# Conservation of Momentum

# Newton: Quantity of Motion

"Forces applied for a period of time change an object's *quantity of motion*."

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}}$$

$$\vec{\mathbf{F}} = m\frac{\Delta\vec{\mathbf{v}}}{t}$$

$$\vec{\mathbf{F}}t = m\Delta\vec{\mathbf{v}} = m\mathbf{v}_f - m\mathbf{v}_i$$

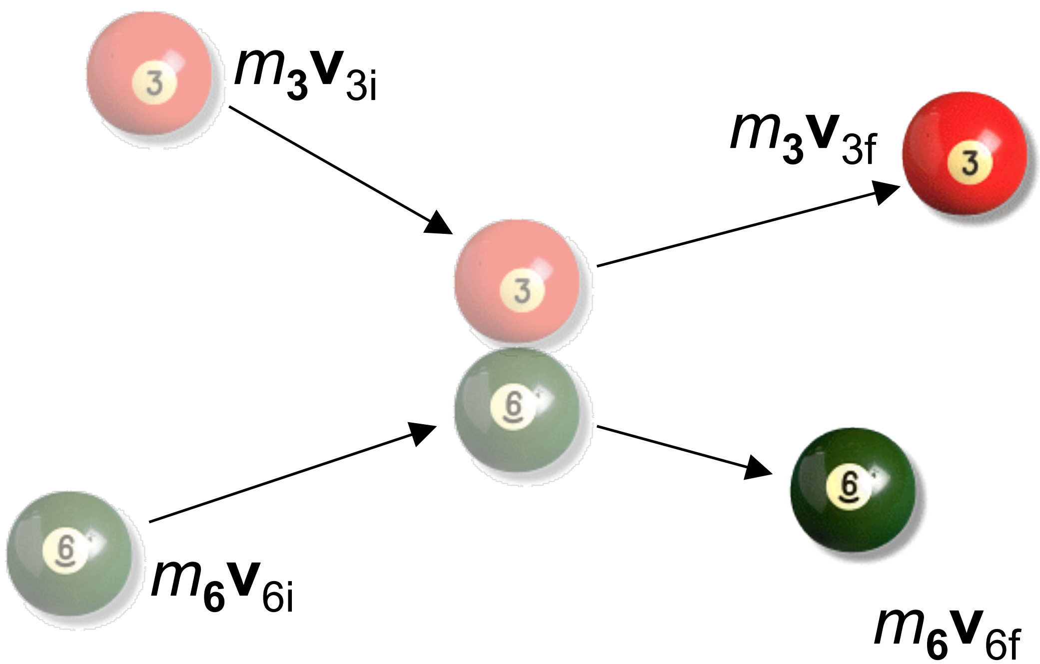
$$\vec{\mathbf{p}} \equiv m\vec{\mathbf{v}}$$

$$\mathbf{F}t = \Delta\mathbf{p}$$

$$\sum\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

# Conservation?

Two particles that are interacting with each other via some force apply equal and opposite forces to each other (Newton's 3rd Law), for some amount of time *t*.




$$F_{36} = -F_{63}$$

$$\frac{dp_3}{dt} = \frac{-dp_6}{dt}$$

$$\frac{d(p_3 + p_6)}{dt} = 0$$

$$p_3 + p_6 = \text{constant}$$

$$p_3 + p_6 = p_3' + p_6'$$

# Law of Conservation of Momentum (isolated sys)

Whenever two or more particles in an *isolated* system interact, their total momentum remains constant.

$$p_1 + p_2 = p_1' + p_2'$$

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

# Example 1

Alex ($m$=75 kg) sits on a 5kg cart with no-friction wheels, and gets hit by a 7.0 kg bowling ball with a velocity of 5.0 m/s.

a) What is Alex's velocity after catching the ball?

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v_{final}$$

$$(7.0kg)(5.0m/s) + (80kg)(0) = (7+80) v_{final}$$

$$v_{final} = \frac{35kg \bullet m/s}{87kg} = 0.40m/s$$

b) What is Alex's velocity after the collision if the bowling ball bounces off with a velocity of -0.50 m/s?

$$m_1 v_1 + m_2 v_2 = m_1 v_{1-final} + m_2 v_{2-final}$$

$$(7.0kg)(5.0m/s) + (80kg)(0) = (7)(-0.5) + (80) v_{final}$$

$$v_{final} = \frac{35 + 3.5}{80} = 0.48m/s$$

# Impulse (non-isolated sys)

= "Change in momentum"

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

$$d\mathbf{p} = \mathbf{F}\ \ dt$$

$$\int d\mathbf{p} = \int_{t_i}^{t_f} \mathbf{F}\ \ dt$$

$$\boxed{\mathbf{J} = \int_{t_i}^{t_f} \mathbf{F}\ \ dt = \Delta\mathbf{p}}$$

Note: for some reason, our textbook uses the symbol **I** to indicate impulse.

# Impulse

Area under the curve of a force-time curve

# Example 2

Alex (still 75 kg) sits on the 5 kg cart and slams into the wall with an initial velocity of 90 km/h. The collision lasts 0.257 seconds.

a) What was the impulse in this collision?

b) What *average* force is exerted on Alex during the collision?

