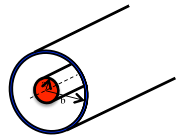
CAPACITANCE OF A COAXIAL CABLE

THINGS TO KNOW ABOUT CAPACITANCE DERIVATIONS:



1.) You will be given a geometry like parallel plates or a sphere or a coaxial cable, but nothing will be said about how much charge is on each plate.

a.) As no charge information is given, YOU have to assume what charge is where.

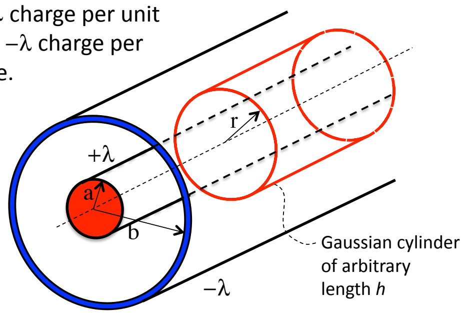
b.) In the case of enclosed geometries (spheres and cylinders), I assume there is positive charge on the inside structure. (That means the electric field will be outward in the direction of the dA vector used in Gauss's Law.)

c.) Don't be surprised to find that you have to use Gauss's Law to determine the electric field magnitude between the plates.

d.) With the E-field determined, you can use $V_c = -\Delta V = \int \vec{E} \bullet d\vec{r}$ to determine *V*, then use *C=Q/V* to get the capacitance of the cap.

What follows is the derivation from scratch:

We will start by assuming there is $+\lambda$ charge per unit length on the inside metal cable and $-\lambda$ charge per unit length on the outside metal pipe.



A Gaussian surface of radius *r* is cylindrical and is shown in red on the sketch. Using Gauss's Law, we can write:

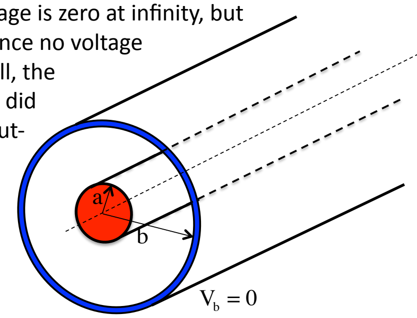
$$\int \vec{E} \bullet d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \quad E(2\pi rL) = \frac{\lambda L}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{\lambda L}{2\pi\varepsilon_o L r}$$

$$\Rightarrow \quad E = \frac{\lambda}{2\pi\varepsilon_o r}$$

What we are about to determine is the voltage difference (the amount of *work per unit charge* AVAILABLE) between the two "plates" of the capacitor. Until now, we have assumed that the voltage is zero at infinity, but because there is no electric field (hence no voltage change) outside the cylinder (after all, the enclosed charge would be zero if we did a Gauss's Law problem outside for out-side the cylinder), we can as well assign the zero voltage point to be at *r=b*. Doing that, we can write:



Noting that the electric field generated between the plates will go from *r=a* to *r=b*, the relationship we want to solve is:

$$V(b) - V(a) = \int_a^b \vec{E} \bullet d\vec{r} = -V_{\text{across the capacitor}}$$

Minor note: V(b)-V(a) = - (voltage across the cap) because if V(b) = 0, the difference will inherently be negative whereas the voltage across the cap is defined as being positive!

Executing the math, we get:

$$V(b) - V(a) = \int_a^b \vec{E} \bullet d\vec{r} = -V_{\text{across the capacitor}}$$

$$\Rightarrow \quad V_{cap} = \int_a^b \vec{E} \bullet d\vec{r}$$

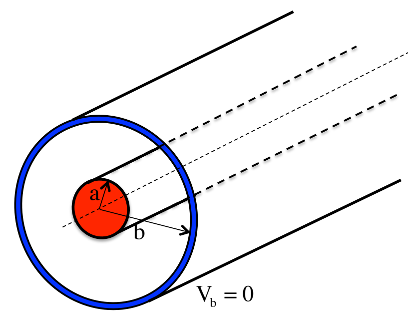
$$\Rightarrow \quad V_{cap} = \int_a^b \left(\frac{\lambda}{2\pi\varepsilon_o r}\right) dr \cos 0^o$$

$$\Rightarrow \quad V_{cap} = \frac{\lambda}{2\pi\varepsilon_o} \int_a^b \left(\frac{1}{r}\right) dr$$

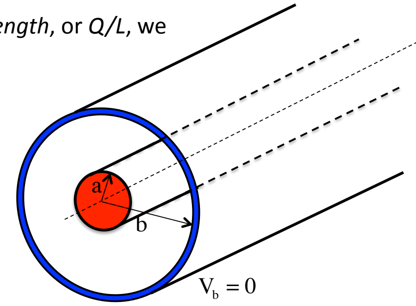
$$\Rightarrow \quad V_{cap} = \frac{\lambda}{2\pi\varepsilon_o} \ln(r)\Big|_a^b$$

$$\Rightarrow \quad V_{cap} = \frac{\lambda}{2\pi\varepsilon_o} \left(\ln(b) - \ln(a)\right)$$

$$\Rightarrow \quad V_{cap} = \frac{\lambda}{2\pi\varepsilon_o} \left(\ln\left(\frac{b}{a}\right)\right)$$



If we remember that λ is *charge per unit length*, or *Q/L*, we can finish off the problem by writing:



$$C = \frac{Q}{V_{cap}}$$

$$\Rightarrow \quad C = \frac{Q}{\left[\frac{\lambda}{2\pi\varepsilon_o}\left(\ln\left(\frac{b}{a}\right)\right)\right]}$$

$$\Rightarrow \quad C = \frac{Q}{\left[\frac{\left(Q/L\right)}{2\pi\varepsilon_o}\left(\ln\left(\frac{b}{a}\right)\right)\right]}$$

$$\Rightarrow \quad C = \frac{2\pi\varepsilon_o L}{\left(\ln\left(\frac{b}{a}\right)\right)}$$

$$\Rightarrow \quad C/L = \frac{2\pi\varepsilon_o}{\left(\ln\left(\frac{b}{a}\right)\right)}$$

In other words, we don't really determine the capacitance of a coaxial cable. What we really derive is the capacitance *per unit length* of a coaxial cable.