The Island Series:

You have been kidnapped by a crazed physics nerd and left on an island with twenty-four hours to solve the following problem. Solve the problem and you get to leave. Don't solve the problem and you don't.

The problem: You are told that a mass will be accelerated, and the question will be, "Will the *velocity change* be relatively big or relatively small." You respond with a, "What the hell, how should I know," at which time your captor says, "Oh, yeah, OK, well, I'll let you ask two questions before giving your answer (but not "is the velocity change big or small") . . . knowing that I (the captor) had a bad experience with kinematics when young and any allusion to that approach will outrage me." What two questions should you ask?

Solution to Island Problem

What *does* govern how much *velocity change* a body experiences under the influence of a force? The two parameters that will matter are:

> *The magnitude of the force* (the bigger the force, the larger the velocity change will likely be); and
>
> *The distance over which the force acts* (the farther the force acts, the more the body will pick up speed);

Except there is a problem with this as stated. We will take a look at what it is shortly.

CHAPTER 7:
Work and Energy

To date, you have seen two approaches to problem-solving in this class:

1.) *Kinematics* says: that if a body's acceleration is constant, look to see what information you are given, look to see what you are trying to determine, then find a kinematic equation that has everything you know along with what you are trying to determine. I call it *idiot physics* because you can be an idiot and do just fine with it. All it really is is pattern recognition.

2.) *Newton's Laws say:* if a body experiences a net force along some line, that force will be proportional to the acceleration of the body along that line with the proportionality constant being the body's *mass*. It is a considerably more powerful approach than kinematics as considerably less information is required to make it work.

We are now ready to look at the world from a completely different perspective, one in which a system's *energy content* is the key. It will begin with a definition, that of *work,* from which all else will follow. First, though, some non-AP related exotica

What is Energy?

(this is not an AP-related topic)

You are out in space and you give a 1.00000 kg object a push with a constant force. What changes?

> *The acceleration won't* change as the force is constant, but the velocity will; time will; momentum will; position will.
>
> *There is* one other things that will change in this case, though not by any amount that you will notice. The body's *mass* will change (remember, *mass* is a relative measure of a body's inertia).

In fact, a velocity/mass breakdown for your 1.0000 kg mass object is found on the next slide;

Velocity	*mass*		
zero	1.00000000000000000000000000000$\overline{0}$ kg		
100 mi/sec	1.000000145 kg		
10,000 mi/sec	1.00145 kg		
100,000 mi/sec	1.186 kg		
170,000 mi/sec	2.46 kg		
180,000 mi/sec	3.97 kg		
185,000 mi/sec	9.66 kg		1
185,900 mi/sec	30.5 kg	(this is .9995c)	3
185,999 mi/sec	304.96 kg		30
185,999.9999 mi/sec	30,496. kg	(this is .9999999995c)	3000
186,000 mi/sec	∞		

Apparently, putting energy *into* a system as low velocities shows itself by changing the body's *energy of motion* (it's *kinetic energy*) whereas putting energy into a system at velocities close to the speed of light *changes the body's mass*.

You are are familiar with the key to this rather bizarre behavior. What was the first relativistic equation you have ever learned from Einstein?

$$E = mc^2$$

It says claims is that mass and energy are different forms of the same thing.

Don't believe me? Take 1.000 grams of hydrogen and fuse it. You will end up with .993 grams of helium. Where did the missing .007 grams go? *Turned into pure energy*, enough energy to send *three-hundred and fifty*, 4000 pound Cadillacs (the old school kind) 100 miles into the atmosphere. The sun fuses 657,000,000 TONS of hydrogen into approximately 653,000,000 tons of helium *every second*. That's how it generates enough energy to heat our planet 93,000,000 miles away.

When Feynman (Nobel laureate from Caltech) was asked by me at a CAIS meeting what energy was, he said, simply, "I have no idea." And in saying that, he spoke for the physicists of the world! We know how to *use energy*, how to *store it*, how to *generate it*, how to *transfer it great distances*, but we have absolutely no idea *what it is*.

Fortunately, you don't need to know *what it is* to use the idea as a problem-solving tool, which is exactly what we are about to do in a non-relativistic setting.

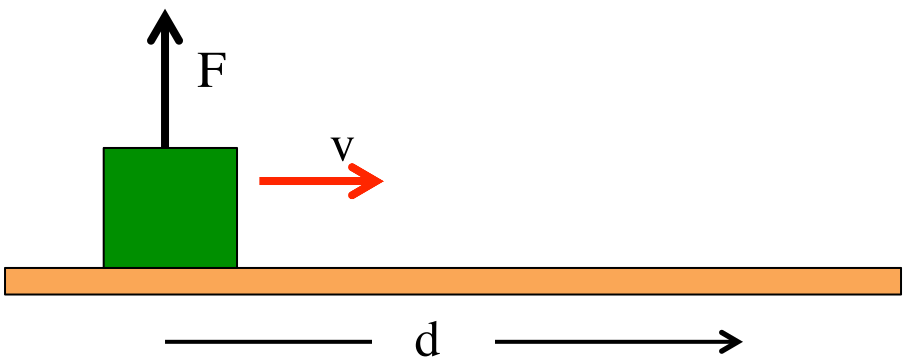
Work

So what does govern the *velocity change* a body experiences under the influence of a force at low velocities? The two parameters that will matter are:

The magnitude of the force (the bigger the force, the larger the velocity change will likely be); and

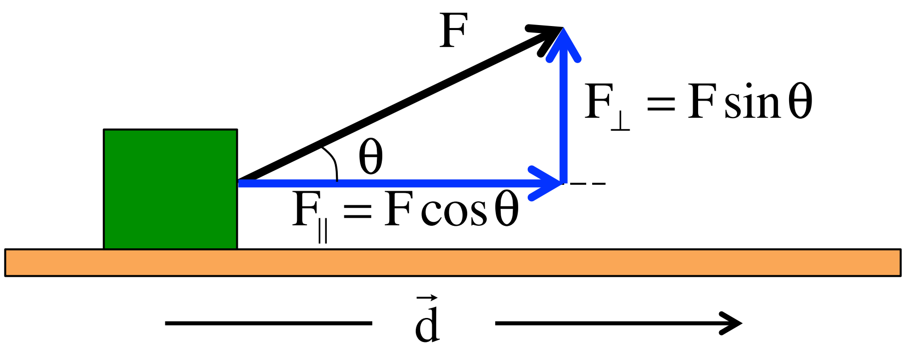
The distance over which the force acts (the farther the force acts, the more the body will pick up speed);

There is a problem with this as stated, though. Consider the following force and displacement . . . will F be changing that body's velocity as it moves across the table?



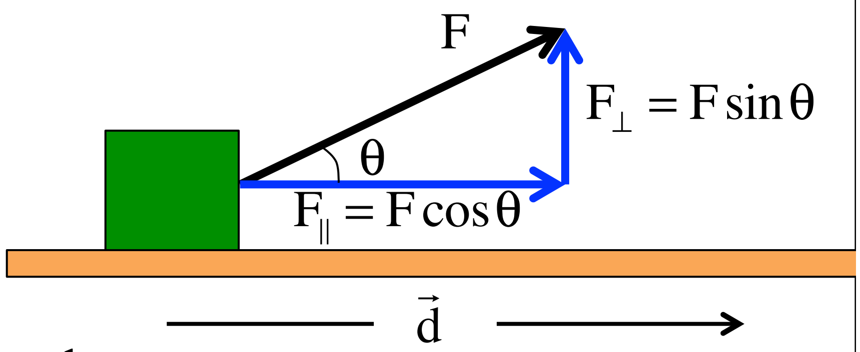
NO!!! This force will generate *NO VELOCITY CHANGE* . . . yet there's a force and displacement involved . . . so what's the deal?

To understand the problem, we need to look at a little more general situation. Consider a force oriented at an angle θ with the displacement vector. In that case, we have:



Clearly, the perpendicular component of the force $F_\perp$ *will do nothing to change the body's velocity* (assuming it doesn't yank the block off the tabletop), whereas the parallel component $F_{||}$ *WILL* effect a velocity change.

In fact, the product of $F_{||}$ and the magnitude of $\vec{d}$ will yield a number that, if large, would suggest a relatively large velocity change, and if small, a would suggest a relatively small velocity change.



This product, the product of the, *magnitude of the component-of-the-force-along-the-line-of-the-displacement* and *the magnitude-of-the-displacement* is important enough to be given a special name. It is called WORK. Formally, it is defined as:

$$W = F_{||}d$$

Note that the units are *newton-meters*, or joules (the units of energy):

Looking at the geometry in the sketch, this is also written as:

$$W = (F\cos\theta)d$$

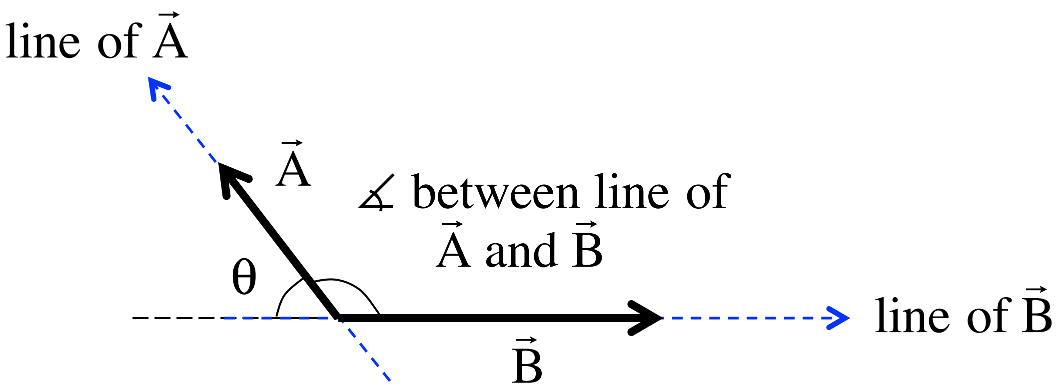
Where θ is the angle between *the line of the force* and *the line of the displacement* and *d* is the *magnitude of the displacement*.

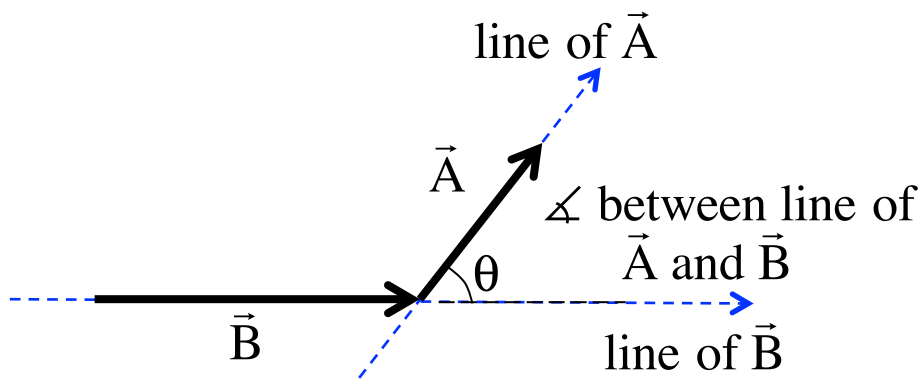
It is also not uncommon to see this quantity written as: $W = |\vec{F}|\ |\vec{d}|\ \cos\theta$

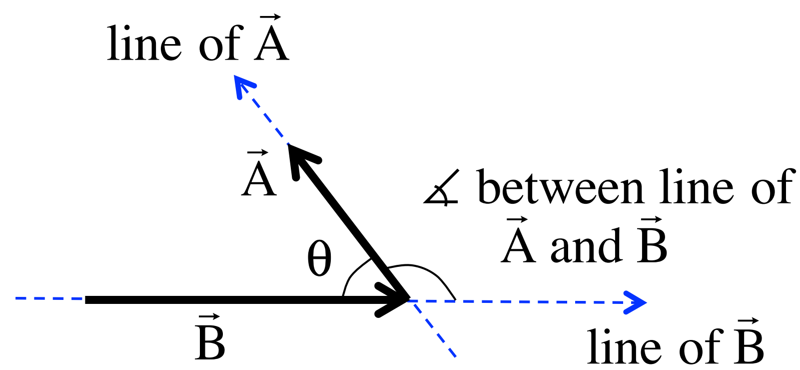
Or the *magnitude of one vector* times the *magnitude of the second vector* times the *cosine of the angle between the line of the two vectors*:

Minor note: It's important to be able to determine the *angle* between the *line of each vector*. Notice that you need a vertex to define an angle . . .