# Example

A 4.00 kg gun with a 80 cm long barrel fires a 50-gram bullet with at a velocity of 400 m/s. Find:

a) recoil velocity of gun
b) impulse on bullet
c) time the bullet accelerated
d) average acceleration of the bullet
e) force applied to the gun by the bullet

# Example

A 4.00 kg gun with a 80 cm long barrel fires a 50-gram bullet with at a velocity of 400 m/s. Find:

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d) average acceleration of the bullet
e) force applied to the gun by the bullet

$$a)\ m_1v_1 + m_2v_2 = m_1v_{1final} + m_2v_{2final}$$
$$0 = (4kg)v_{1final} + (0.050kg)(400m/s)$$
$$v_{1final} = -5.0m/s$$

$$b)\ \text{Impulse} = Ft = m\Delta v$$
$$= (0.050kg)(400-0) = 20kg \bullet m/s$$

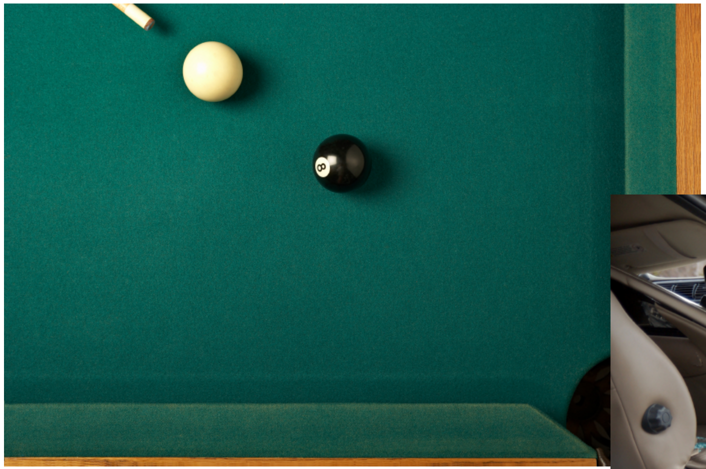
$$c)\ t = \frac{distance}{speed} = \frac{0.80m}{200m/s} = 0.004s$$

$$d)\ a = \frac{v_f - v_i}{t} = \frac{400m/s - 0}{0.004s} = 1.0e5m/s^2$$

$$e)\ F = ma = (0.050kg)(1.0e5m/s^2)$$
$$F = 5000N$$

# Collisions

## Elastic



## Inelastic

## Perfectly inelastic

# Collisions

Momentum (in isolated system) is always conserved

$$\Sigma \Delta p=0, \text{ or } p_1+p_2=p_1'+p_2'$$

Energy (in isolated system) is always conserved

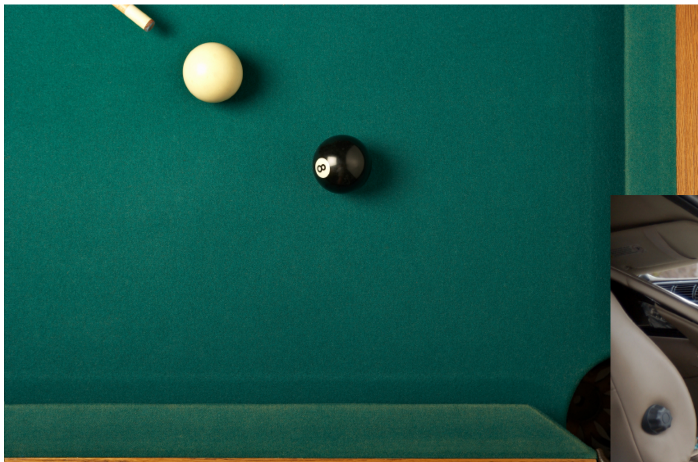
$$\Sigma \Delta E=0, \text{ or } \Sigma E_i=\Sigma E_f$$

In *some* collisions, there is very little energy "lost" to heat (sound, deformation). In these *elastic collisions*, kinetic energy is conserved:

$$K_1+K_2=K_1'+K_2'$$

# Collisions

## Elastic



## Inelastic

## Perfectly inelastic

# Collisions

- *Elastic collisions*: $K_1+K_2=K_1'+K_2'$ & $\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_1'+\mathbf{p}_2'$

$$\frac{1}{2}m_1v_1^2+\frac{1}{2}m_2v_2^2=\frac{1}{2}m_1v_1'^2+\frac{1}{2}m_2v_2'^2 \quad \&$$

$$m_1\mathrm{v}_1+m_2\mathrm{v}_2=m_1\mathrm{v}_1'+m_2\mathrm{v}_2'$$

- *Inelastic collisions:*

$$\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_1'+\mathbf{p}_2' \quad \text{or}$$

