up, we can write:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

$$0 + m_1g(h\sin\theta) + 0 = \left[\frac{1}{2}m_1v^2 + \frac{1}{2}m_hv^2 + \frac{1}{2}I_{pully}\omega^2\right] + m_hgh$$

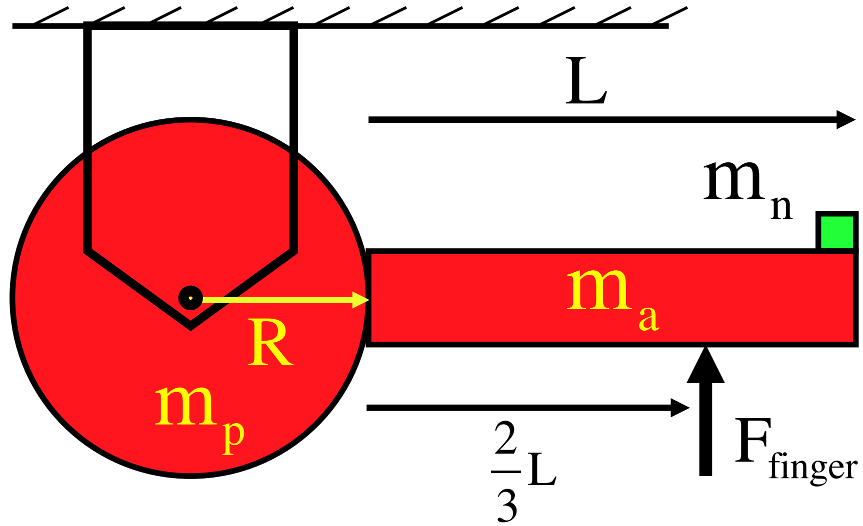
$$\Rightarrow\ m_1g(h\sin\theta) = \frac{1}{2}m_1v^2 + \frac{1}{2}m_hv^2 + \frac{1}{2}\left(\frac{1}{2}m_p\cancel{R^2}\right)\left(\frac{v^2}{\cancel{R^2}}\right) + m_hgh$$

$$\Rightarrow\quad v = \frac{2m_1g(h\sin\theta) - 2m_hgh}{m_1 + m_h + \frac{1}{2}m_p}$$

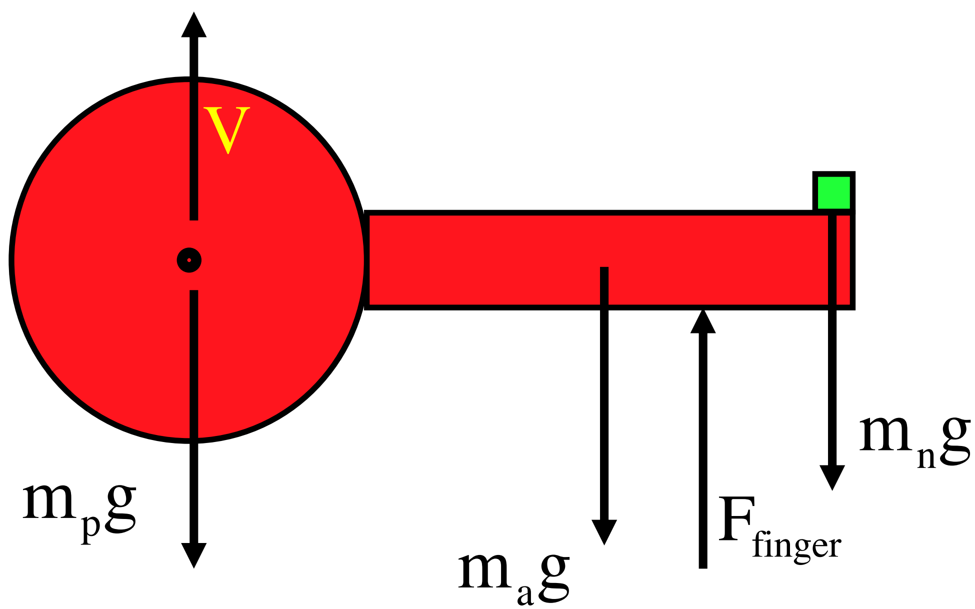
Problem: An arm of length "L" is welded to a pulley of radius R. The system is pinned at the pulley's center of mass. A lump is glued to the pulley/arm's end whose length is 2R. The system is initially stationary. Known:

$$m_n,\ m_a,\ m_p,\ R,\ g,\ L,\ I_{\text{arm's c. of m.}} = \frac{1}{12} m_a L^2$$

$$\text{and } I_{cm,pully} = \frac{1}{2} m_p R^2$$

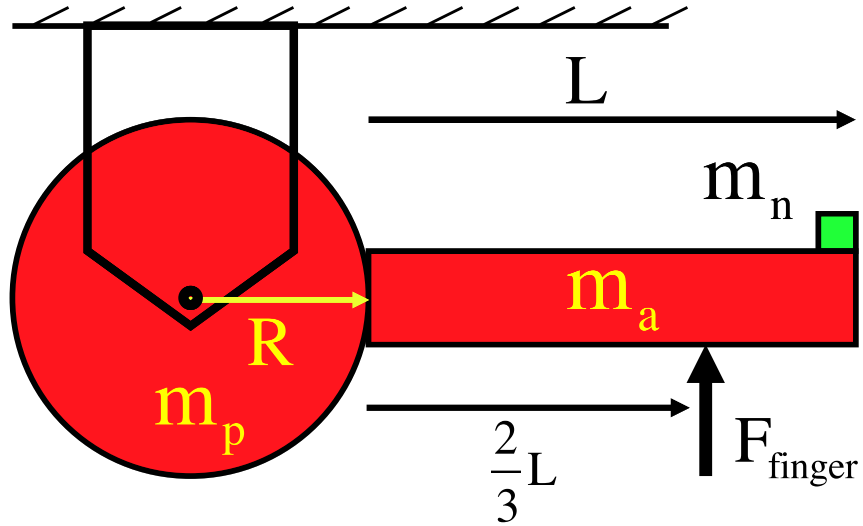


a.) Draw a f.b.d. for the forces acting on the pulley/arm system. (Hint: Note that as there is only a vertical force initially acting at the pin, and there are 5 of these forces).