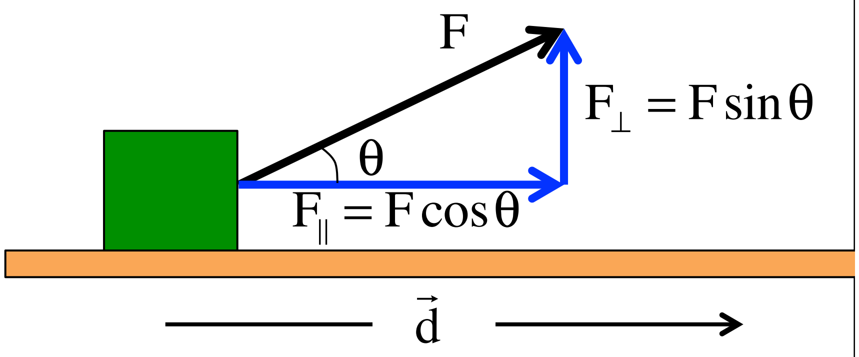
Examples:







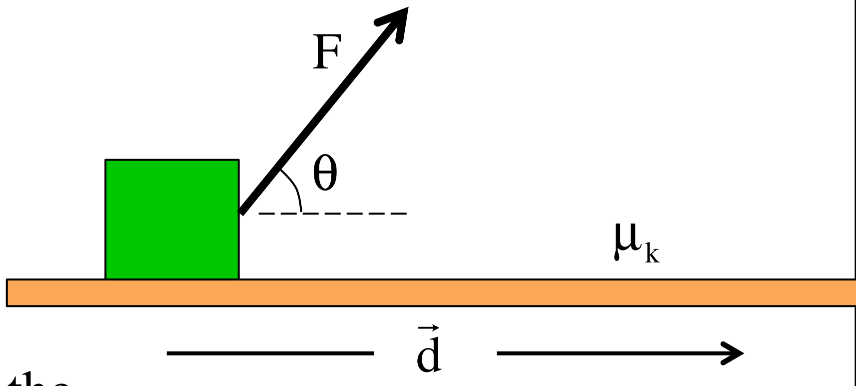
Because this kind of operation is used so often in physics (that is, multiplying the *magnitude of one vector* times the *component of the second vector along the line of the first*), the operation is given a special name and designation. It is called a DOT PRODUCT, and it's use allows us to write out the work relationship as:



$$W = \vec{F} \bullet \vec{d}$$

This is not as spooky as it looks. It is just a mathematical operation. The point is that the *dot product* between a force and displacement tells you something about how that force is motivating the body to *change its motion*.

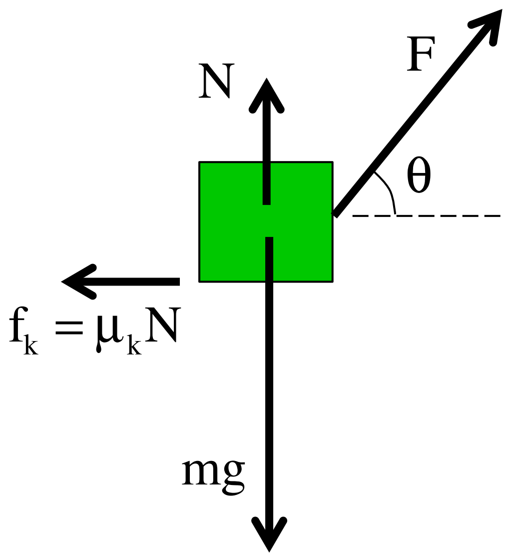
Consider: a 100-N force F applied at 60^o above the horizontal drags a 20-kg box 3.0 meters across a rough surface whose *coefficient of kinetic friction* is $\mu_k = .15$.



Derive an expression for the work done by the various forces acting on the box.

gavity:

$$\begin{aligned} W_g &= \vec{F}_g \bullet \vec{d} \\ &= \left|\vec{F}_g\right|\left|\vec{d}\right|\cos\phi \\ &= (mg)d\left(\cos 90^o\right) \\ &= 0 \end{aligned}$$

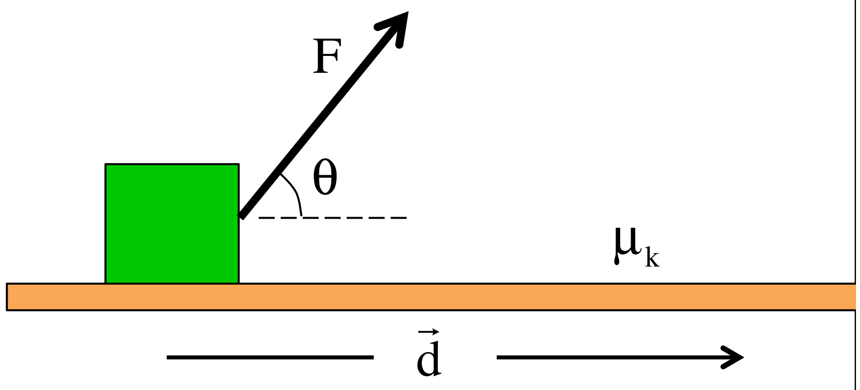


normal:

$$\begin{aligned} W_N &= \vec{F}_N \bullet \vec{d} \\ &= \left|\vec{F}_N\right|\left|\vec{d}\right|\cos\phi \\ &= (N)d\left(\cos 90^o\right) \\ &= 0 \end{aligned}$$

friction:

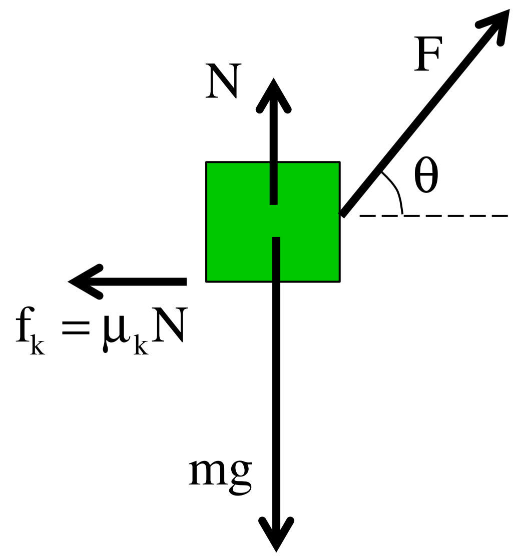
To deal with friction, we need the normal force. Using N.S.L. in the vertical:



$$\sum F_y :$$

$$N - mg + F\sin\theta = m\cancel{a_y}^{\,0}$$

$$\Rightarrow \quad N = mg - F\sin\theta$$



So $\quad W_f = \vec{F}_f \bullet \vec{d}$

$$= |\vec{F}_f||\vec{d}|\cos\phi$$

$$= (\mu_k N)d(\cos 180^\circ)$$

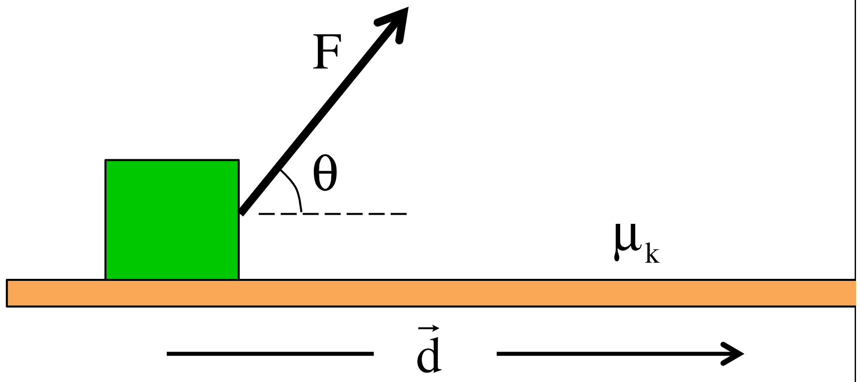
$$= (\mu_k (mg - F\sin\theta))d(\cancel{\cos 180^\circ}^{\,-1})$$

$$= -((.15)((20\text{ kg})(9.8\text{ m/s}^2) - (100\text{ N})\sin 60^\circ))(3.0\text{ m})$$

$$= -49.23\text{ nt} \bullet \text{m}$$

$$= -49.23\text{ joules}$$

Notice the units of work are *newton-meters*, which are *joules*, an energy quantity. So what must the significance of a negative work calculation like friction's be?



Negative work acts to *pull energy out of the system* (which means that positive work must act to put energy *into* a system).

F's work:

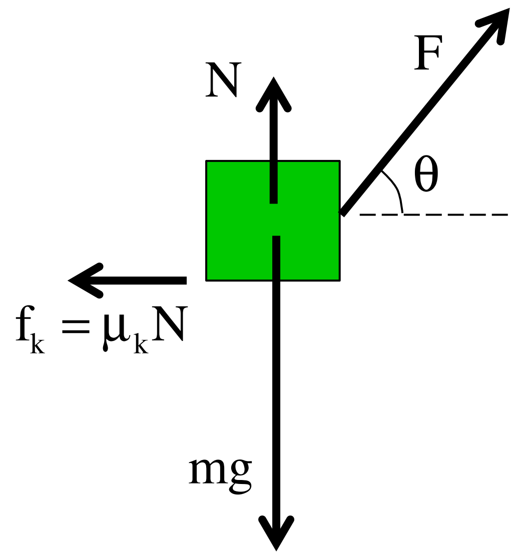
$$W_F = \vec{F} \bullet \vec{d}$$

$$= |\vec{F}||\vec{d}| \cos\phi$$

$$= (F)d(\cos 60^\circ)$$

$$= (100\text{ N})(3\text{ m})(\cos 60^\circ)$$

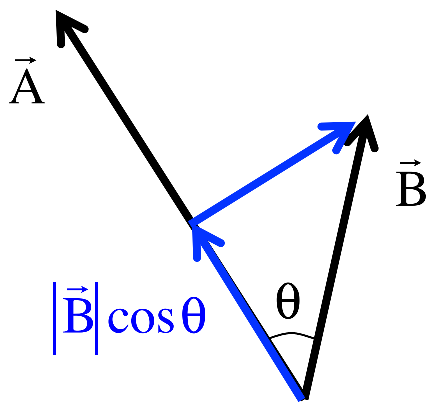
$$= 150 \text{ joules}$$



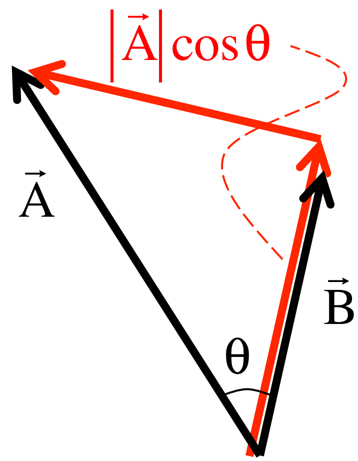
Summary of dot product characteristics:

A dot product is equal to the magnitude of one vector times the component of the second vector along the line of the first vector.

$$\vec{A}\bullet\vec{B}=\left|\vec{A}\right|\ \left(\left|\vec{B}\right|\cos\theta\right)$$



$$\vec{A}\bullet\vec{B}=\left|\vec{B}\right|\ \left(\left|\vec{A}\right|\cos\theta\right)$$



Or, simply, the magnitudes of one vector times the magnitude of the other vector times the cosine of the angle between the vectors.

But everything we've done to date has been in terms of unit vectors.

How about a dot product if the vectors are in unit vector notation? In that case:

$$\vec{A} \bullet \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

which is to say, the sum of the products of like component.

Justification:

Let $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$$\vec{A} \bullet \vec{B} = \left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right) \bullet \left(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}\right)$$

$$= A_x B_x \cancelto{1}{\cos 0^\circ} + A_x B_y \cancelto{0}{\cos 90^\circ} + \text{etc.}$$

Notice the like-terms stay and the off-terms go away, so extrapolating:

$$\vec{A} \bullet \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

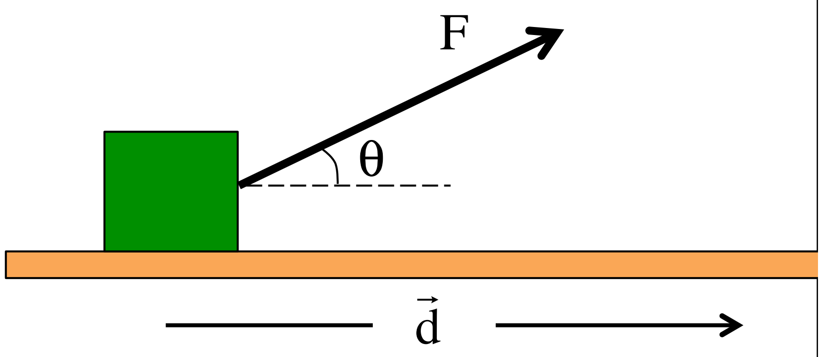
Example: if $\vec{A} = -3\hat{i} + 0\hat{j} + 5\hat{k}$ and $\vec{B} = -2\hat{i} + 3\hat{j} - 5\hat{k}$

$$\vec{A} \bullet \vec{B} = (-3)(-2) + (0)(3) + (5)(-5)$$

$$= -19$$

Work and Variable Forces

Consider a force that changes both its *direction* and *magnitude* as a body moves along the x-axis. Specifically, assume that $\vec{F} = x\hat{i} + y\hat{j}$. How much work does the force do as the body travels from x = 1 to x = 4 meters?