$$m_1\mathbf{v}_1+m_2\mathbf{v}_2=m_1\mathbf{v}_1'+m_2\mathbf{v}_2'$$

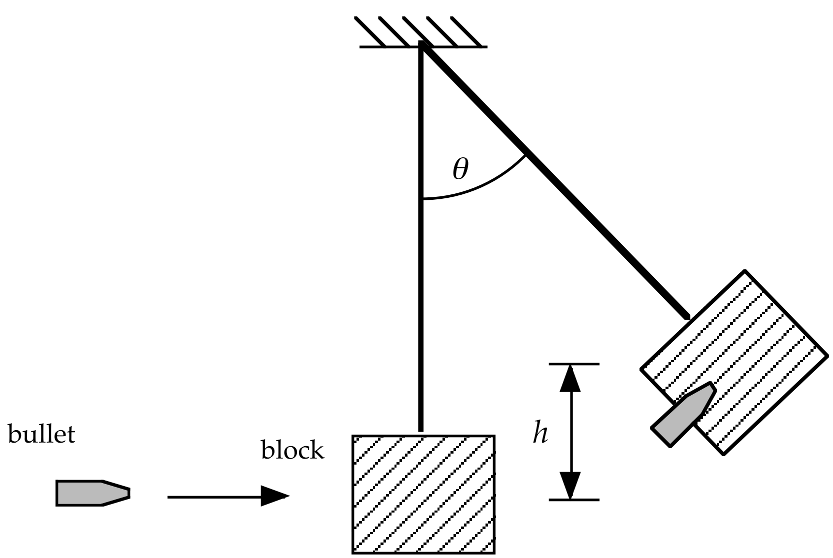
- ***Perfectly*** *inelastic collisions:*

$$\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_1'+\mathbf{p}_2' \quad \text{or}$$

$$m_1\mathbf{v}_1+m_2\mathbf{v}_2=(m_1+m_2)\mathbf{v}_2'$$

# Example 4

A ballistic pendulum is used to measure the speed of a projectile: a 5-gram bullet is fired into a 1-kg block of wood, which swings up to a height of 5-cm.




Find the initial speed of the projectile, and the energy lost in the collision.

For pendulum swinging up, use Cons. of Energy:

$$K_i = U_f$$

$$\frac{1}{2} m v_{bottom}{}^2 = mgh$$

$$v_{bottom} = \sqrt{2gh} = \sqrt{2(9.8 m/s^2)(0.05 m)}$$

$$v_{bottom} = 0.99 m/s$$

For collision, use Cons. of Momentum:

$$m_{bullet} v_{bullet} + m_{block} v_{block} = \left(m_{bullet} + m_{block}\right) v'$$

$$(0.005 kg) v_{bullet} + 0 = (1.005 kg)(0.99 m/s)$$

$$v_{bullet} = 199 m/s$$

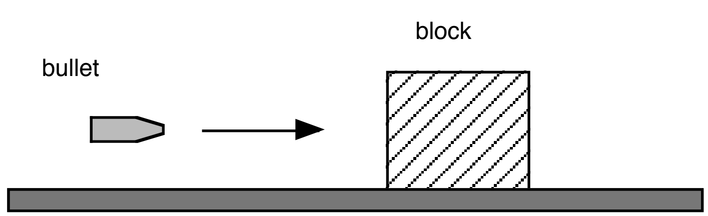
# Example 5

Tarzan (69kg) is on the ground, about to be eaten by a tiger. Jane (62 kg) climbs 14 meters up a nearby tree and swings down towards the ground on a vine. She swings past Tarzan, grabbing him as she passes by, and they swing together up into another tree.

What is the maximum height that Jane and Tarzan can reach on their upswing?

# Example 6

A 100 gram rubber bullet is fired horizontally at an 800 gram block of wood which is sitting on top of a flat surface. The coefficient of kinetic friction between the block of wood and the surface is 0.70., and the block slides 50 cm before coming to rest.




If the bullet collided *elastically* with the block, what were its initial and final velocities?

# Example 6

$$\tfrac{1}{2}m_1 v_{1i}^{\ 2} + \tfrac{1}{2}m_2 v_{2i}^{\ 2} = \tfrac{1}{2}m_1 v_{1f}^{\ 2} + \tfrac{1}{2}m_2 v_{2f}^{\ 2}$$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$m_1(v_{1i}^{\ 2} - v_{1f}^{\ 2}) = m_2(v_{2f}^{\ 2} - v_{2i}^{\ 2})$$

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i})$$

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i})$$

Divide eqns to get :

$$\boxed{(v_{1i} + v_{1f}) = (v_{2f} + v_{2i})}$$

Useful relationship for 1-d elastic collision problems

$$F_{net} = ma,\ F_f = ma,\ \mu mg = ma,\ a = (0.7)(9.80) = 6.86 m/s^2 (a\ \ of\ \ block)$$

$$v_f^{\ 2} = v_i^{\ 2} + 2a\Delta x,\ v_i = \sqrt{2a\Delta x} = \sqrt{2(6.86)(.50)} = 2.62 m/s\ (v_i\ \ of\ \ block)$$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