$$m_n,\ m_a,\ m_p,\ R,\ g,\ L,\ I_{\text{arm's c. of m.}} = \frac{1}{12}m_aL^2$$

$$\text{and } I_{cm,pully} = \frac{1}{2}m_pR^2$$



b.) Determine the moment of inertia of the system about the pulley's pin.

$$I_{pin} = I_{nub} + I_{pulley} + I_{arm}$$

parallel axis theorem

$$= \left(m_nr^2\right) + \left(\frac{1}{2}m_pR^2\right) + \left(I_{arm,cm} + m_{arm}d^2\right)$$

$$= \left(m_n(R+L)^2\right) + \left(\frac{1}{2}m_pR^2\right) + \left[\left(\frac{1}{12}m_aL^2\right) + m_a\left(R+\frac{L}{2}\right)^2\right]$$

$$= \left(m_n(R+2R)^2\right) + \left(\frac{1}{2}m_pR^2\right) + \left[\left(\frac{1}{12}m_a(2R)^2\right) + m_a\left(R+\frac{(2R)}{2}\right)^2\right]$$

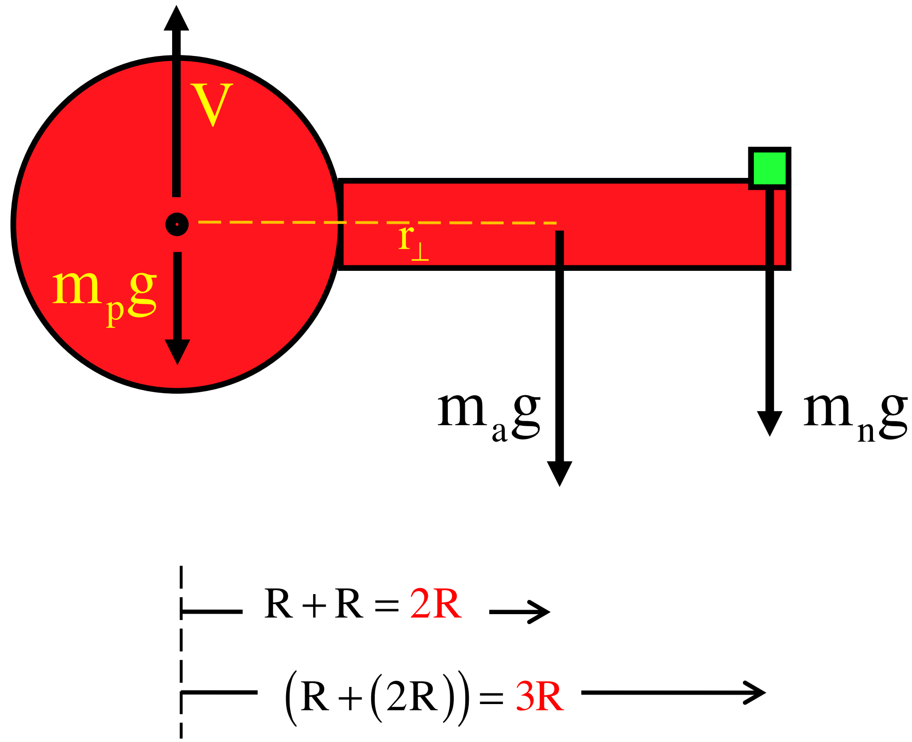
$$= \left(3m_n + \frac{1}{2}m_n + \frac{13}{3}m_n\right)R^2$$

$$m_L,\ m_a,\ m_p, R, g, L, I_{\text{arm's c. of m.}} = \frac{1}{12} m_a L^2$$
$$\text{and } I_{cm,pully} = \frac{1}{2} m_p R^2$$

c.) How would you determine how large a force the finger would have to be to keep the system in equilibrium?

Sum the torques about the pin . . .

d.) The finger is removed and the system released. Derive an expression for its initial *angular acceleration*?