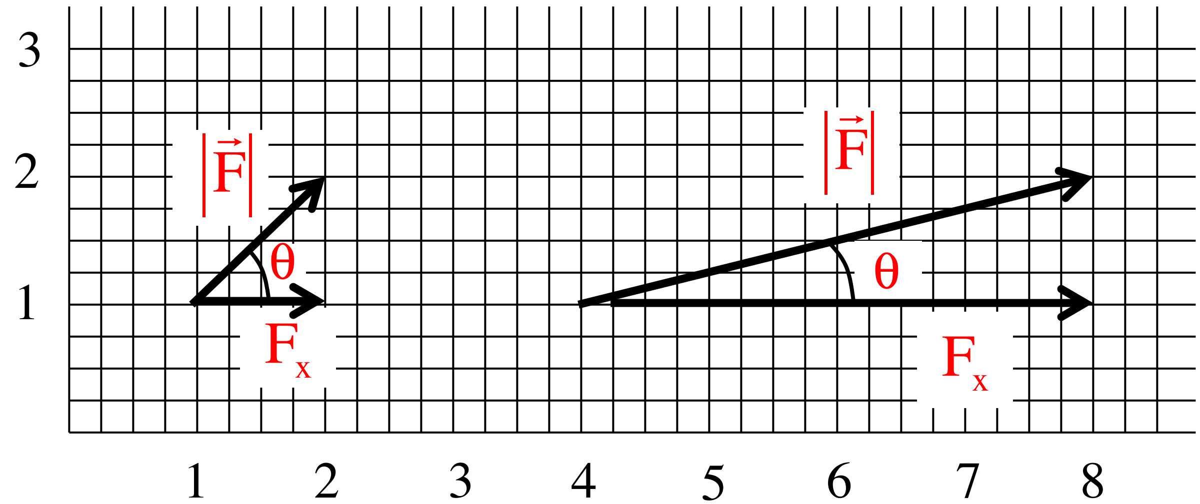


To begin with, because the force changes continuously as the body moves from point to point, we need to determine the "differential work dW" (that is, the tiny bit of the work) that is generated when the body moves through "differentially small displacement dx" along the x-axis (notice that if the displacement was large, we'd use Δx , but because it's edging toward infinitesimally small, we use "dx" instead).

We can do this one of two ways: via unit vector notation or via polar notation. We'll do both (good review as to how to do a *dot product*).

Note that a general differential displacement is denoted as: $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

Kindly notice that the force $\vec{F} = x\hat{i} + y\hat{j}$ at x=1,y=1 and x=4,y=1 looks like:



So both the angle and the *magnitude of the force* are changing as the body moves along its path. With the angle between the force and the displacement along the x-direction at any point defined as θ, we can write:

$$\cos\theta = \frac{F_x}{|\vec{F}|} \qquad \text{with} \qquad |\vec{F}| = \left(x^2 + y^2\right)^{1/2} \qquad \text{and} \quad |d\vec{r}| = dx$$

$$= \frac{x}{\left(x^2 + y^2\right)^{1/2}}$$

So using polar notation, we can write:

$$|\vec{F}| = \left(x^2 + y^2\right)^{1/2}$$

$$\cos\theta = \frac{x}{\left(x^2 + y^2\right)^{1/2}}$$

$$|d\vec{r}| = dx$$

$$\begin{aligned}
W &= \int dW \\
&= \int_{x=1}^{x=4} \vec{F} \bullet d\vec{r} \\
&= \int_{x=1}^{x=4} |\vec{F}||d\vec{r}|\cos\theta \\
&= \int_{x=1}^{x=4} \cancel{\left(x^2 + y^2\right)^{1/2}} dx \left(\frac{x}{\cancel{\left(x^2 + y^2\right)^{1/2}}} \right) \\
&= \int_{x=1}^{x=4} x\,dx \\
&= \frac{x^2}{2}\Big|_{x=1}^{x=4} \\
&= \frac{(4)^2}{2} - \frac{(1)^2}{2} = 7.5
\end{aligned}$$

And using unit vector notation, we can write:

$\vec{F} = x\hat{i} + y\hat{j}.$

$d\vec{r} = dx\hat{i}$ for motion soley in the x-direction

$$W = \int dW$$
$$= \int_{x=1}^{x=4} \vec{F} \bullet d\vec{r}$$
$$= \int_{x=1}^{x=4} \left(x\hat{i} + y\hat{j}\right) \bullet \left(dx\hat{i}\right)$$
$$= \int_{x=1}^{x=4} x\,dx$$
$$= \frac{x^2}{2}\Big|_{x=1}^{x=4}$$
$$= \frac{(4)^2}{2} - \frac{(1)^2}{2} = 7.5$$

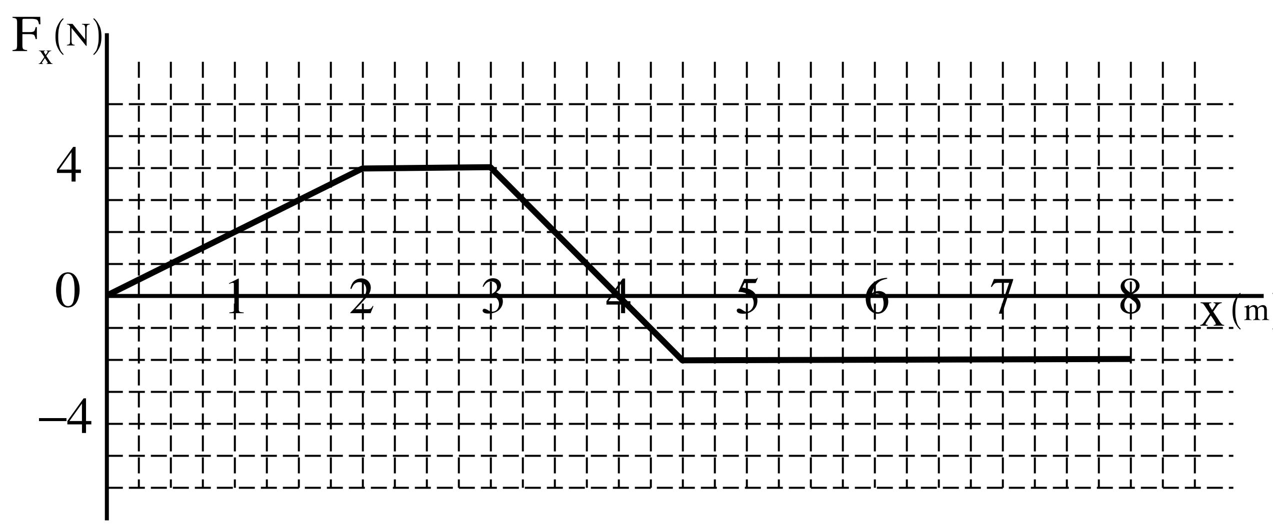
Great jumping huzzahs . . . same work value using either notation! Aint physics grand!

Force vs Displacement Graphs

As a small aside, how are *force versus displacement* graphs related to *work* quantities?

Noting that the last problem used $W = \int_{x=1}^{x=4} \vec{F} \cdot d\vec{r}$ to come to solutions, and give that integrals yields *areas under curves*, it pretty much makes sense that the amount of work a force does as a body moves in, say, the x-direction is equal to the *area under* the *force versus x-displacement* graph.

So how much work does the force graphed to the right do as the body moves from x = 1 meter to x = 6 meters?



Solution: the area under the graph above the axis is positive and below is negative, so the solution is 22 joules.

A Useful Observation

In the Island Series question, what you hopefully realized was this. First, because the question was couched in terms of *per unit mass*, the mass had nothing to do with the solution. You can as easily have had two *equal masses* in the experiment. So letting the masses be the same for simplicity, the bigger the net force along the line of motion, the bigger the velocity change, and the bigger the distance over which the net force acts, the bigger the velocity change. From those two observation should come your two questions.

Put in a little different light, what you were really asking about was the amount of WORK $F_x\left|\vec{d}\right|$ each net force did on the mass: *the greater the work*, *the greater the velocity change*! we are about to derive.

If that be the case, there must be a quantitative relationship between velocity-change and net work. What is what we are about to derive.

Derivation of the WORK/ENERGY THEOREM

It's time to link up *work* with *velocity change*. In it's most general, differential form, *net work* is defined as:

$$W_{net} = \int \vec{F}_{net} \bullet d\vec{r}$$

But with Newton's Second Law, we can write:

$$\begin{aligned} W_{net} &= \int \vec{F}_{net} \bullet d\vec{r} \\ &= \int (m\vec{a}) \bullet d\vec{r} \end{aligned}$$

Assuming, for simplicity, that the body is moving in the x-direction and that the the force is also in that direction, and noting the acceleration is the time derivative of velocity, we can expand the dot product to read:

$$\begin{aligned} W_{net} &= \int (m\vec{a}) \bullet d\vec{r} \\ &= \int \left(m \frac{dv}{dt} \right) (dx) \cancelto{1}{\cos 0^\circ} \end{aligned}$$

Noticing that the displacement *dx* equals the velocity *dx/dt* time the time interval *dt* over which the displacement occurs, we can write:

$$W = \int \left(m\frac{dv}{dt} \right)(dx)$$
$$= \int \left(m\frac{dv}{dt} \right)\left[\frac{dx}{dt} dt \right]$$

Executing the ever present *canceling of variables* (a ghastly moan coming out of the math department as we do), noting that dx/dt is “v” and integrating over a beginning and ending velocity, we can write:

$$W = \int \left(m\frac{dv}{\cancel{dt}} \right)\left[\frac{\cancel{dx}}{\cancel{dt}} \cancel{dt} \right] \quad (\tfrac{dx}{dt} = v)$$
$$= \int_{v=v_1}^{v_2} mv\,dv$$
$$= m\frac{v^2}{2}\bigg|_{v=v_1}^{v_2}$$
$$= \frac{1}{2}m{v_2}^2 - \frac{1}{2}m{v_1}^2$$

This $\frac{1}{2}mv^2$ *quantity* is deemed important enough to be given a special name. It is called *KINETIC ENERGY*, and the relationship between it and the net work done on a body by all the forces acting on the body as it moves is called the *WORK/ENERGY THEOREM.*

THE WORK/ENERGY THEOREM'S many forms:

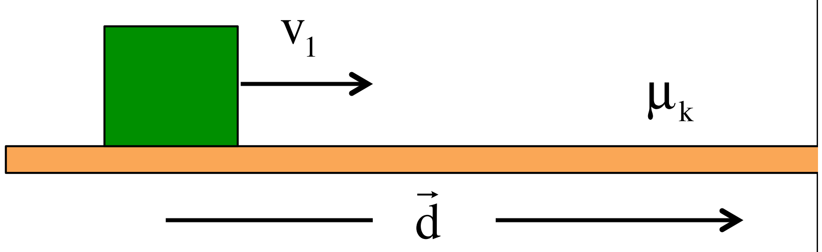
$$W_{net} = \Delta KE$$

$$W_{net} = KE_2 - KE_1$$

$$W_{net} = \frac{1}{2}mv_2{}^2 - \frac{1}{2}mv_1{}^2$$

Classic Work/Energy Problem

How far will a block of mass *m* moving with velocity v_1 travel over a frictional surface whose coefficient of kinetic friction is μ_k before coming to rest?