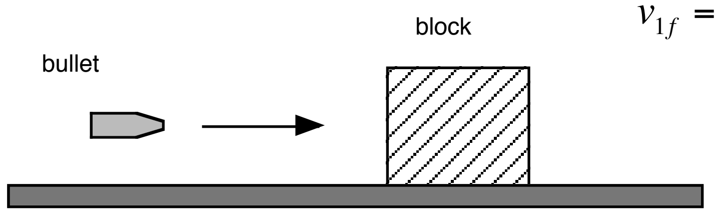
$$0.1v_{1i} + 0 = 0.1v_{1f} + (0.8)(2.62) \rightarrow v_{1i} = v_{1f} + 20.96$$

$$(v_{1i} + v_{1f}) = (v_{2f} + v_{2i})$$

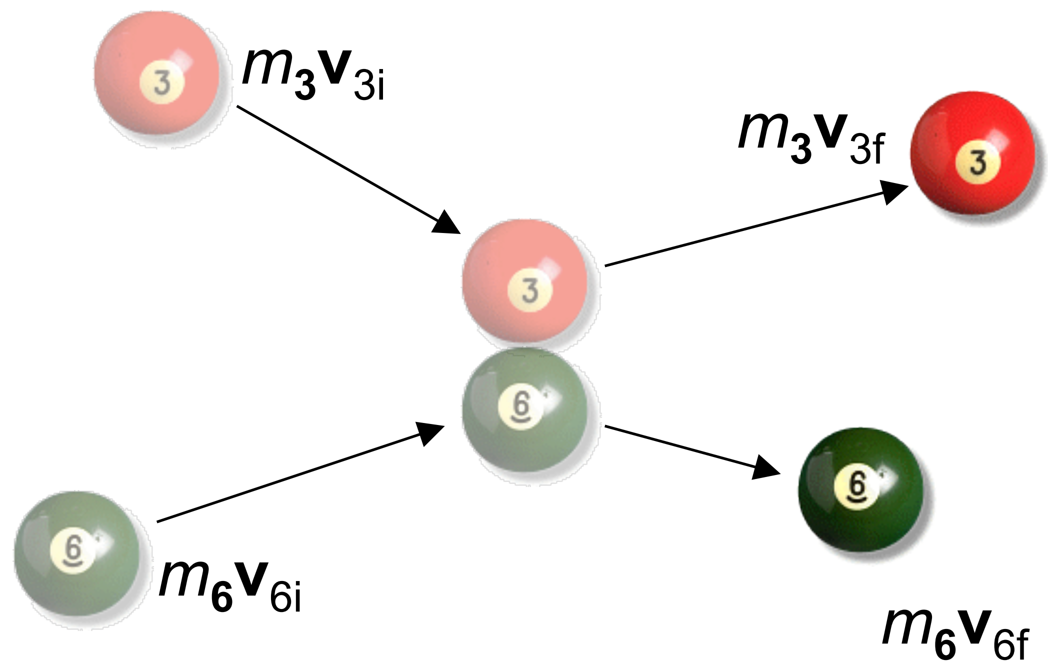
$$v_{1i} + v_{1f} = (2.62 + 0)$$

Combine eqns to get

$$v_{1f} = -9.17 m/s,\ v_{1i} = 11.79 m/s$$



# 2-D Collisions




$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f}$$

$$x: \quad m_1\mathbf{v}_{1ix} + m_2\mathbf{v}_{2ix} = m_1\mathbf{v}_{1fx} + m_2\mathbf{v}_{2fx}$$

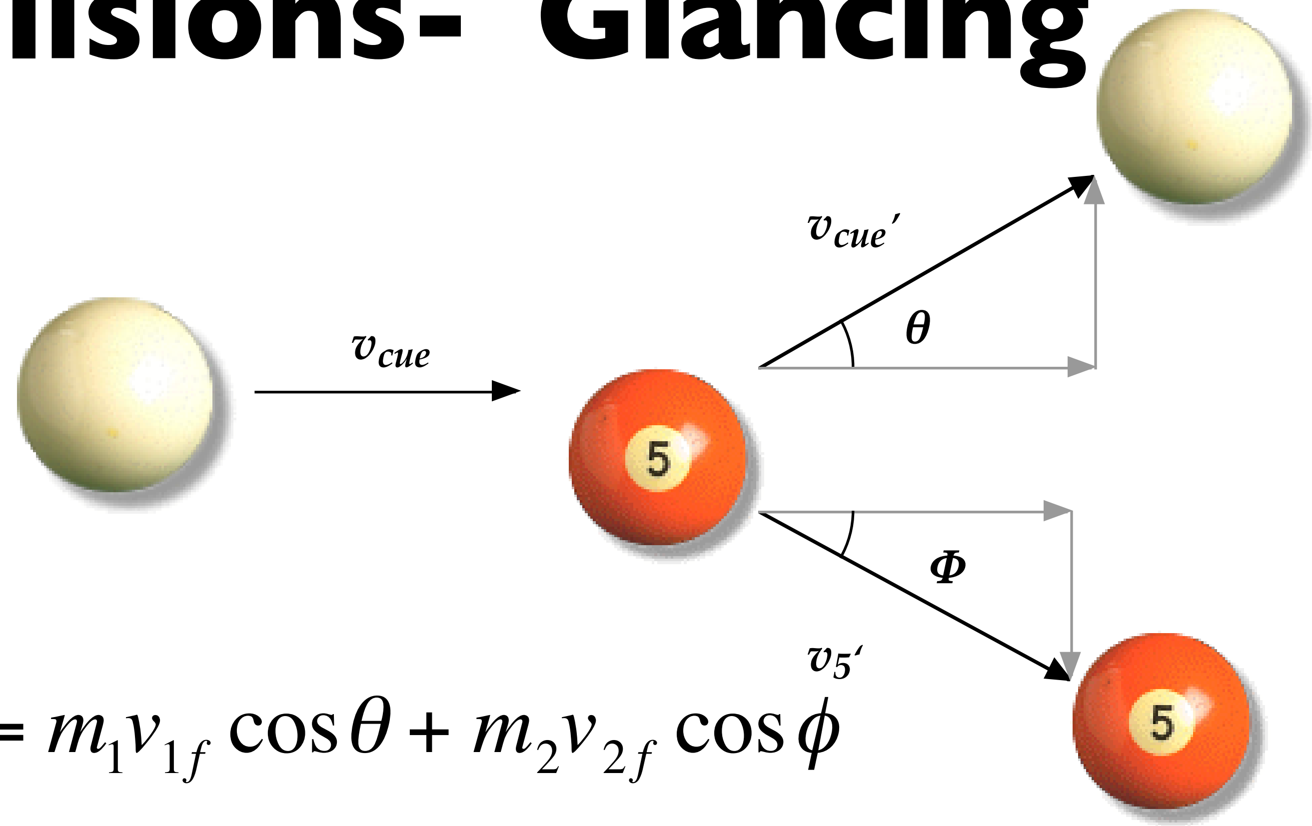
$$y: \quad m_1\mathbf{v}_{1iy} + m_2\mathbf{v}_{2iy} = m_1\mathbf{v}_{1fy} + m_2\mathbf{v}_{2fy}$$