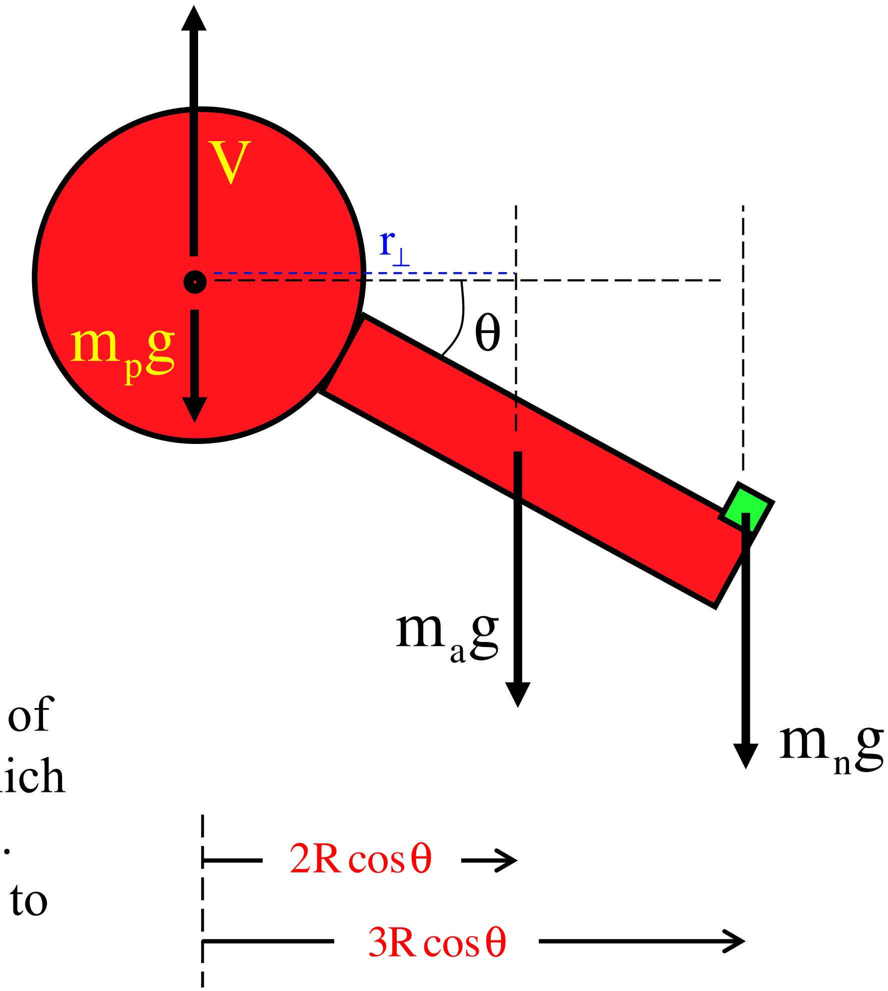


$$\sum \Gamma_{pin}:$$
$$-m_a g(2R) - m_n g(3R) = -I_{pin}\alpha$$
$$\Rightarrow \quad \alpha = \frac{(2m_a + 3m_n)gR}{I_{pin}}$$

$$m_L,\ m_a,\ m_p, R, g, L, I_{\text{arm's c. of m.}} = \frac{1}{12} m_a L^2$$
$$\text{and } I_{cm,pully} = \frac{1}{2} m_p R^2$$

e.) Is the angular acceleration constant? That is, if you wanted to determine, say, the angular velocity of the ensemble as it swung down through its lowest point, could you use rotational kinematics?

To see, we need to determine the angular acceleration expression for an arbitrary orientation. Consider:



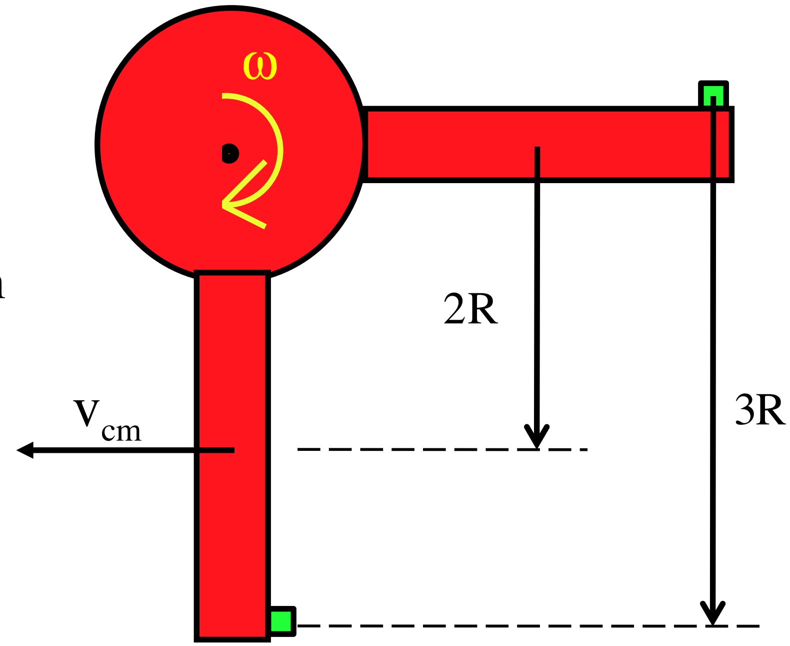
$\sum \Gamma_{pin}$:

$$-m_a g(2R\cos\theta) - m_n g(3R\cos\theta) = -I_{pin}\alpha$$

$$\Rightarrow \quad \alpha = \frac{(2m_a + 3m_n)gR}{I_{pin}}\cos\theta$$

Apparently the angular acceleration is a function of the angular displacement from the horizontal, which means the angular acceleration is not constant . . . which means you can't use *rotational kinematics* to solve for anything . . .

f.) The entire system rotates down with the the arm passing through the vertical. At that point, what is the system's *angular velocity*?



We need to identify the potential energy in the system to start with. As there are no rotational potential energy functions, all we need to worry about is gravitational PE for each piece of the system. From there, we can either approach this as a pure rotation of the entire system, or as individual pieces rotating with one another. We'll do both, starting with the latter:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

$$0 + \left[m_b g(2R) + m_{lump} g(3R)\right] + 0 = \left[KE_{lump} + KE_{arm} + KE_{pulley}\right] + 0$$