Using the *Work/Energy Theorem*: Noting that the frictional force will be $\mu_k N$, where *N* is just *mg* and where that force is directed *opposite the direction of motion* (this means the work friction does will be negative) and the final velocity is *zero*, we can write:

$$W_{net} = KE_2 - KE_1$$

$$\Rightarrow \quad W_f = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$\Rightarrow \quad \vec{f} \bullet \vec{d} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

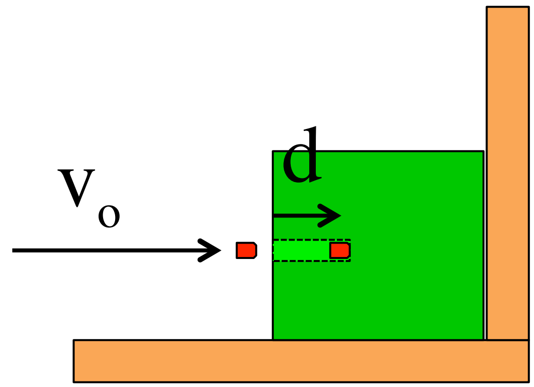
$$\Rightarrow \quad (\mu_k mg)d(\cos 180^\circ) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \quad (\cos 180^\circ = -1,\ v_2 = 0)$$

$$\Rightarrow \quad -(\mu_k mg)d = -\frac{1}{2}mv_1^2$$

$$\Rightarrow \quad d = \frac{v_1^2}{2\mu_k g}$$

Note: when the net work done is *NEGATIVE*, energy is pulled *out of the system* and the body slows down.

Another Problem: An 8 gram bullet moving with velocity $v_o = 550$ m/s embeds itself into a block of wood burying itself 4.5 centimeters before coming to rest.



a.) Using the idea of *energy*, how much force, on average, was required to do this?

$$W_{net} = KE_2 - KE_1$$

$$\Rightarrow \quad \vec{f}_{collision} \cdot \vec{d} = \frac{1}{2}m\cancelto{0}{v_2}^2 - \frac{1}{2}mv_1^2$$

$$\Rightarrow \quad fd\left(\cancelto{-1}{\cos 180^\circ}\right) = -\frac{1}{2}mv_1^2$$

$$\Rightarrow \quad f = \frac{mv_1^2}{2d}$$

b.) Was the work done on the bullet positive or negative? (negative)

c.) Where did the energy of motion go as the bullet came to a stop?

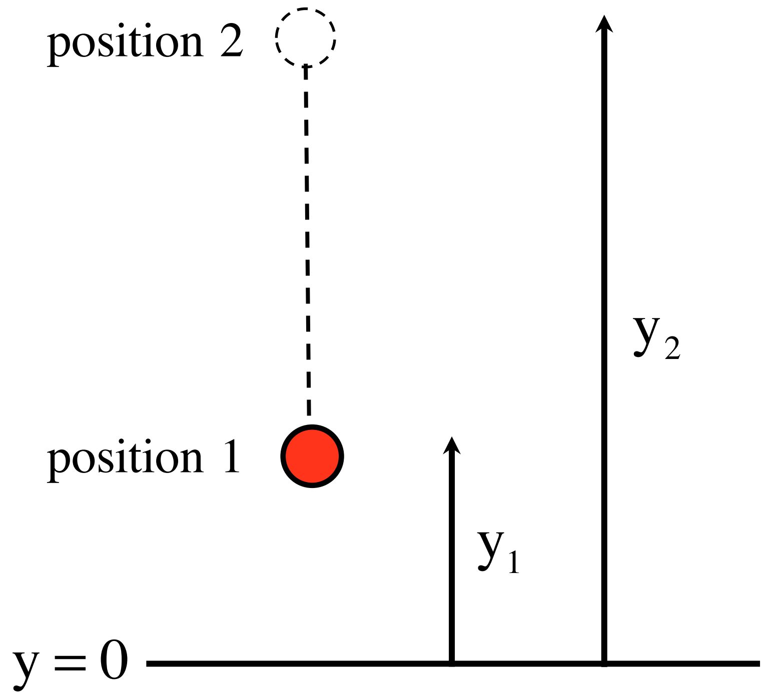
(burrowing hole, sound and heat)

Bit of Slight of Hand

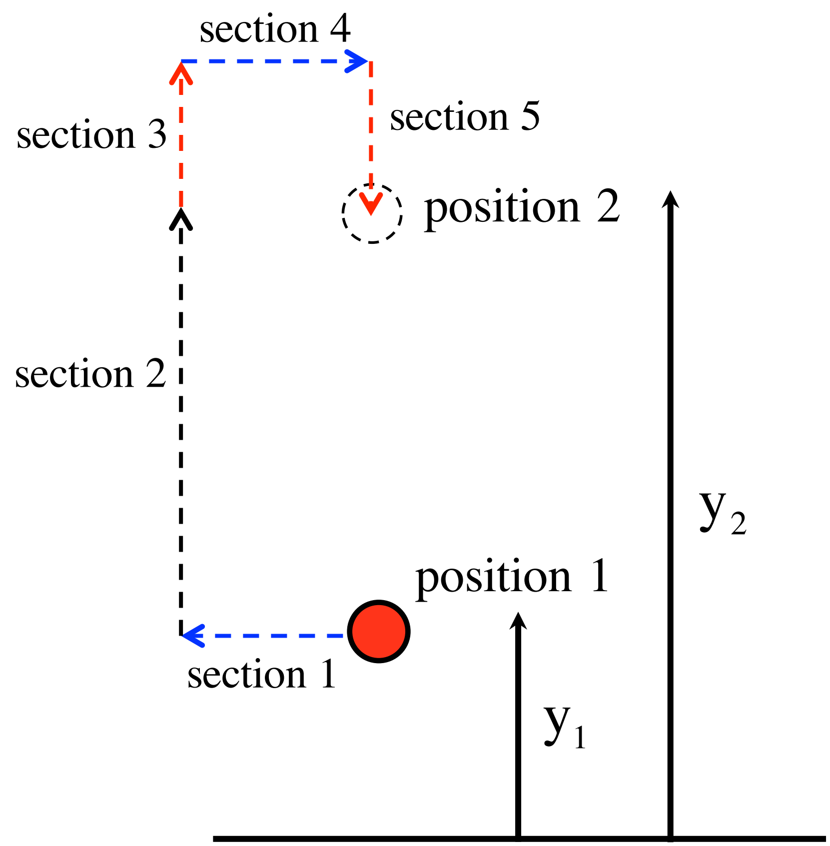
A ball moves from *position 1* at y_1 to *position 2* at y_2 as shown on the sketch. How much *work* does *gravity do* in the process?

We can derive the work quantity using the standard $\vec{F} \cdot \vec{d}$ approach and noting that the direction of the gravitational force is *opposite* the direction of the displacement of the ball as it moves upward. Doing so yields:

$$
\begin{aligned}
W_{grav} &= \vec{F}_g \cdot \vec{d} \\
&= \left|\vec{F}_g\right|\left|\vec{d}\right| \cos\theta \\
&= (mg)(y_2 - y_1)\cancelto{-1}{\cos 180^\circ} \\
&= -(mgy_2 - mgy_1)
\end{aligned}
$$



We next need to derive the amount of work gravity does on the ball as it traveled from *position 1* to *position 2* along the convoluted path shown to the right. Note that I've broken the sections into mini-sections for easy analysis.



--in section 1 and 4, gravity does no work done as the force is perpendicular to the displacement through those sections;

--in section 3, gravity's work is negative (force down, motion up) whereas *in section 5,* its work is the same magnitude but positive, so these two add to zero.

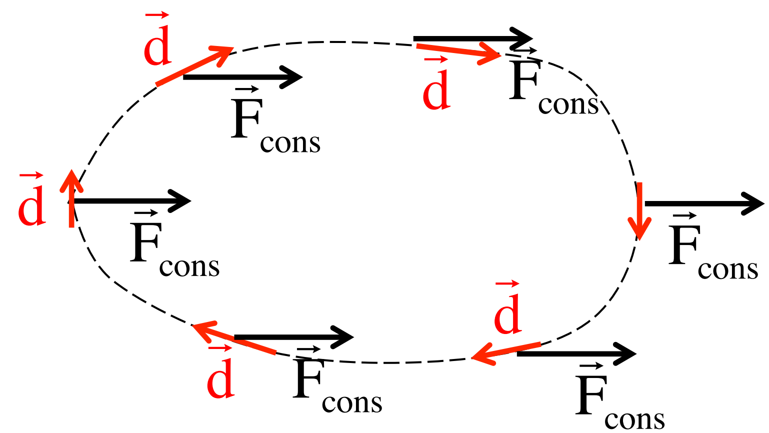
--in section 2, gravity does negative work in the amount calculated in the previous section . . .

Conclusion: *All that matters* when doing a work calculation with gravity is where you start and where you end with the *path making no difference at all*.

Force fields that act this way, whose work calculations are PATH INDEPENDENT, are called *CONSERVATIVE FORCES*.

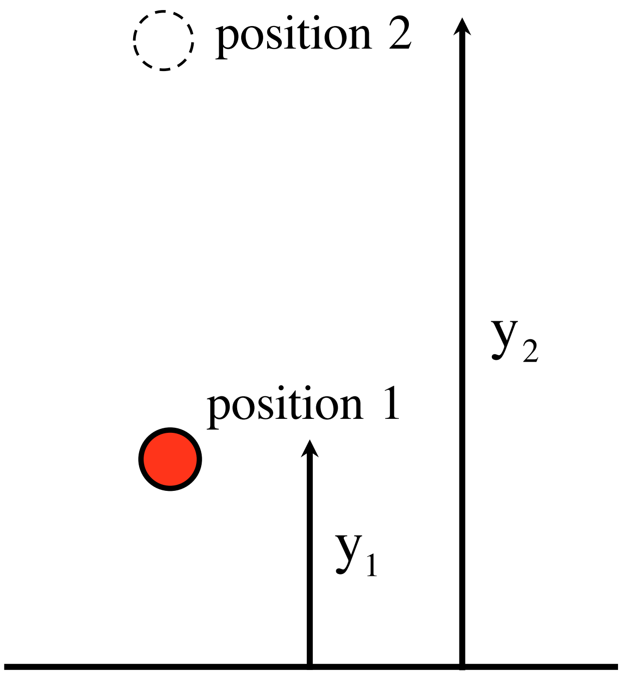
Note that a *conservative force* does NO NET WORK around a closed path (another example beyond the visual shown to the right is the work gravity would do as the ball went from position 1 *out and back* to position 1).

This means non-conservative forces do work that *is not* path dependent. The classic example of this is *friction*.



Pick any point and begin summing the work quantities back around to that point. If the force is *conservative,* the sum will be zero (same amount of positive as negative work done, net)

If all that matters is where you start and where you end when doing a work calculation for a *conservative forces* like gravity, wouldn't it be cool if we could define a function U that would assign a number to y_1 and a number to y_2 that would be such that when you took the difference between the two, you got the amount of work done by the field as you proceeded between the two points? (And the answer to that question is, "Yes, definitely very cool.")

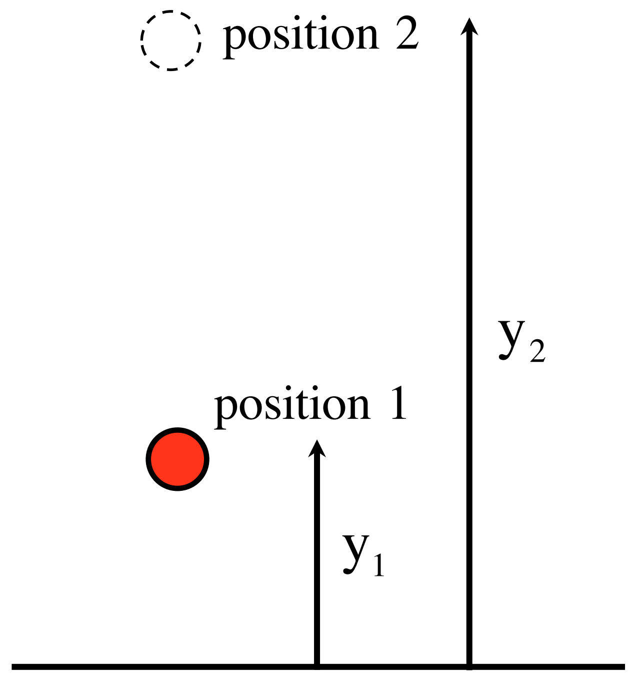


Well, that's exactly what we stumbled onto when we did the work calculation using the conventional $\vec{F} \cdot \vec{d}$ approach on on gravity. Because looking at our solution, we got:

$$W_{grav} = -(mgy_2 - mgy_1)$$

with mgy looking very much like such a function (almost—there's a negative sign out in front of the difference, but that's OK, the general idea still holds)

Functions that do this kind of thing, that allow you to determine how much work a conservative force field does on a body moving from one point to another in the field simply by evaluating the function at the endpoints of the displacement, are called *POTENTIAL ENERGY FUNCTIONS*.