$$\tfrac{1}{2}m_1 v_{1i}{}^2 + \tfrac{1}{2}m_2 v_{2i}{}^2 = \tfrac{1}{2}m_1 v_{1f}{}^2 + \tfrac{1}{2}m_2 v_{2f}{}^2$$

# Example 7

A 1000 kg car traveling north at 4 m/s collides with a 800 kg car traveling east at 3 m/s. After the masses of the two cars have locked together, what is the final velocity of the wreckage just after the collision?

# 2-D Collisions-"Glancing"



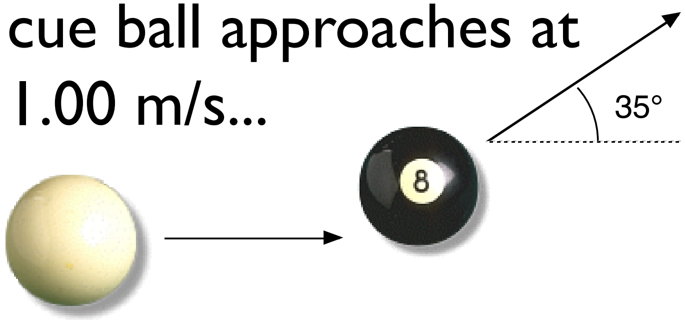

$$x:\quad m_1 v_{1ix} + 0 = m_1 v_{1f}\cos\theta + m_2 v_{2f}\cos\phi$$

$$y:\quad 0 + 0 = m_1 v_{1f}\sin\theta - m_2 v_{2f}\sin\phi$$

$$K:\quad \tfrac{1}{2} m_1 {v_{1i}}^2 + 0 = \tfrac{1}{2} m_1 {v_{1f}}^2 + \tfrac{1}{2} m_2 {v_{2f}}^2$$

# Example 8

In a billiard game, a player needs to sink the eight-ball into a corner pocket as shown. If the cue ball approaches at 1.00 m/s...




...what angle is the cue ball deflected? What are the final velocities of each ball? (Assume same mass, elastic collision.)