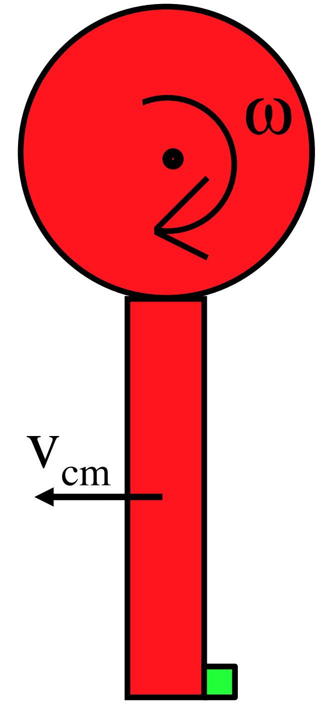
$$= \left[\left(\frac{1}{2}m_{lump} v_{lump}^2\right) + \left(\frac{1}{2} I_{beam,pin}\,\omega^2\right) + \left(\frac{1}{2} I_{pulley}\,\omega^2\right)\right] + 0$$

$$= \left[\left(\frac{1}{2}m_{lump}\left((3R)\omega\right)^2\right) + \left(\frac{1}{2}\left(\frac{13}{3}m_a R^2\right)\omega^2\right) + \left(\frac{1}{2}\left(\frac{1}{2}m_p R^2\right)\omega^2\right)\right]$$

(treated like point mass)

("I" from parallel axis theorem)

g.) (con't.) Equating the right and left side, the beast yields an angular velocity of:



$$\omega = \sqrt{\frac{2m_b gR + 3m_{lump} gR}{\frac{3}{2}m_{lump}R^2 + \frac{1}{2}\left(\frac{13}{3}m_a R^2\right) + \frac{1}{2}\left(\frac{1}{2}m_p R^2\right)}}$$

$$= \sqrt{\frac{4m_b g + 6m_{lump} g}{\left(3m_{lump} + \frac{13}{3}m_a + \frac{1}{2}m_p\right)R}}$$

If we had treated this as a pure rotation of the entire ensemble, it would have looked like:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

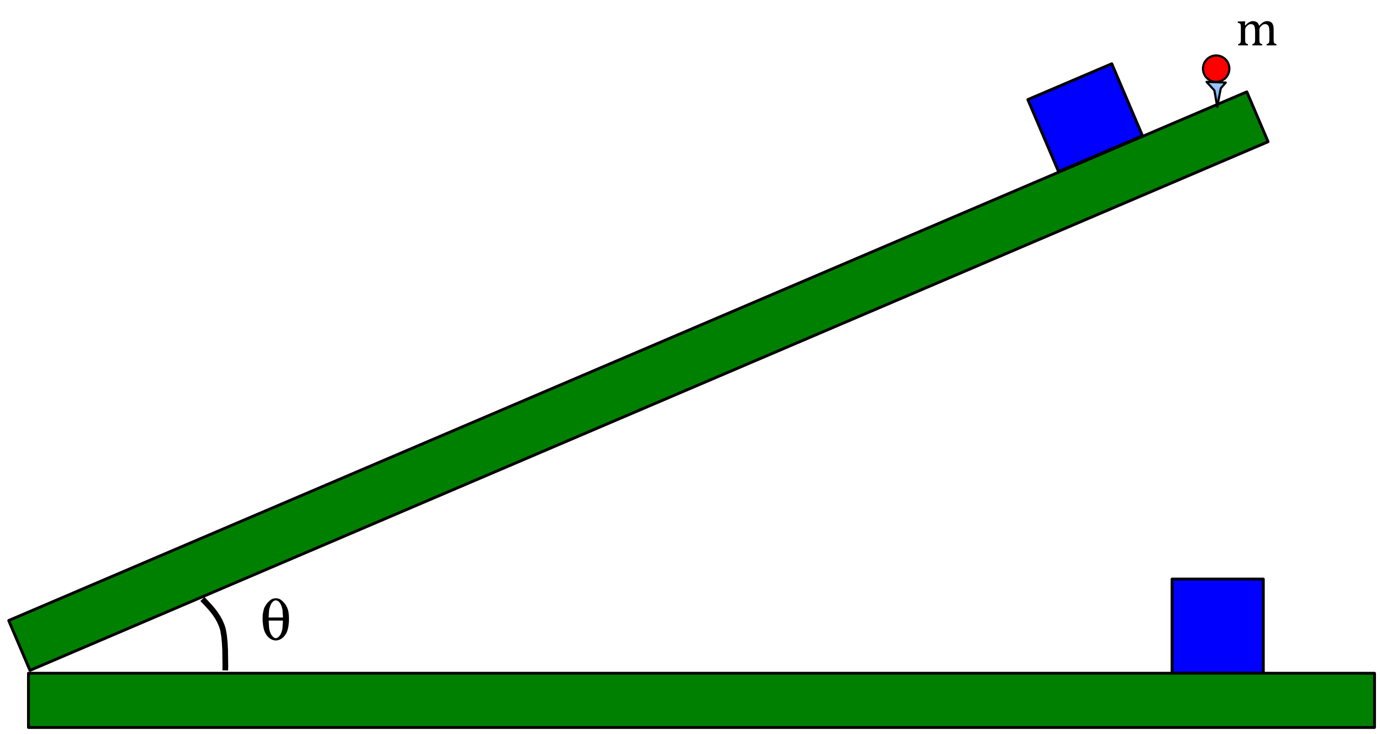
$$0 + \left[m_b g(2R) + m_{lump} g(3R)\right] + 0 = KE_{rot} + 0$$

$$\Rightarrow \quad m_b g(2R) + m_{lump} g(3R) = \frac{1}{2} I_{system,pin}\,\omega^2 + 0$$

h.) Lastly, what would the nub's velocity be at the bottom of the motion?