The symbol used for potential energy functions is a U, and the *potential energy function* for gravity when near the earth's surface is:

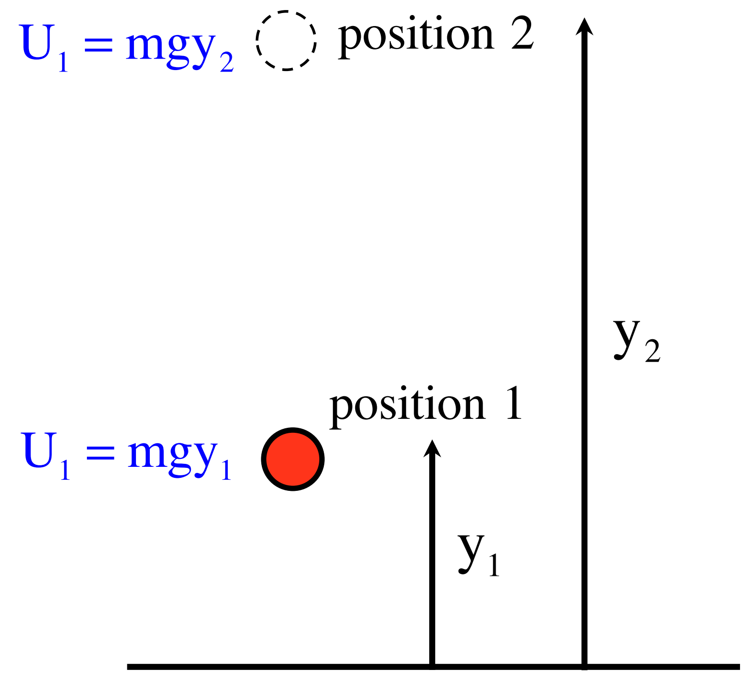
$$U_{grav} = mgy$$

Furthermore, the relationship between *potential energy functions* and *work* calculations is:

$$W_{cons.force} = -\Delta U$$

So using our *potential energy function* for gravity (near the surface of the earth) on our ball problem, we could write:

$$\begin{aligned} W_{grav} &= -\Delta U \\ &= -(U_2 - U_1) \\ &= -(mgy_2 - mgy_1) \end{aligned}$$

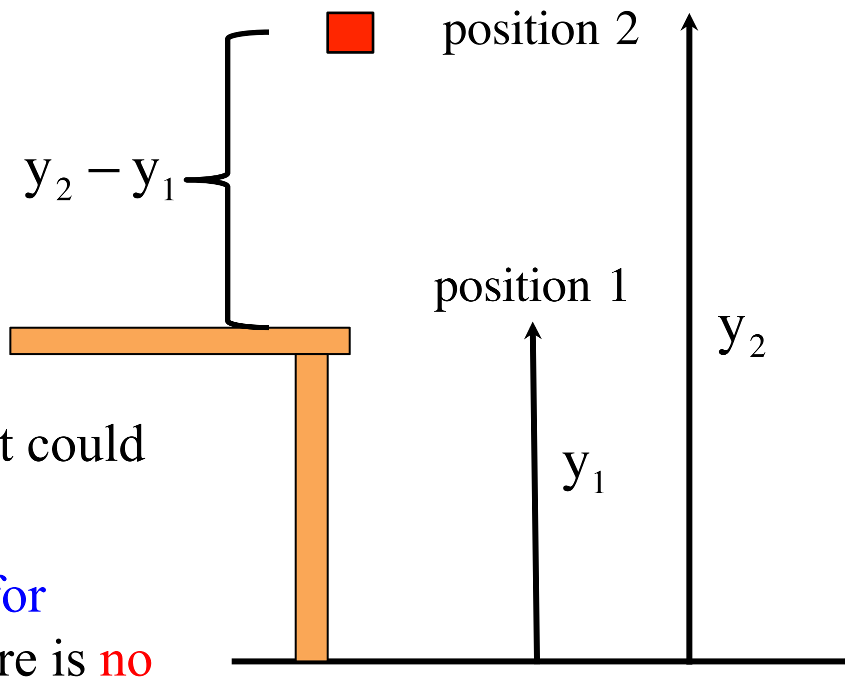


In short, if you know a force field's *potential energy function*, in almost all instances it's a LOT easier to do a work calculation for the force using its potential energy function than trying to get the result using $\vec{F} \cdot \vec{d}$.

What's important to understand is that *potential energy* is a mathematical contrivance created to do ONE THING and ONE THING ONLY—*to determine how much work a conservative force does* on a body moving from one point to another in the field.

The ONLY THING you will ever use a *potential energy function* for will be to do WORK CALCULATIONS, and the only way you will ever use that function will be by taking the difference of the *evaluation of the function* at the motion's endpoints, then sticking a negative sign in front of that value.

So let's see how well you've understood this. A block is about to fall from the position shown to the level of the table top. Little Billy says the block starts out with potential energy mgy_2. Little Missie says, "Oh, no, it has potential energy equal to $mg(y_2 - y_1)$. Who is right?

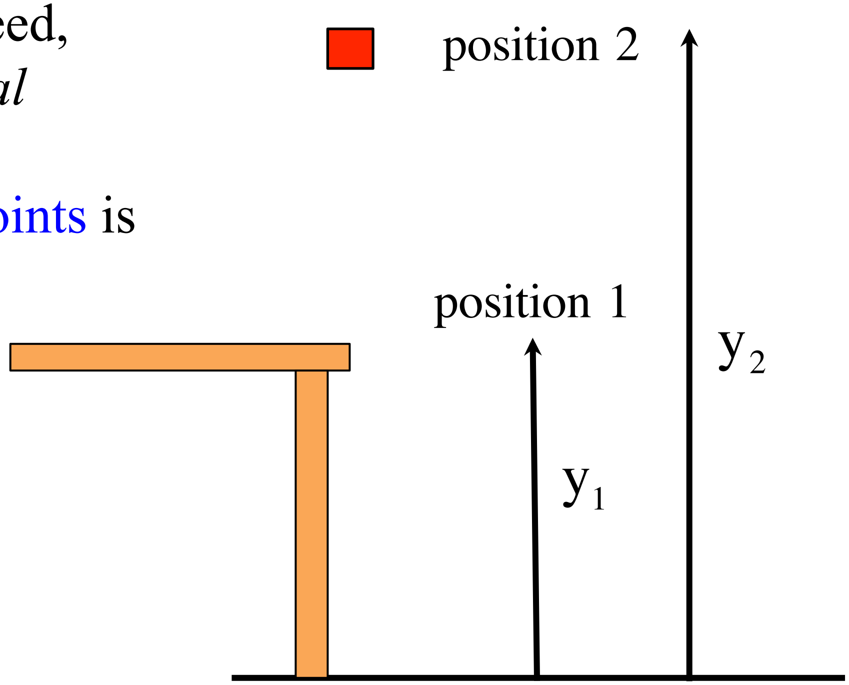


Answer: Both are making statements that could be correct. How so?

As there is no preferred *F = 0* position for gravity near the surface of the earth, there is no preferred position to make the gravitational potential energy equal to zero. You can see that in its definition. *Mgy* is dependent upon where you put $y = 0$, which is dependent upon where how you define your coordinate axis.

As the axis is currently set up with y = 0 at the floor, the potential energy at *position 2* is, indeed, $U_2 = mgy_2$ as Little Billie stated. But *potential energy functions* are only meaningful in pairs because *minus* their difference between two points is what is related to *work* calculations.

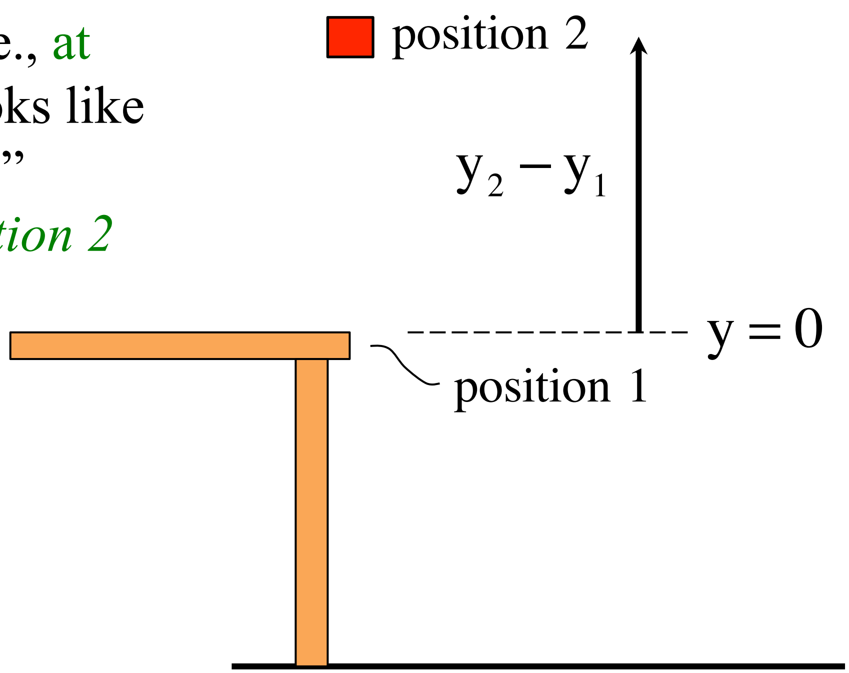
Using Little Billie's defined value as it was meant to be used, then, the work done on the body as it went from *position 2* to *position 1* becomes:



$$W_{grav} = -(U_2 - U_1)$$
$$= -(mgy_1 - mgy_2)$$
$$= (mgy_2 - mgy_1) \qquad \text{(done for comparison later)}$$

Little Billie seems vindicated.

But Little Suzie is an iconoclast. She decides to make her $y = 0$ position the top of the table (i.e., at position 1). So in her rendition, her sketch looks like the one shown to the right and her "work done" calculation for the body as it moves from *position 2* to *position 1* looks like:

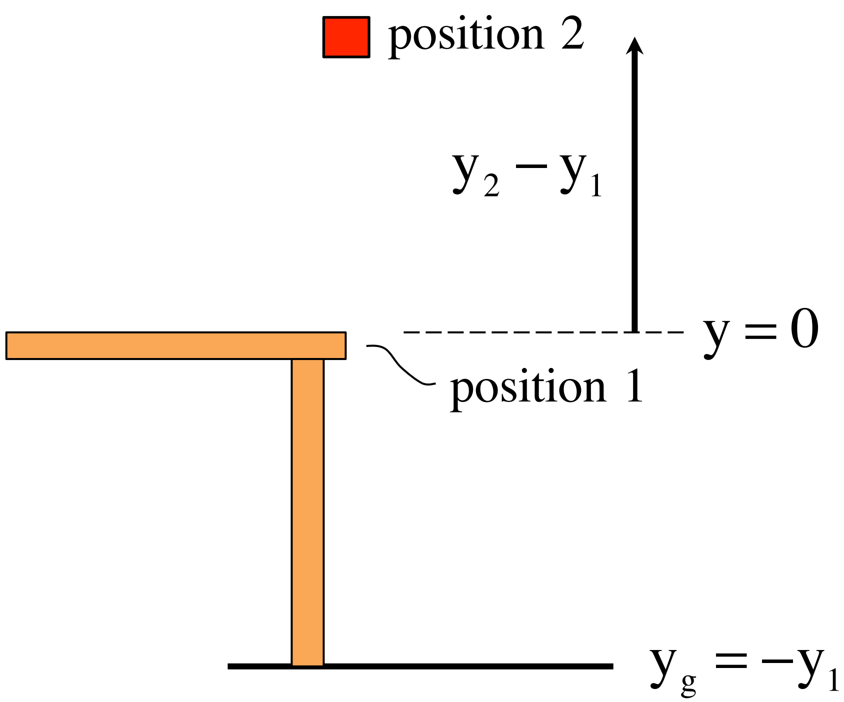


$$
\begin{aligned}
W_{grav} &= -(U_2 - U_1) \\
&= -(0 - mg(y_2 - y_1)) \\
&= (mgy_2 - mgy_1)
\end{aligned}
$$

This is the same value Little Billie got in his calculation (shouldn't be surprising as the motion hasn't changed even if the coordinate axis has).

Conclusions:

1.) Although it is commonly done, making statements like, "The body at *position 2* has potential energy in the sense that it can potentially pick up kinetic energy" is a little bit dangerous. Why? Because if we take the zero level to be the tabletop, the gravitational potential energy at that point, BY DEFINITION, is ZERO. With that, it makes no sense to claim that there is NO potential for a body to pick up kinetic energy if it were to drop from that level to the ground. In fact, that work calculation would look like:



$$
\begin{aligned}
W_{grav} &= -\left(U_g - U_1\right) \\
&= -\left(mg\left(-y_1\right) - 0\right) \\
&= mgy_1
\end{aligned}
$$

It's time to talk energy concepts:

1.) From a purely theoretical perspective, idea of *potential energy* is a mathematical contrivance. It is used to aid in work calculations and that's it.

