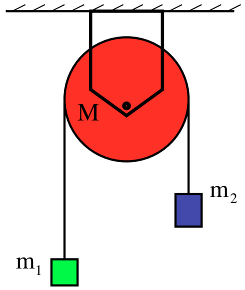
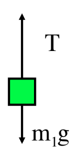
ATWOOD MACHINE WITH MASSIVE PULLEY

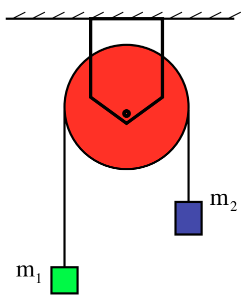
A mass m_1 is attached to a rope that is threaded over a massive pulley and attached to a second mass m_2. If the pulley's mass is "M," its radius "R" and its moment of inertia about its center of mass is $.5MR^2$, determine both the angular acceleration of the pulley and the acceleration of each of the masses.



To get a feel for the intricacies of this problem, let's do it first on the assumption the the pulley is NOT massive. In that case, Newton's Second Law applied to each mass and we can write:

f.b.d. on m_1:





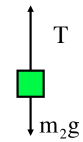
So:

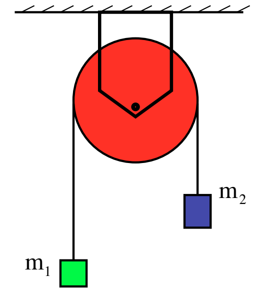
$$\sum F_{1,y}$$

$$T - m_1 g = m_1 a$$

$$\Rightarrow \quad T = m_1 g + m_1 a$$

f.b.d. on m_2:





So:

$$\sum F_{2,y}$$

$$T - m_2 g = -m_2 a$$

From the previous page, $T = m_1 g + m_1 a$, so we can write:

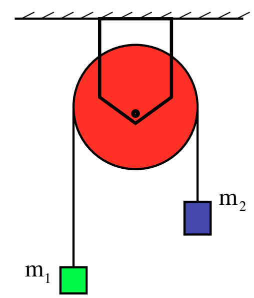
$$(T) - m_2 g = -m_2 a$$

$$\Rightarrow \quad (m_1 g + m_1 a) - m_2 g = -m_2 a$$

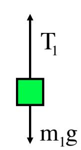
$$\Rightarrow \quad a = \frac{m_2 - m_1}{m_1 + m_2}$$

Notice that we are able to assume the tension on both sides of the pulley is the same. This is the consequence of the fact that the pulley is assumed to be *massless*. If that hadn't been the case, a net torque would have been required to make the pulley rotate. That could only come if the tension forces on either side of the pulley were imbalanced.

Now let's look at the situation assuming the pulley is massive. In that case, the only difference is that the tensions are different on either side of the pulley (this has to be so so the torque sum about the pulley's center of mass is not zero). Writing, we get:



f.b.d. on m_1:



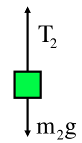
So:

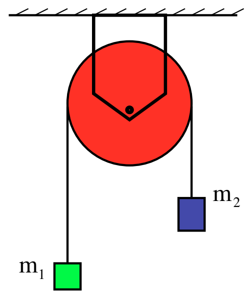
$$\sum F_{1,y}$$

$$T_1 - m_1 g = m_1 a$$

$$\Rightarrow \quad T_1 = m_1 g + m_1 a$$

f.b.d. on m_2:





So:

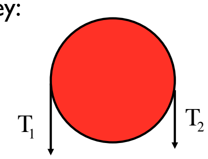
$$\sum F_{2,y}$$

$$T_2 - m_2 g = -m_2 a$$

$$\Rightarrow \quad T_2 = m_2 g - m_2 a$$

At this point, we have three unknowns, the two tensions and the acceleration "a." We need another equation. ENTER SUMMING THE TORQUES ABOUT THE PULLEY'S CENTER OF MASS.

f.b.d. on pulley:



So:

$$\sum \Gamma_{cm}$$

$$T_1 R - T_2 R = -I_{cm}\alpha$$

At this point, we have FOUR unknowns, the two tensions, the acceleration "a" and the angular acceleration α. Once again, we need another equation. That relationship connects the angular acceleration about the pulley's center of mass to the translational acceleration of a point on the pulley's edge (this will be the same as the translational acceleration of the string and, hence, the masses). In other words, we need:

$$a_{cm} = R\alpha$$

We now have four equations:

$$T_1 - m_1 g = m_1 a$$

$$\Rightarrow \boxed{T_1 = m_1 g + m_1 a \quad \text{Equ. A}}$$

$$T_2 - m_2 g = -m_2 a$$

$$\Rightarrow \boxed{T_2 = m_2 g - m_2 a \quad \text{Equ. B}}$$

$$\boxed{T_1 R - T_2 R = -I_{cm}\alpha \quad \text{Equ. C}}$$

$$\boxed{a_{cm} = R\alpha \quad \text{Equ. D}}$$

Substituting Equ. A, B and D into C, we get:

$$\underbrace{(m_1 g + m_1 a)}_{T_1}R - \underbrace{(m_2 g - m_2 a)}_{T_2}R = -\underbrace{\left(\frac{1}{2}MR^2\right)}_{I_{cm}}\underbrace{\left(\frac{a}{R}\right)}_{\alpha}$$

$$\Rightarrow \quad a = \frac{m_2 g - m_1 g}{\left(m_1 + m_1 + \frac{M}{2}\right)}$$

Note that with the exception of the presence of the "M" term, this is exactly the same relationship you got with the massless pulley analysis.