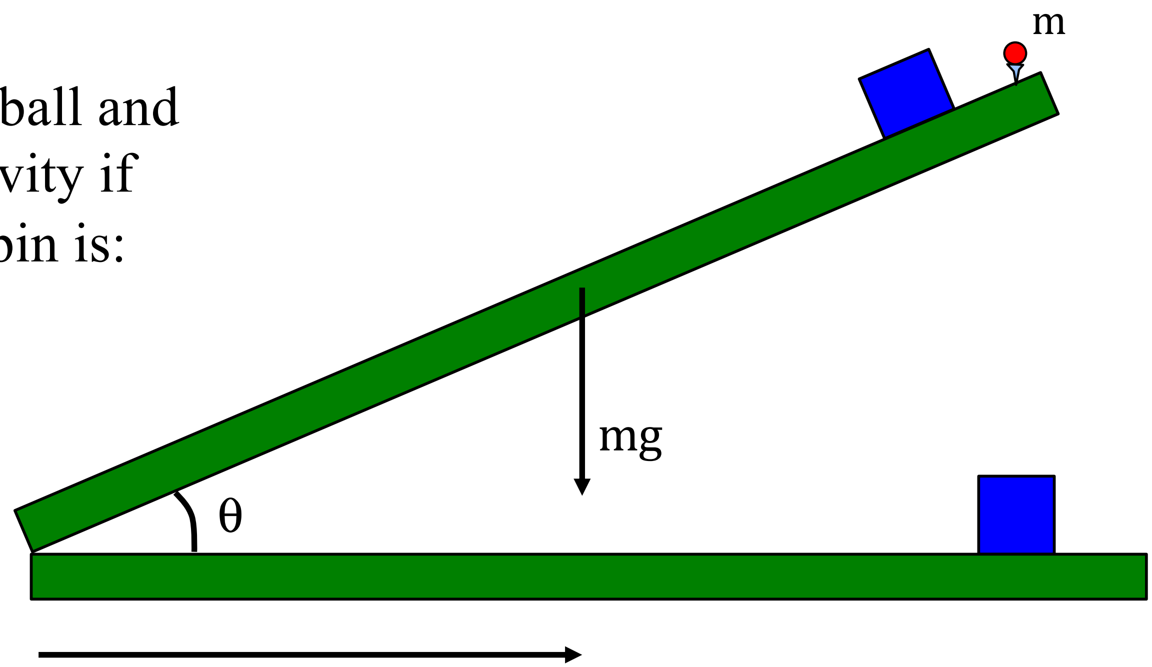
$$v_{nub} = (3R)\omega$$

Example 14: Hinged beam and ball demo: plank hinged at one end rotating down with end achieving acceleration greater than *g*. (demo)



Ignore the mass of the catch, ball and tee. The moment arm for gravity if the torque is taken about the pin is:

$$r_\perp = \frac{L}{2}\cos\theta$$



shortest distance between pin and line of force

so N.S.L. yields:

$$\sum \tau_{pin}:$$

$$(r_\perp)\;(mg) = I_{pin}\;\alpha$$

$$\left(\frac{L}{2}\cos\theta\right) mg = \left(\frac{1}{3}mL^2\right)\alpha$$

$$\Rightarrow \quad \alpha = \frac{3}{2}\frac{g}{L}\cos\theta$$

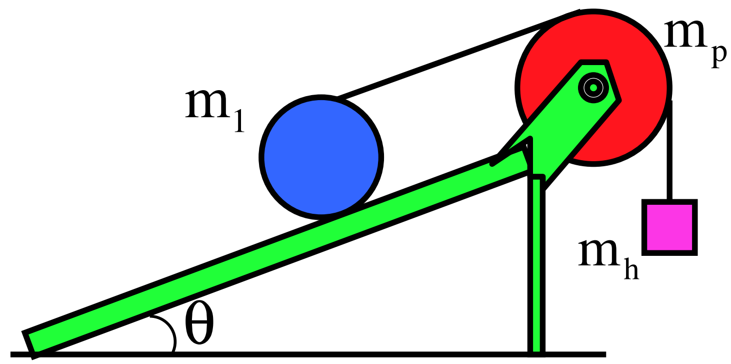
and the end's acceleration is:

$$a = L\alpha = L\left(\frac{3}{2}\frac{g}{L}\cos\theta\right)$$

$$= \frac{3}{2}g\cos\theta$$

Note: for $\theta = 0$ when board horizontal, the acceleration of the end of the board is "1.5g", which is greater than "g."

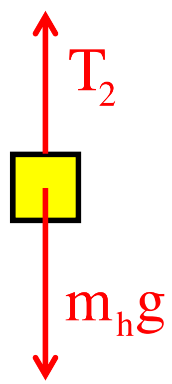
Example 15: Lastly, a stinker: A hanging mass is attached to a string which is threaded over a massive pulley of radius *R*, and wound around a ball of radius *R/2* sitting on an incline. The pulley is positioned so as to let the string always be parallel to the plane of the incline. We know:



$$m_b,\ m_h,\ m_p, R, g, \theta, I_{pulley} = \frac{1}{2} m_p R^2, \text{ and } I_{\text{cm of ball}} = \frac{2}{3} m_b R^2$$

a.) Derive an expression for the hanging mass's *acceleration* when the system is released.