2.) At least in situations in which there is no preferred *F = 0 point* (like gravity near the surface of the earth), the coordinate axis can be set ANYWHERE CONVENIENT, which means trying to place physical significance on the quantity can get you in trouble (hence, the last example).

3.) *That doesn't mean* you don't know anything about a system when given potential energy information.

a.) *What do you know* if I tell you the potential energy difference between two points in a force field is 6 joules?

You know the field will do -6 joules of work on any body moving between the points. How so? Because you know that $W_{consforce} = -\Delta U = -6$ joules.

b.) *What do you know* if I tell you the potential energy at a point in a force field is -6 joules?

Nothing unless you know the field has a dedicated U = 0 point. If then, the negative sign may tell you the body is "bound" to the system (see energy diagrams), and even then there are bounded systems that don't act like that.

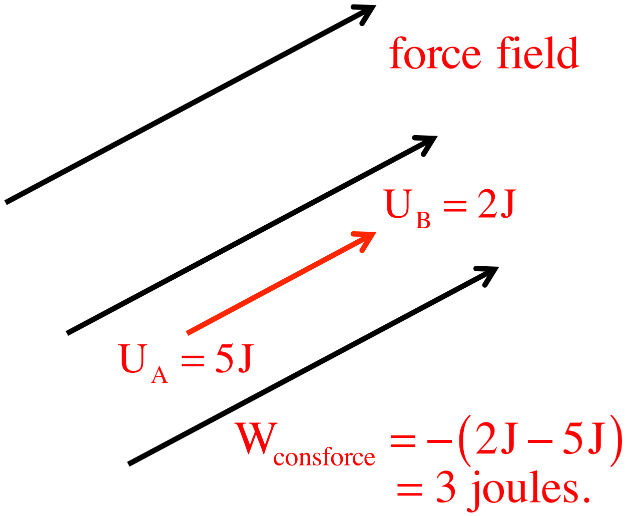
3.) That doesn't mean you don't know anything about a system when given potential energy information (con't).

c.) We know that if left to their own devices, systems in nature migrate from higher potential energy states to lower potential energy states.

--Electrons in atomic orbitals respond to electrical forces by spontaneously cascading from higher energy state "outer orbitals" to lower energy state "inner orbitals" naturally;

--Water runs down hill.

d.) We know from the math that when a body moves with a force field, it moves from higher potential energy to lower potential energy with the field doing *positive work* (hence the need for the negative sign in the work/potential energy relationship).



--An interesting observation is that when positive work is done and energy is put into the system (in most cases showing itself with an increase of energy of motion, or kinetic energy), the potential energy of the body goes DOWN. From the perspective of energy transfer, it is as though the potential energy converts itself into kinetic energy.

Deriving a Potential Energy Function from a Force Field Function

Think back to how we got the *potential energy function* for gravity. We determine the amount of work done by gravity as a body moved through a gravitational field using $\vec{F} \bullet \vec{d}$:

$$\begin{aligned} W_{grav} &= \vec{F}_g \bullet \vec{d} \\ &= \left|\vec{F}_g\right|\left|\vec{d}\right|\cos\theta \\ &= (mg)(y_2 - y_1)\cos 180^o \\ &= -(mgy_2 - mgy_1) \end{aligned}$$

then compared that final result to

$$\updownarrow$$

$$W_{grav} = -(U_2 - U_1)$$

To conclude that:

$$\updownarrow$$

$$U_{grav} = mgy$$

Putting everything together, we effectively executed the following operation:

$$W = \vec{F}_g \bullet \vec{d}$$
$$= -\Delta U$$

Rearrange, we could write this out as:

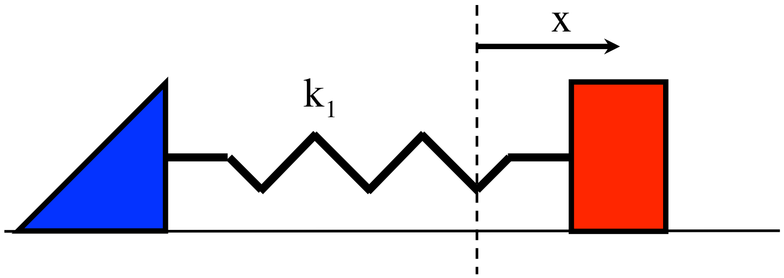
$$\Delta U = -\vec{F}_g \bullet \vec{d}$$

This is fine and swell as long as the angle between $\vec{F}$ and $\vec{d}$ doesn't change, and as long as $\vec{F}$ is constant. If, on the other hand, there is any variability, you have to use:

$$\Delta U = -\int \vec{F} \bullet d\vec{r}$$

Example 1: An ideal spring oriented along the x-axis carries a force function:

$$\vec{F} = -k_1 x\hat{i}$$



where k_1 is the spring's *spring constant*—this tells you how much *force per unit distance* is required to elongate or compress the spring . . . a big spring constant means a stiff spring—and *x* is the displacement of the spring from its equilibrium position. In the case of a spring, equilibrium is where the spring force is ZERO.

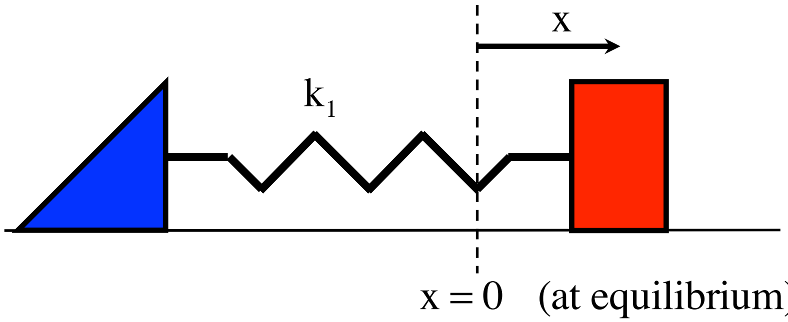
Because we have a preferred position where the spring force $F = 0$ (i.e., at $x = 0$), *that* is where we will define our *potential energy function* to be ZERO. Determining the work the force does as we go from that point to an arbitrary point looks like:

$$\Delta U = -\int \vec{F} \bullet d\vec{r}$$

$$\Rightarrow \quad U(x) - \underbrace{U(x=0)}_{\equiv 0} = -\int_{x=0}^{x} \left(-k_1 x\hat{i}\right) \bullet \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$\Rightarrow \quad U(x) = \int_{x=0}^{x} (k_1 x)\,dx$$

Doing: the integral yields a potential energy function for an ideal spring of:



$$U(x) = \int_{x=0}^{x} (k_1 x)dx$$
$$= \frac{1}{2}kx^2$$

You would use this as you would any other potential energy function:

Example: Determine how much work the spring does to the mass as the spring moves the mass from A/2 to –A/4, where A is the maximum displacement.

$$W_{spring} = -(U_2 - U_1)$$
$$= -\left(\frac{1}{2}k_1\left(-\frac{A}{4}\right)^2 - \frac{1}{2}k_1\left(\frac{A}{2}\right)^2\right)$$
$$= -\frac{1}{2}k_1\left(-\frac{3A^2}{16}\right) = \frac{3}{32}k_1A^2$$

Determination of Spring Constant

Side note: How do you determine a *spring constant*? (This is something you will need to be able to do in the future.)

The spring constant tells you how much force is required to elongate or depress a spring a given amount, so apply a force, measure the amount of elongation produced by the force and divide the one into the other. That is:

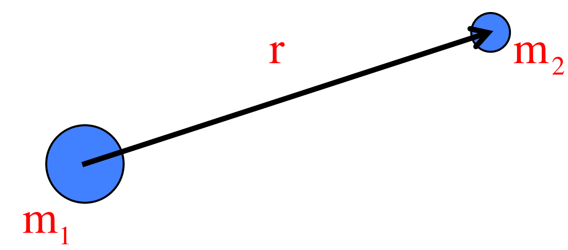
1.) Attach a spring balance (a force balance) to the spring, pull until the spring is a predetermined length, record the force required, then divide that force by the distance of elongation to get k. Or:

2.) Hang a mass m from the spring in the vertical. Divide the distance the spring elongates into the mass's weight (mg). That will also give you the amount of *force per unit distance required to elongate the spring*.

Example: What's the spring constant if a .4 kg hanging mass elongates a spring 1.2 meters?

$$k = \frac{F}{y} = \frac{mg}{y} = \frac{(.4\text{ kg})(9.8\text{ m/s}^2)}{(1.2\text{ m})} = 3.27\text{ N/m}$$

A second example: (Note that this is given as a *radial* vector.) Away from the earth's surface, the gravitational force that m_1 exerts on m_2 is given by :



$$\vec{F}_g = -G\frac{m_1 m_2}{r^2}\hat{r}$$

where G is a constant. Determine the force's *potential energy function*.

The first thing to do is to decide if there is a preferred F = 0 point, as that should be where you would put your potential energy equal to zero point.

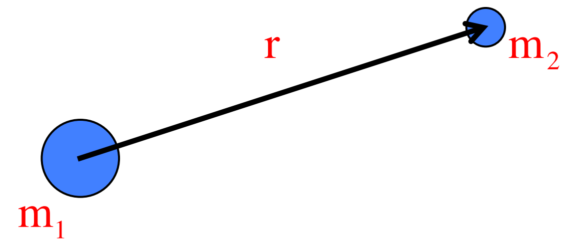
In this case, this is out at infinity. Additionally noting that in *radial unit vector notation*, $d\vec{r} = (dr)\hat{r}$, we can write:

$$\Delta U = -\int \vec{F} \bullet d\vec{r}$$

$$\Rightarrow \quad U(r) - \underbrace{U(r = \infty)}_{\equiv 0} = -\int_{r=\infty}^{r}\left(-G\frac{m_1 m_2}{r^2}\vec{r}\right)\bullet (dr\hat{r})$$

$$\Rightarrow \quad U(r) = \int_{x=0}^{x}\left(G\frac{m_1 m_2}{r^2}\right)dr$$

As before, doing the integral yields a *potential energy function* for gravity far from the earth's surface of:



$$U(r) = \int_{r=\infty}^{r} \left(G\frac{m_1 m_2}{r^2} \right) dr$$

$$= -G\frac{m_1 m_2}{r}\Big|_{r=\infty}^{r}$$

$$= -\left(G\frac{m_1 m_2}{r} - G\frac{m_1 m_2}{\infty} \right)$$

$$= -G\frac{m_1 m_2}{r}$$

Interesting note: This potential energy function is NEGATIVE. No worries. As long as you use it the way potential energy functions are supposed to be used (i.e., to calculate *work* quantities), it will work just fine . . . no pun intended.

Deriving a Force Function *from* a Potential Energy Function

According to the math, if a body moves a *differentially small distance* over a path in a force field, it will have a differentially small amount of work *dW* done to it equal to:

$$dW = \vec{F} \bullet d\vec{r} = -dU$$

If both the force and motion happen to be in the x-direction, this would mean that

$$\vec{F} \bullet d\vec{r} = -dU$$

$$\Rightarrow \quad F_x dx = -dU$$

It would also mean that:

$$F_x = -\frac{dU}{dx}$$

This is the relationship uses to determine a *potential energy's force function*.

Except life is never that simple. Forces aren't just in the x-direction. So how do we notationally deal with multiple-directional forces?

We want a force expression that, in its most general form, looks like:

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

According to our potential energy relationship with force, this will become:

$$\vec{F} = \left(-\frac{\partial U}{\partial x}\right)\hat{i} + \left(-\frac{\partial U}{\partial x}\right)\hat{j} + \left(-\frac{\partial U}{\partial x}\right)\hat{k}$$

Where the symbol $\partial U/\partial x$ is called the *partial derivative with respect to "x,"* and is used when taking a derivative of a multi-dimensional function (U) with respect to ONE VARIABLE ONLY—in this case, with respect to a change in the x-direction.

And this is where it gets freaky, and non-AP (though you will certainly see it if you ever take a physics course in college). There is a differential vector operator called *the del operator* whose symbol looks like $\vec{\nabla}$ and that is equal to:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

The beauty of this operator, being a vector, is that you can use it to operate on other vectors. So, for instance, use can *cross* a del operator into a vector (the cross product being an operation you will learn more about when we talk about torque and rotational motion)

$$\vec{\nabla} x \vec{F}$$

producing what is called *the curl*, a vector unto itself that tells you something about the way vector F *circulates* around a point of interest.