$$x: m_{cue}v_{cue-x} + m_8 v_{8-x} = m_{cue}v'_{cue-x} + m_8 v'_{8-x}$$

$$(1) + (0) = v'_{cue}\cos 55 + v'_8 \cos 35$$

$$y: m_{cue}v_{cue-y} + m_8 v_{8-y} = m_{cue}v'_{cue-y} + m_8 v'_{8-y}$$

$$(0) + (0) = -v'_{cue}\sin 55 + v'_8 \sin 35$$

$$v'_{cue}\sin 55 = v'_8 \sin 35$$

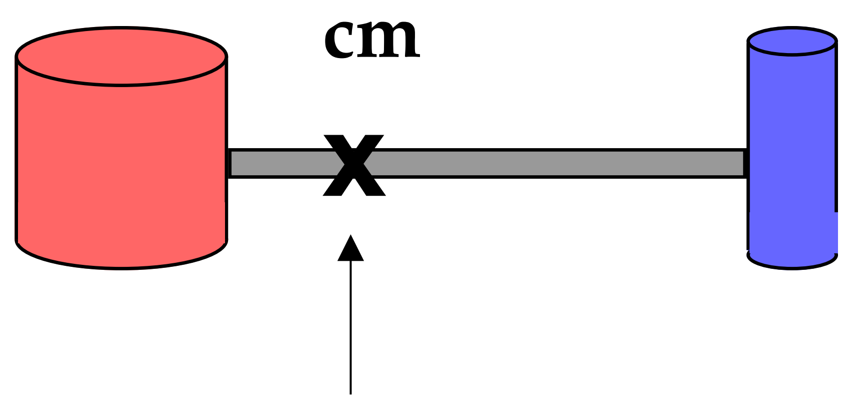
$$v'_{cue} = 0.700 v'_8$$

# Center of Mass

# Center of Mass

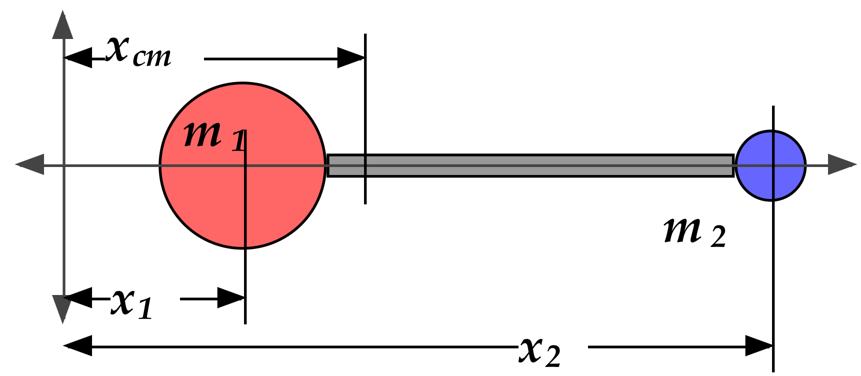
A force applied at the center of mass will cause the system of total mass *m* to accelerate in the direction of F, without rotation.

The center-of-mass is located at the "weighted average position of the system's mass."



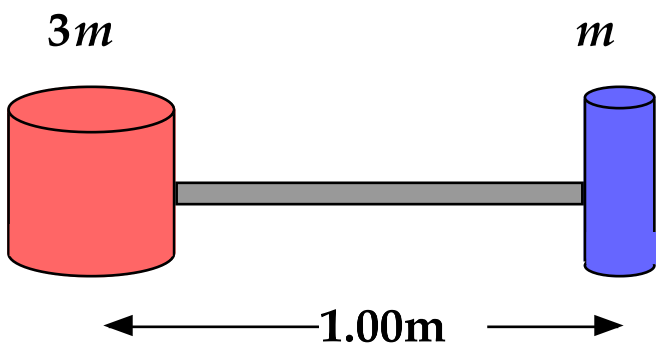
# Center of Mass

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$



# Example 9

Where is the center of mass in this system?




$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$x_{cm} = \frac{3m(0) + m(1)}{3m + m}$$

$$x_{cm} = \frac{1}{4}$$

# Complex cm?

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \ldots m_n x_n}{m_1 + m_2 + \ldots m_n} = \frac{\sum m_i x_i}{\sum m_i}$$

$$y_{cm} = \frac{\sum m_i y_i}{\sum m_i}$$

$$z_{cm} = \frac{\sum m_i z_i}{\sum m_i}$$

$$\vec{\mathbf{r}}_{cm} = x_{cm}\vec{\mathbf{i}} + y_{cm}\mathbf{j} + z_{cm}\vec{\mathbf{k}}$$

$$\vec{\mathbf{r}}_{cm} = \frac{\sum m_i x_i \vec{\mathbf{i}} + \sum m_i y_i \vec{\mathbf{j}} + \sum m_i z_i \vec{\mathbf{k}}}{M}$$

$$\vec{\mathbf{r}}_{cm} = \frac{\sum m_i \vec{\mathbf{r}}_i}{M}$$

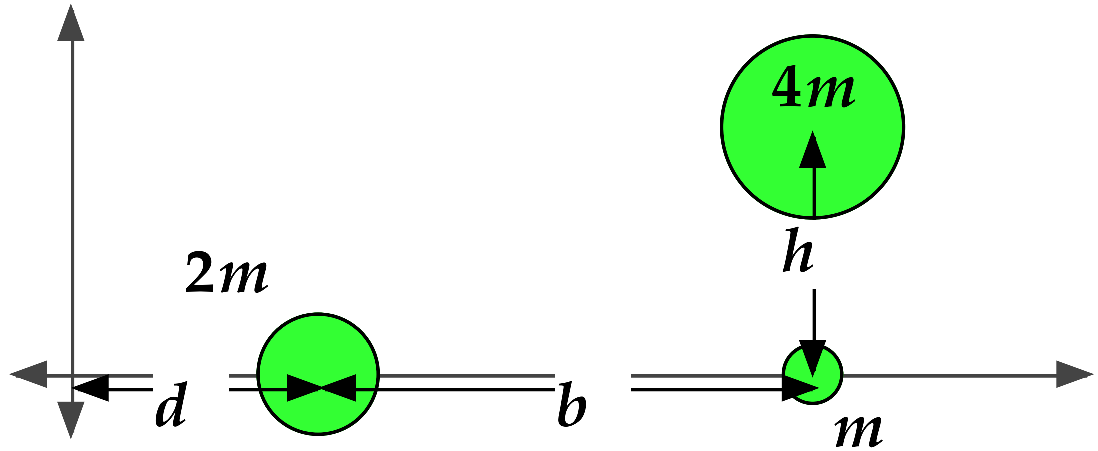
# Example 10

A 2 kg mass is located at position (2,3), and a 4 kg mass is located at position (-2,-1). Sketch this two-mass system, and calculate the position of its center of mass.