For hanging mass (assuming acceleration *downward*):



$$\sum F_y :$$

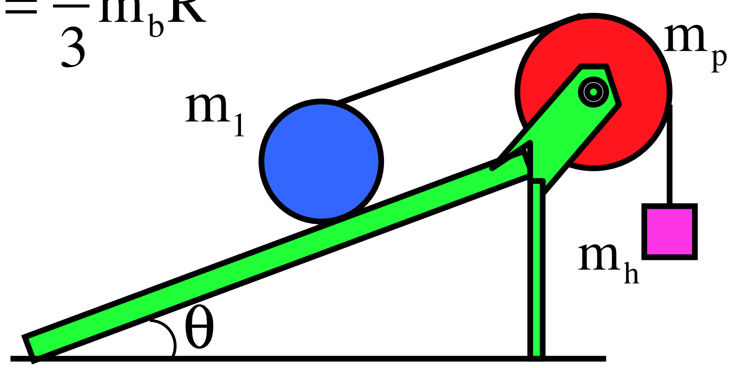
$$T_2 - m_h g = -m_h a_{hangingmass}$$

$$\Rightarrow \quad T_2 = m_h g - m_h a_{hangingmass}$$

For reasons of clarity, I'm going to redefine this acceleration as the *acceleration of the string* a_{string} and rewrite this N.S.L. equation as:

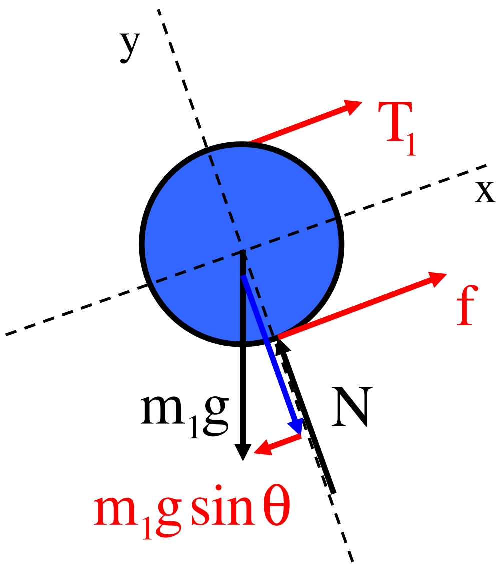
$$T_2 = m_h g - m_h a_{string}$$

$$m_b,\ m_h,\ m_p,\ R,\ g,\ \theta,\ I_{pulley} = \frac{1}{2}m_pR^2, \text{ and } I_{cm\ of\ ball} = \frac{2}{3}m_bR^2$$



The ball is experiencing both an *acceleration* of its *center of mass* AND and *angular acceleration* about it's *center of mass*. Noting that friction is *up* the incline (if the ball broke traction, it would spin clockwise), we can take each type of motion, translational and rotational, as a separate entity.

For ball's translational motion (assuming acceleration *upward*) with components:

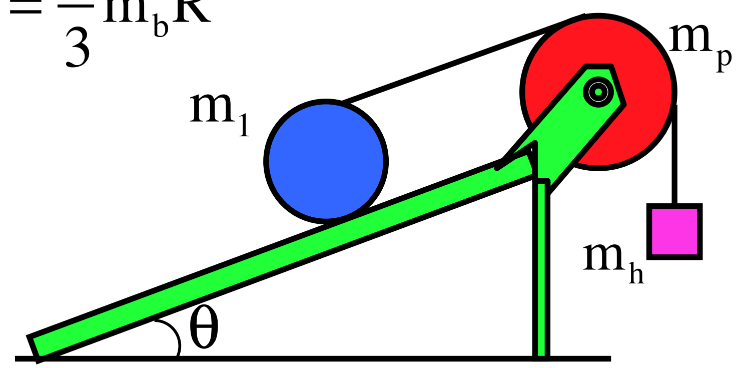


$$\sum F_{m_1,x}$$

$$T_1 + f - m_b g\sin\theta = m_b a_{ballcm}$$

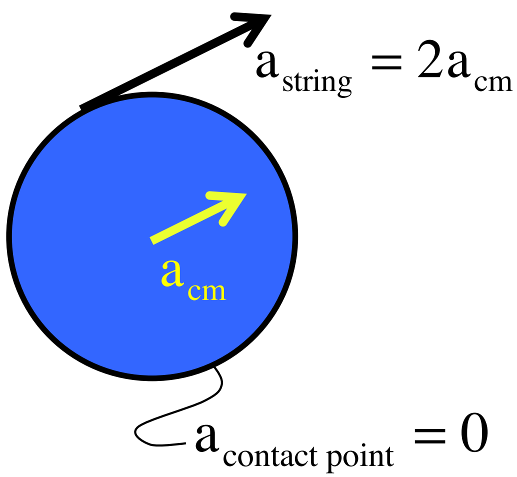
$$\Rightarrow \quad T_1 = -f + m_b g\sin\theta + m_b a_{ballcm}$$

$$m_b,\ m_h,\ m_p,\ R,\ g,\ \theta,\ I_{pulley} = \frac{1}{2}m_pR^2,\ \text{and } I_{cm\ of\ ball} = \frac{2}{3}m_bR^2$$



The problem with this expression is that we have the hanging mass's *acceleration* in terms of what the string is doing, and the ball's *acceleration* in terms of what its center of mass is doing. To relate the two, notice that if the contact point has zero instantaneous acceleration, we can write:

instantaneously



$$\Rightarrow\ a_{cm} = \frac{1}{2}a_{string}$$

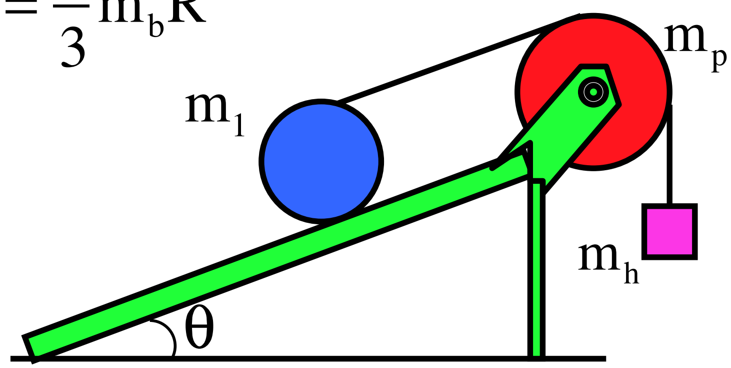
so

$$T_1 = -f + m_b g\sin\theta + m_b a_{ballcm}$$

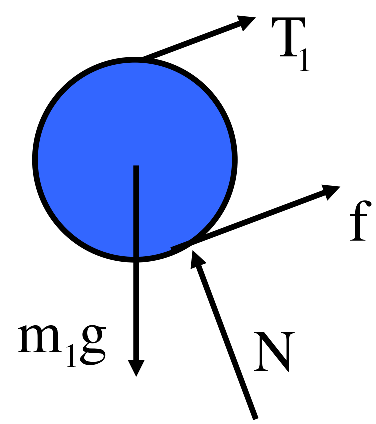
$$\Rightarrow\quad T_1 = -f + m_b g\sin\theta + m_b\left(a_{string}/2\right)$$

$m_b, m_h, m_p, R, g, \theta, I_{pulley} = \frac{1}{2}m_pR^2$, and $I_{cm\ of\ ball} = \frac{2}{3}m_bR^2$

a.) con't



For ball's rotation (assuming angular acceleration *clockwise* and summing torques about the center of mass):



$$\sum \tau_{ball,cm}$$

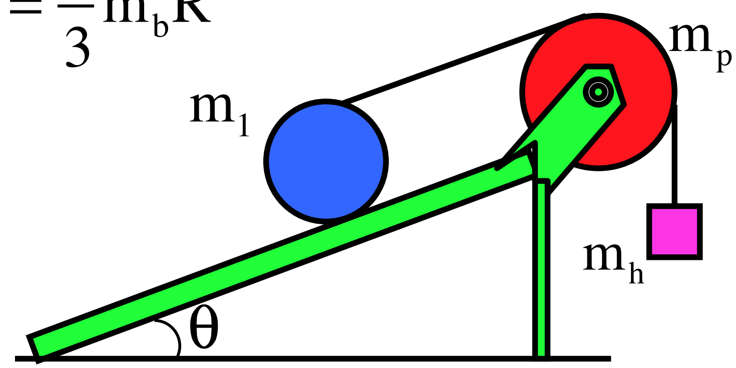
$$f\left(R/2\right) - T_1\left(R/2\right) = -I_{ball,cm}\alpha_{ball}$$

$$f\left(R/2\right) - T_1\left(R/2\right) = -\left(\frac{2}{3}m_bR^2\right)\alpha_{ball}$$

$$\Rightarrow \qquad f = T_1 - \left(\frac{4}{3}m_bR\right)\alpha_{ball}$$

Problem? We need an expression for the *angular acceleration of the ball* in terms of the *acceleration of the string*.

m_b, m_h, m_p, R, g, θ, $I_{pulley} = \frac{1}{2} m_p R^2$, and $I_{cm\ of\ ball} = \frac{2}{3} m_b R^2$

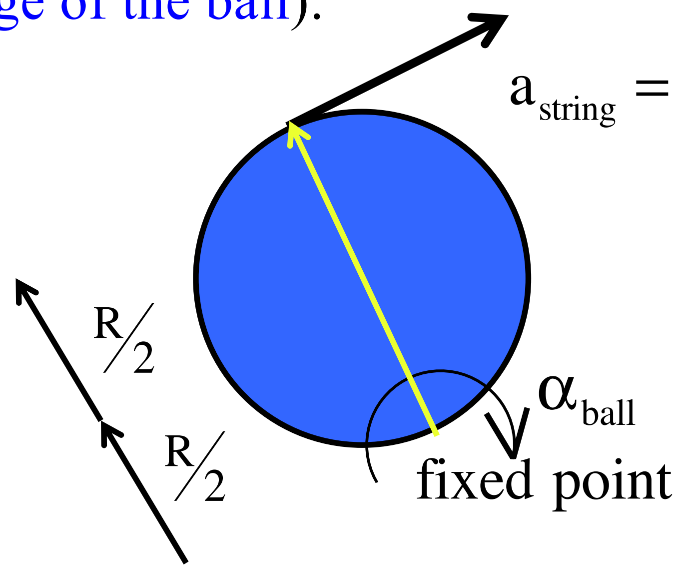


a.) con't

For the angular acceleration of the ball in terms of the *acceleration* of the string (which is to say, the acceleration of a point on the outer edge of the ball):

$$a_{string} = (R)\alpha_{ball}$$

instantaneously



$$\Rightarrow \quad \alpha_{ball} = \frac{a_{string}}{R}$$

With that:

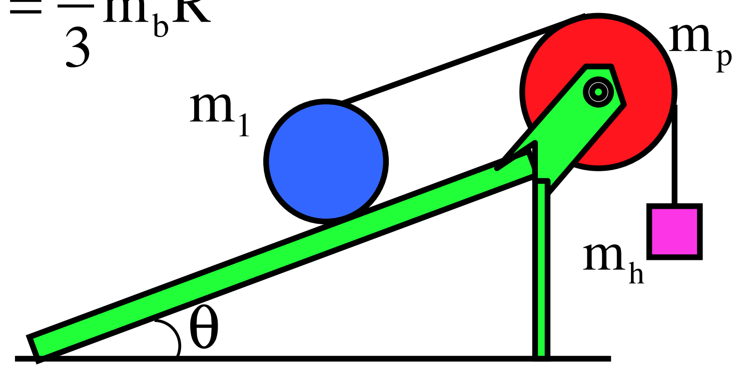
$$f = T_1 - \left(\frac{4}{3} m_b R\right)\alpha_{ball}$$