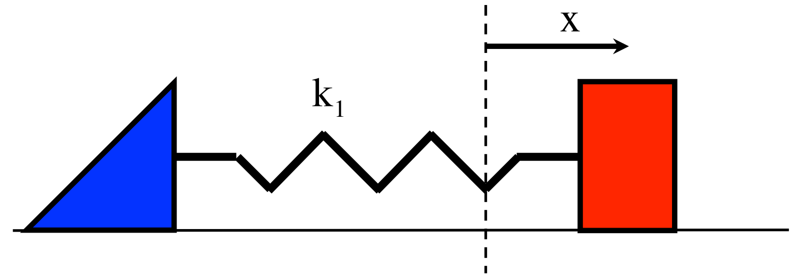
Our needs are simpler. All we require is what is called *the gradient*. That is the del operator acting on a *scalar function* (like our *potential energy function* U). It produces a vector whose magnitude is the *maximum SPATIAL rate of change of the scalar* and whose direction is the *direction* of that maximum change. In any case, using:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right)$$

we can write:

$$\begin{aligned}
\vec{F} &= \left(-\frac{\partial U}{\partial x} \right)\hat{i} + \left(-\frac{\partial U}{\partial y} \right)\hat{j} + \left(-\frac{\partial U}{\partial z} \right)\hat{k} \\
&= -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) U \\
&= -\vec{\nabla} U
\end{aligned}$$

Example 3: An ideal spring oriented along the x-axis has a potential energy function of: $U = \frac{1}{2}k_1x^2$



where k_1 is the spring's *spring constant*—this tells you how much *force per unit distance* is required to elongate or compress the spring . . . a big spring constant means a stiff spring—and x is the displacement of the spring from its equilibrium position. In the case of a spring, equilibrium is where the spring force is ZERO.

What is the force function that goes along with this *potential energy function*?

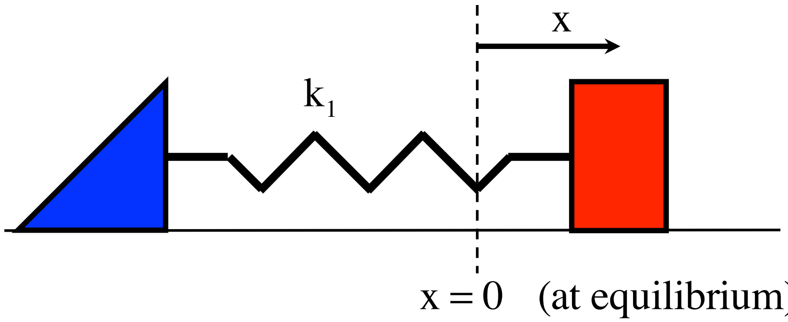
Doing this the reasonable (and acceptable) way:

$$F_x = -\frac{\partial U}{\partial x}\hat{i}$$

$$= -\frac{\partial\left(\frac{1}{2}k_1x^2\right)}{\partial x}\hat{i}$$

$$= -k_1x\hat{i}$$

Doing this the formal way utilizing the *del operator* idea:



$$\vec{F} = -\vec{\nabla}U$$

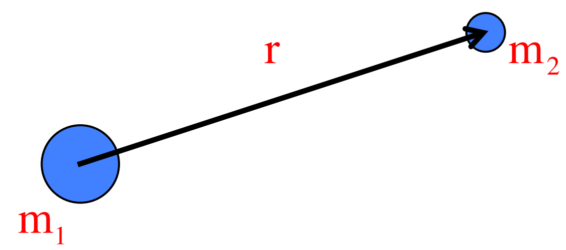
$$= -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{1}{2}k_1x^2\right)$$

$$= -\left(\frac{\partial\left(\frac{1}{2}k_1x^2\right)}{\partial x}\hat{i} + \underbrace{\frac{\partial\left(\frac{1}{2}k_1x^2\right)}{\partial y}}_{0}\hat{j} + \underbrace{\frac{\partial\left(\frac{1}{2}k_1x^2\right)}{\partial z}}_{0}\hat{k}\right)$$

$$= -k_1x\hat{i}$$

This is clearly cumbersome in comparison the first way, which is the kind of AP problem you will undoubtedly get, but it is the way you'd do the problem for more complex potential energy function.

Example 4: The gravitational potential energy function for a mass far from its surface is:



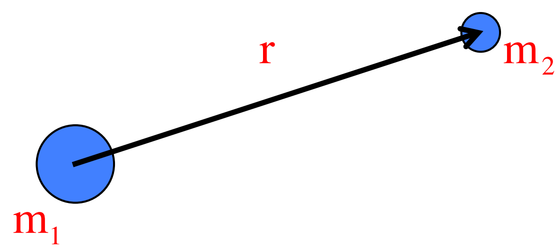
$$U = -G\frac{m_1 m_2}{r}$$

where G is a constant. Derive an expression the force that goes with that *potential energy function*.

Observation: From our previous example and all the theory that has proceeded this problem, it should be reminded that the relationship between a force and its potential energy function is that the force magnitude is related to the *maximum rate at which the energy content of the system changes spatially* at a given point (that's what $\partial U/\partial x$ is if the force is only in the x-direction), with a direction that is related to the *direction* of MAXIMUM change (that might not be obvious, but that's what a *gradient* does).

If the potential energy function is radial in nature, *force* will tell you something about the way energy changes *radially* as you move out away from the force producing object.

With this in mind, it shouldn't be too much of a stretch to see that:



$$F_r = -\frac{\partial U}{\partial r}\hat{r}$$
$$= -\frac{\partial\left(-G\frac{m_1 m_2}{r}\right)}{\partial r}\hat{r}$$
$$= -G\frac{m_1 m_2}{r^2}\hat{r}$$

as expected.

NOTE: In polar-spherical notation, the del operator is a horror. We don't need to go there for the calculation of the work gravity does on objects far away (translation: be happy with what you see above—it's more than enough for our needs).

Summary

a.) Work calculations: $W_F = \vec{F} \cdot \vec{d}$ or $W_F = \int \vec{F} \cdot d\vec{r}$

b.) Work/Energy Theorem: $W_{net} = \Delta KE$, where $KE = \frac{1}{2}mv^2$

c.) Potential Energy functions:

$U_{near\ earth\ grav} = mgy$, where y is the distance above the $y = 0$ level

$U_{spring} = \frac{1}{2}kx^2$, where x is the displacement from equilibrium

d.) Use of Potential Energy functions: $W_{cons.force} = -\Delta U_{for\ force\ fld}$

e.) Deriving U function from Force F function:

$$U(x) - U(\text{zero pt}) = -\int_{\text{zero pt}}^{x} \vec{F} \cdot d\vec{r}$$

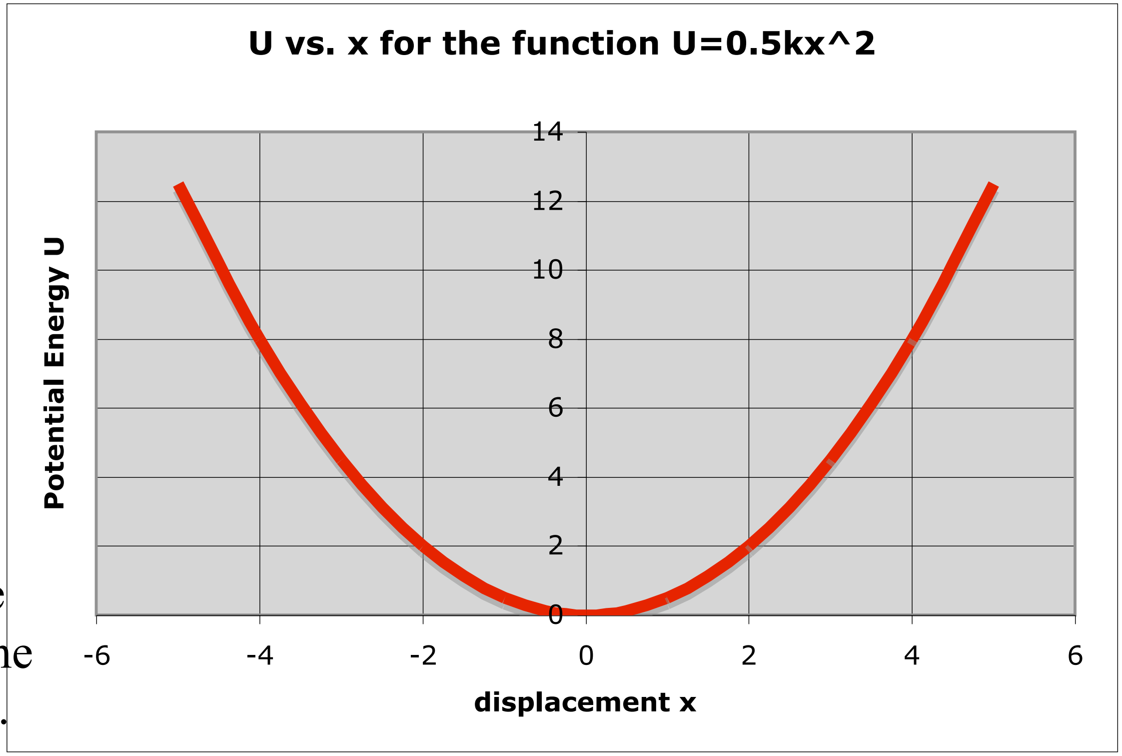
f.) Deriving Force F function from U function:

$$\vec{F} = -\vec{\nabla}U \quad \text{or} \quad \vec{F} = -\frac{\partial U}{\partial x}\hat{i} \quad \text{or} \quad \vec{F} = -\frac{\partial U}{\partial r}\hat{r}$$

Energy Diagrams I

a.) What does the slope of this curve represent?

As $\vec{F} = -\frac{\partial U}{\partial x}$, it represents a *conservative force function*



b) What is happening at the bottom of the curve?

This is a point of *stable equilibrium*. Place a body there and the force will be zero, which means the body won't be motivated to move.

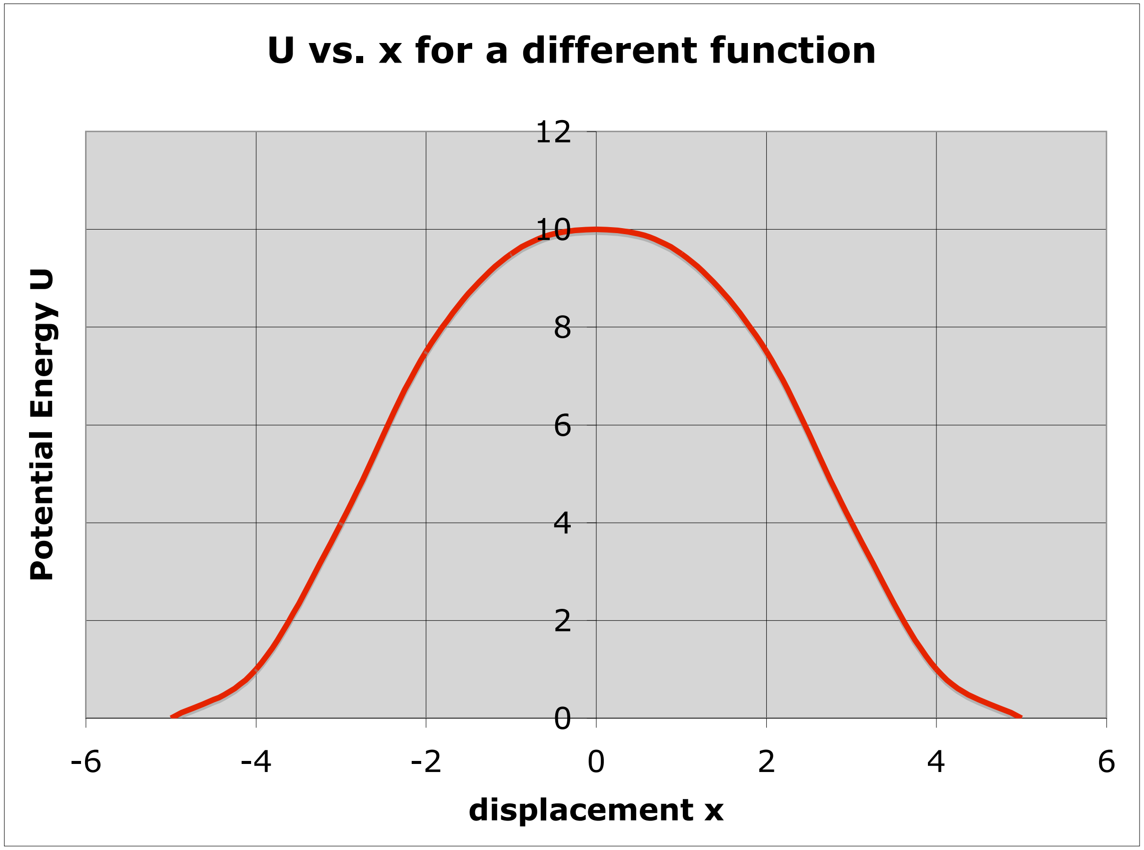
c) What happens if the mass is given an initial displacement *x*?