-4/6, 1/3

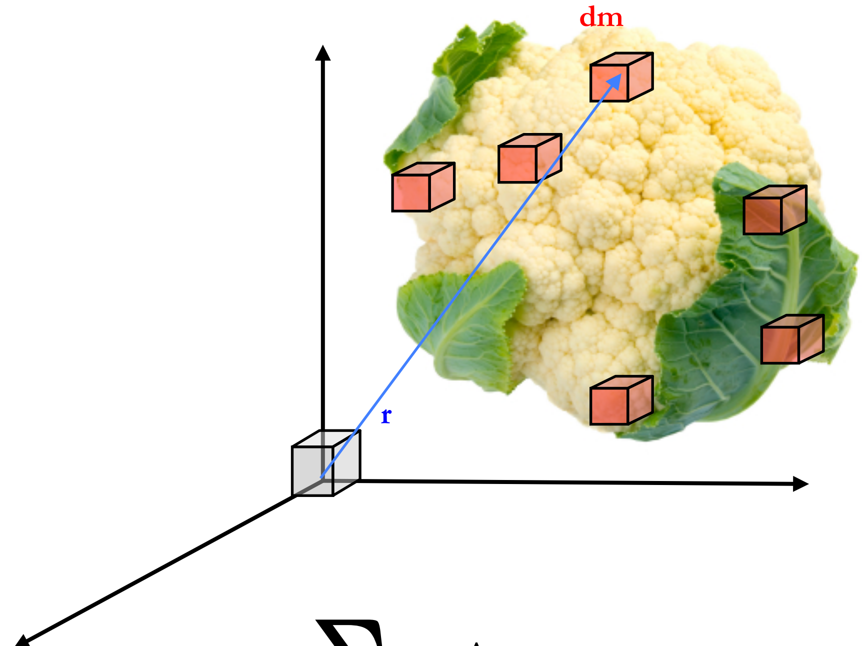
# Example 11

Calculate the center-of-mass for this 3-particle system.



# cm Continuous

Locating the center of mass for a large, extended object can be a little trickier, but the same basic principles apply: large objects have lots of particles, all located various distances from the **cm**.




$$x_{cm} = \lim_{\Delta m_i \to 0} \frac{\sum x_i \Delta m_i}{M} = \frac{1}{M}\int x \; dm$$

$$y_{cm} = \frac{1}{M}\int y \; dm$$

$$\vec{\mathbf{r}}_{cm} = \frac{1}{M}\int \vec{\mathbf{r}} \; dm$$

$$z_{cm} = \frac{1}{M}\int z \; dm$$

# Densities

For these analyses, we're going to need some way of relating *dm* to the location of that mass.
We can do this by considering the

- linear density $\lambda$,
- surface area density $\sigma$, or
- volume density $\rho$

of the object.

$$\vec{\mathbf{r}}_{cm} = \frac{1}{M}\int \vec{\mathbf{r}}\ dm$$

$$\lambda = \frac{m}{L} = \frac{dm}{dL} \rightarrow dm = \lambda\ dx$$

$$\sigma = \frac{m}{A} = \frac{dm}{dA} \rightarrow dm = \sigma\ dA$$

$$\rho = \frac{m}{V} = \frac{dm}{dV} \rightarrow dm = \rho\ dV$$

# Example 12

Assuming that a long, thin, rod has a uniform linear density λ=M/L, show that the center of mass for the rod, with mass M and length L, is in the middle.

$$x_{cm} = \frac{1}{M}\int x\,dm$$

$$dm = \lambda \bullet dx$$

$$x_{cm} = \frac{1}{M}\int x\,\lambda \bullet dx$$

$$x_{cm} = \frac{\lambda}{M}\int_0^L x \bullet dx$$

$$x_{cm} = \frac{\lambda}{M}\frac{1}{2}L^2$$

$$\lambda = \frac{M}{L}$$

$$x_{cm} = \frac{L}{2}$$

# Example 13

Assuming that a long, thin, rod has a changing linear density $\lambda = \alpha x$, calculate the location of the center of mass.

$$x_{cm} = \frac{1}{M}\int x \;\; dm$$

$$dm = \lambda \bullet dx$$

$$x_{cm} = \frac{1}{M}\int x \;\; \lambda \bullet dx$$

$$x_{cm} = \frac{1}{M}\int_0^L \alpha x^2 \bullet dx$$

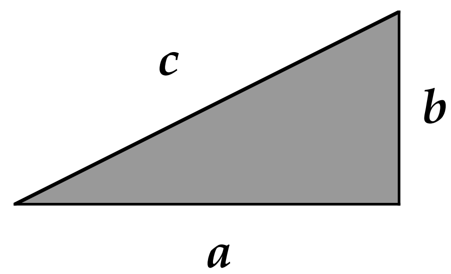
$$x_{cm} = \frac{\alpha}{M}\frac{1}{3}L^3$$

$$M = \int dm = \int_0^L \lambda \bullet dx = \int_0^L \alpha x \bullet dx = \frac{1}{2}\alpha L^2$$

$$x_{cm} = \frac{\alpha}{\left(\frac{1}{2}\alpha L^2\right)}\frac{1}{3}L^3 = \frac{2}{3}L$$