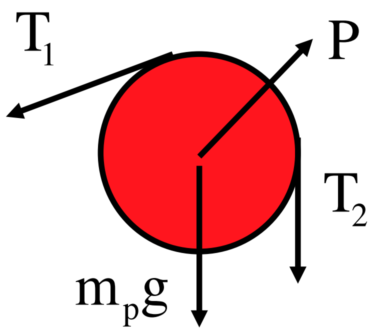
$$\Rightarrow \quad f = T_1 - \left(\frac{4}{3} m_b R\right)\left(\frac{a_{string}}{R}\right)$$

$m_b,\ m_h,\ m_p,\ R,\ g,\ \theta,\ I_{pulley} = \frac{1}{2}m_pR^2,$ and $I_{cm\ of\ ball} = \frac{2}{3}m_bR^2$

a.) con't

For pulley (assuming angular acceleration clockwise):





$$\sum \tau_{pin}$$

$$-T_2R + T_1R = -I_{pulley}\alpha$$

$$\Rightarrow \quad -T_2R + T_1R = -\left(\frac{1}{2}m_pR^2\right)\left(\frac{a_{string}}{R}\right)$$

So we have everything in terms of *the string*. It's time to compile.

$$m_1,\ m_h,\ m_p,\ R,\ g,\ \theta,\ I_{pulley} = \frac{1}{2}m_pR^2,\ \text{and } I_{cm\ of\ ball} = \frac{2}{3}m_bR^2$$

a.) con't

(from angular motion of ball) $f = T_1 - \left(\frac{4}{3}m_bR\right)\left(\frac{a_{string}}{R}\right)$

(from translation of ball) $T_1 = -f + m_b g\sin\theta + m_b\left(a_{string}/2\right)$

$$= -\left(T_1 - \left(\frac{4}{3}m_bR\right)\left(\frac{a_{string}}{R}\right)\right) + m_b g\sin\theta + m_b\left(a_{string}/2\right)$$

$$\Rightarrow\quad 2T_1 = +\left(\frac{4}{3}m_bR\right)\left(\frac{a_{string}}{R}\right) + m_b g\sin\theta + m_b\left(a_{string}/2\right)$$

$$\Rightarrow\quad T_1 = \left(\frac{2}{3}m_b + m_b\right)a_{string} + 2m_b g\sin\theta$$

$$= \left(\frac{5}{3}m_b\right)a_{string} + 2m_b g\sin\theta$$

(from hanging mass) $T_2 = m_h g - m_h a_{string}$

$$m_1,\ m_h,\ m_p,\ R,\ g,\ \theta,\ I_{pulley}=\frac{1}{2}m_pR^2,\ \text{and}\ I_{cm\ of\ ball}=\frac{2}{3}m_bR^2$$

a.) con't

with $T_2=\left(m_hg-m_ha_{string}\right)$ and $T_1=\left(\left(\frac{5}{3}m_b\right)a_{string}+2m_bg\sin\theta\right)$

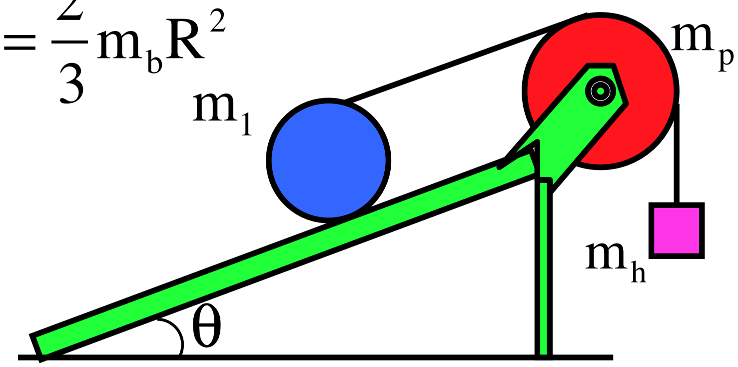
(from pulley) $\quad -T_2R+T_1R=-\left(\frac{1}{2}m_pR^2\right)\left(\frac{a_{string}}{R}\right)$

$$\Rightarrow\quad -\left(m_hg-m_ha_{string}\right)R+\left(\left(\frac{5}{3}m_b\right)a_{string}+2m_bg\sin\theta\right)R=-\left(\frac{1}{2}m_pR^2\right)\left(\frac{a_{string}}{R}\right)$$

$$\Rightarrow\quad \left(m_ha_{string}\right)+\left(\frac{5}{3}m_b\right)a_{string}+\left(\frac{1}{2}m_p\right)a_{string}=m_hg-2m_bg\sin\theta$$

$$\Rightarrow\quad a_{string}=\frac{m_hg-2m_bg\sin\theta}{\left(m_h\right)+\left(\frac{5}{3}m_b\right)+\left(\frac{1}{2}m_p\right)}$$

$m_b, m_h, m_p, R, g, \theta, I_{pulley} = \frac{1}{2}m_pR^2$, and $I_{cm\ of\ ball} = \frac{2}{3}m_bR^2$



b.) Derive an expression for the hanging mass's velocity after dropping a distance *h*.

This is very much like a standard *conservation of energy* problem with the exception that you have to keep track of subscripts and make the appropriate substitutions when it's time to solve. I'll not solve it, but writing it out looks like:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$

$$0 + [m_hgh] + 0 = \left[\left(\frac{1}{2}m_1v_{b,cm}^{\ 2} + \frac{1}{2}I_{ball,cm}\omega_b^{\ 2}\right) + \frac{1}{2}m_hv_{string}^{\ 2} + \frac{1}{2}I_{pulley}\omega_{pulley}^{\ 2}\right] + [m_1g(h\sin\theta)]$$