A displacement in the positive *x* direction results in a conservative "restoring force" in the –*x* direction. If the mass is released, it will oscillate back and forth, with U+K = constant. Visualize this oscillating system as a "marble in a bowl."

Energy Diagrams II

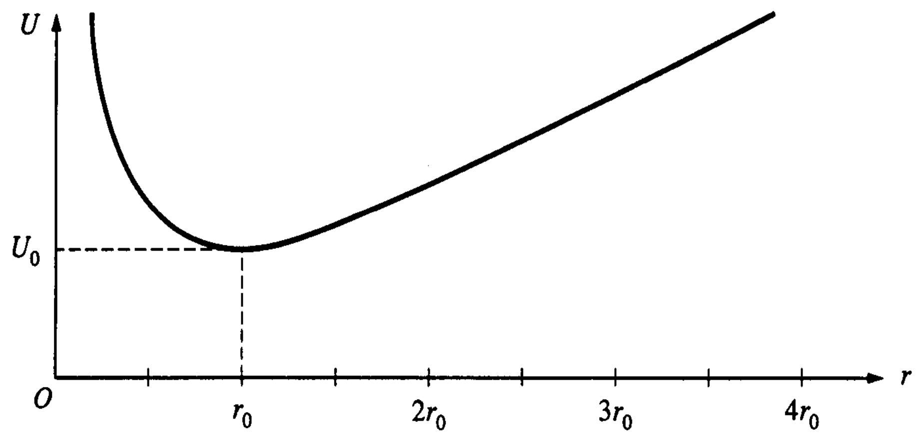
How is this potential energy diagram different?

There is a point of *unstable equilibrium* at x = 0 . . . Put a body there and the body won't feel a force (so it will sit stationary), but displace it slightly and it will accelerate picking up kinetic energy rapidly . . .



AP Problem

1995 M2. A particle of mass *m* moves in a conservative force field described by the potential energy function $U(r) = a(r/b + b/r)$, where *a* and *b* are positive constants and *r* is the distance from the origin. The graph of $U(r)$ has the following shape.



a.) In terms of the constants *a* and *b*, determine the following.

i.) The position r_o at which the potential energy is a minimum?

To get a minimum, take the derivative of the function and put equal to zero

$$\frac{dU}{dr} = \frac{d\left(\frac{ar}{b} + \frac{ab}{r}\right)}{dr} = \frac{d\left(\frac{ar}{b} + abr^{-1}\right)}{dr}$$

$$\Rightarrow \text{for equilibrium} \quad \frac{a}{b} - abr_o^{-2} = 0$$

$$\Rightarrow \quad r_o = b$$

ii.) The minimum potential energy U_o?

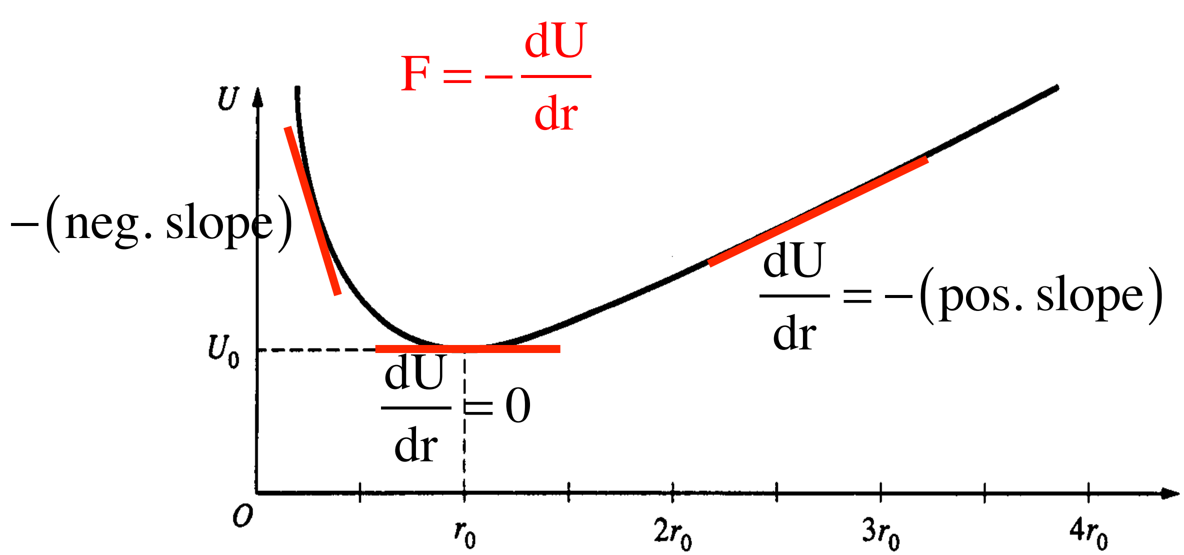
$$U_o = \frac{ar_o}{b} + \frac{ab}{r_o}$$

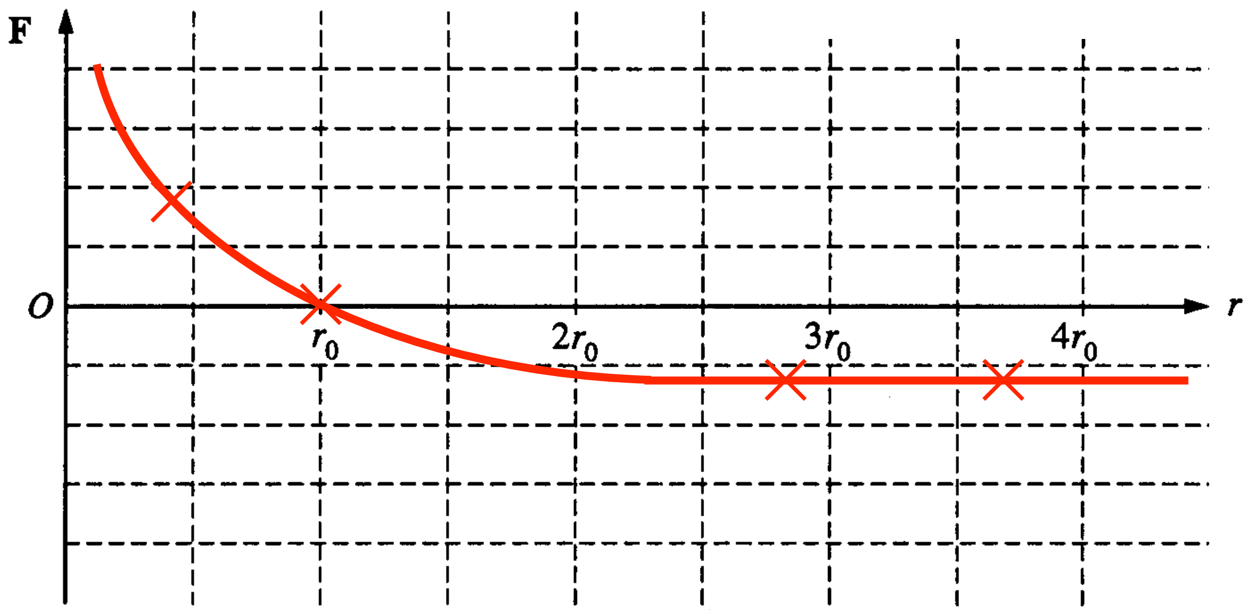
$r_o = b$

$$= \frac{ab}{b} + \frac{ab}{b} = 2a$$

AP Problem (con't)

b. Sketch the net force on the particle as a function of r on the graph below, considering a force directed away from the origin to be positive, and a force directed toward the origin to be negative.





AP Problem (con't)

The particle is released from rest at $r = r_o/2$.

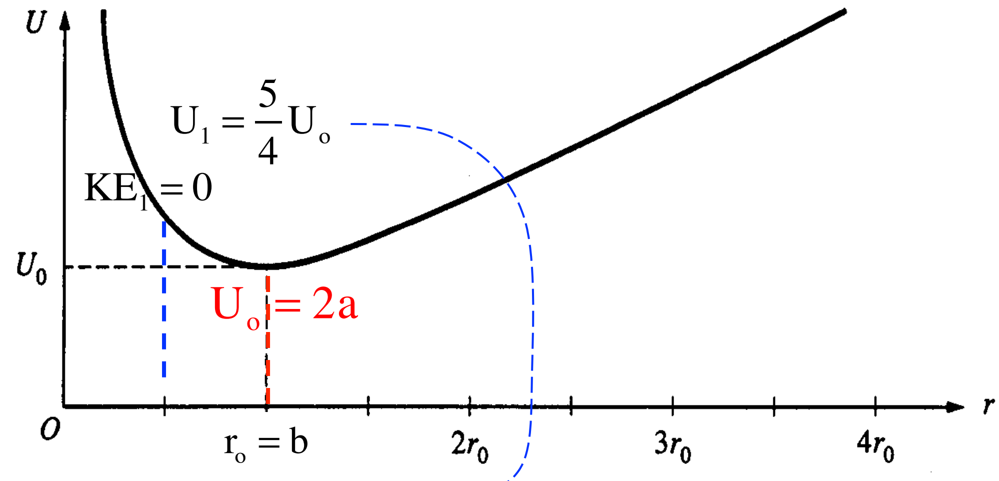
c. In terms of U_o and m, determine the speed of the particle when it is at $r = r_o$.

$$W_{net} = -\left(U_{final} - U_{initial}\right) = KE_2 - KE_1$$

$$\Rightarrow \quad -\left(U_o - U(r_o/2)\right) = \frac{1}{2}mv_2^{\ 2} - 0$$

$$-\left(U_o - \frac{5}{4}U_o\right) = \frac{1}{2}mv_2^{\ 2}$$

$$\Rightarrow \quad v = \left(\frac{U_o}{2m}\right)^{1/2}$$



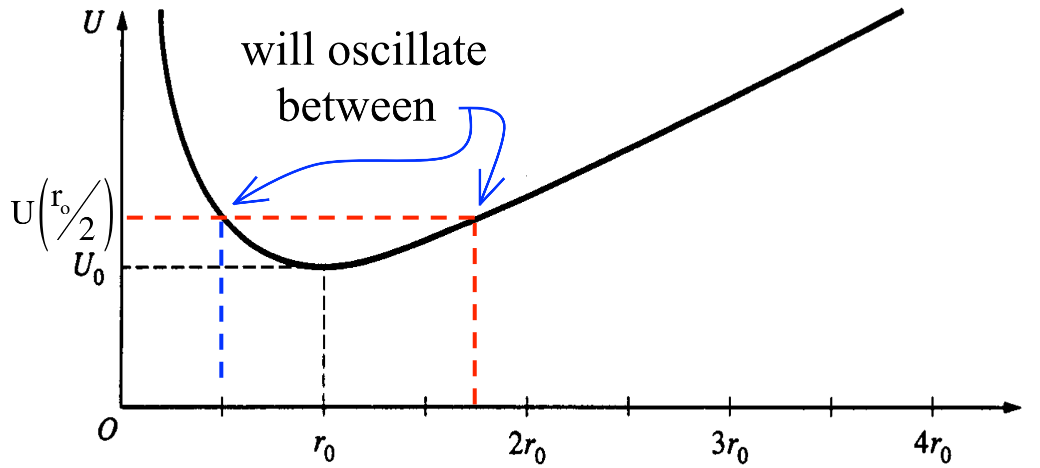
$$U\left(\frac{r_o}{2}\right) = \left(\frac{a\left(r_o/2\right)}{b} + \frac{ab}{\left(r_o/2\right)}\right) = \left(\frac{a\left(b/2\right)}{b} + \frac{ab}{\left(b/2\right)}\right)$$

$$= \left(\frac{a}{2} + 2a\right) = 2.5a = \frac{5}{4}U_o$$

AP Problem (con't)

The particle is released from rest at $r = r_o/2$.

d. Write the equation or equations that could be used to determine where, if ever, the particle will again come to rest. It is not necessary to solve for this position.



Assuming no energy is lost in the motion, the body will come back to rest when it returns to it's start point or at it's other extreme. In other words, wherever $U(r) = U\left(r_o/2\right)$.

e. Briefly and qualitatively describe the motion of the particle over a long period of time.

The particle will oscillate with no energy loss.