# Example 14

Find the center of mass of the 2-d right triangle shown here, of constant density.




$$x_{cm} = \frac{1}{M}\int x \;\; dm$$

$$dm = \sigma \;\; dA$$

$$x_{cm} = \frac{1}{M}\int x \;\; \sigma \bullet dA$$

$$\mathrm{y} = \frac{\mathrm{b}}{\mathrm{a}}\mathrm{x} \text{ is the height of a column}$$

$$dA = \frac{b}{a}x \;\; dx$$

$$x_{cm} = \frac{1}{M}\int_0^a x \;\; \sigma \;\; \frac{b}{a}x \;\; dx$$

$$x_{cm} = \frac{\sigma b}{Ma}\left(\frac{1}{3}a^3\right)$$

$$\sigma = \frac{M}{A} = \frac{M}{\frac{1}{2}ab}$$

$$x_{cm} = \left(\frac{M}{\frac{1}{2}ab}\right)\frac{b}{Ma}\left(\frac{1}{3}a^3\right) = \frac{2}{3}a$$

# System in Motion

What happens if we take the time-derivative of $r_{cm?}$

$$\vec{\mathbf{r}}_{cm} = \frac{\sum m_i \vec{\mathbf{r}}_i}{M}$$

$$\frac{d}{dt}\vec{\mathbf{r}}_{cm} = \frac{d}{dt}\frac{\sum m_i \vec{\mathbf{r}}_i}{M}$$

$$\vec{\mathbf{v}}_{cm} = \frac{\sum m_i \vec{\mathbf{v}}_i}{M}$$

$$M\vec{\mathbf{v}}_{cm} = \sum m_i \vec{\mathbf{v}}_i = \sum \vec{\mathbf{p}}_i$$

$$M\vec{\mathbf{v}}_{cm} = \vec{\mathbf{p}}_{total}$$

What happens if we take the time-derivative of $v_{cm?}$

$$\frac{d}{dt}\vec{\mathbf{v}}_{cm} = \frac{d}{dt}\frac{\vec{\mathbf{p}}_{total}}{M}$$

$$\vec{\mathbf{a}}_{cm} = \frac{\mathbf{F}_{total}}{M}$$

# Example 15

An exploding hockey puck of mass *m* is moving at 10 m/s in the +*x* direction when it explodes into two pieces, one of them with mass *m*/3 traveling in the +*x* direction at 15m/s.

a. What is the final velocity of the second piece?

$$(m_1 + m_2)v = m_1 v_1' + m_2 v_2'$$

$$m(+10) = \frac{m}{3}(+15) + \frac{2}{3} m v_2'$$

$$v_2' = +7.5 m/s$$

b. Where is the center of mass of the system 3 seconds after the explosion?

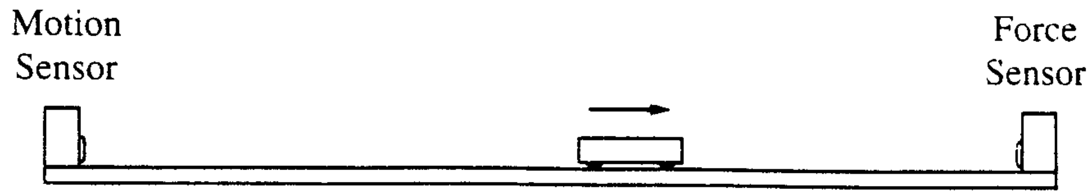
$$v_{cm} = 10 m/s$$

$$x = vt = (10 m/s)(3s) = 30m$$

*or*

Find location of two particles and calculate cm manually.

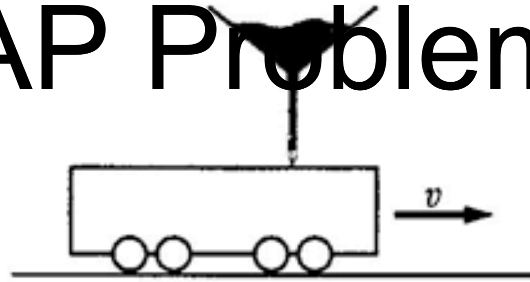
# AP Problem




Mech 1.

A motion sensor and a force sensor record the motion of a cart along a track, as shown above. The cart is given a push so that it moves toward the force sensor and then collides with it. The two sensors record the values shown in the following graphs.

# AP Problem




Mech 2. An open-top railroad car (initially empty and of mass $M_0$) rolls with negligible friction along a straight horizontal track and passes under the spout of a sand conveyor. When the car is under the conveyor, sand is dispensed from the conveyor in a narrow stream at a steady rate $\Delta M/\Delta t = C$ and falls vertically from an average height $h$ above the floor of the railroad car. The car has initial speed $v_0$ and sand is filling it from time $t = 0$ to $t = T$. Express your answers to the following in terms of the given quantities and $g$.