The Island Series:

You have been kidnapped by a crazed physics nerd and left on an island with twenty-four hours to solve the following problem. Solve the problem and you get to leave. Don't solve the problem and you don't.

The problem: You have *fifty-meters of wire*, a *powerful horseshoe magnet* and a *small light bulb* (like the kind that goes into a flashlight). You are told there is a *book on the island* that will mysteriously open at exactly 10 PM, and will stay open for *30 seconds*. In it are written *3-words*. If you know what those words are when the helicopter arrives on the island the next day, you will be allowed to leave. There is *no moon*, so there will be *no ambient light* at 10 PM, and there are no vegetables or fruit on the island (you can't make a battery, not that you could anyway—you'd need two different kinds of wire to make that work). *How do you generate the light needed to read the book when it briefly opens at 10 PM?*

Solution to Island Problem

Wind the wire into a coil, attach the coil to the light bulb, and repeatedly move the coil into and out-of the magnetic field as rapidly as you can.

The real question is, "Why does this work?"

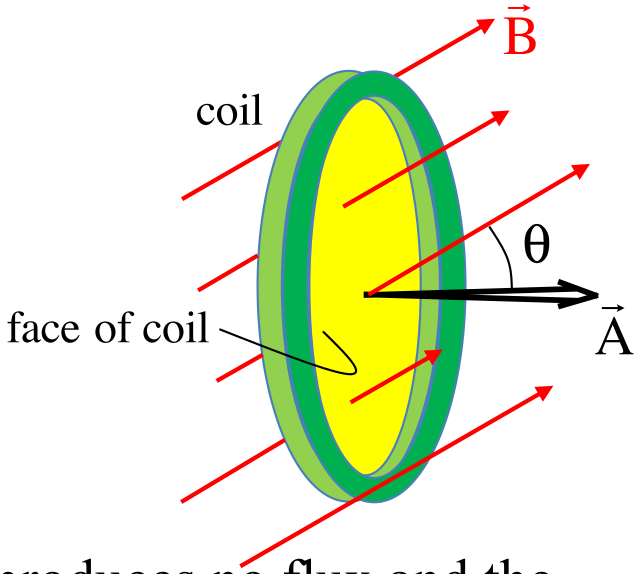
CHAPTER 31-32:
Faraday's Law and Inductance

photo courtesy of
Mr. White

Magnetic Flux

If you'll remember, when you have a surface in a vector field, a certain amount of the field will pass through the surface (we first talked about this with Gauss's Law where we had electric field lines passing through a closed surface).

The idea of flux is a mathematical way to measure how much of the field passes through the surface.



Magnetic flux through a coil is shown to the right.

So let's define an area vector $\vec{A}$ whose magnitude is the area of the face of the coil and whose direction is perpendicularly out from the face.

Noting that the component of $\vec{B}$ parallel to the face produces no flux and the component perpendicular to the face (along the line of $\vec{A}$) does produce a flux we realize a dot product will do the desired math, so

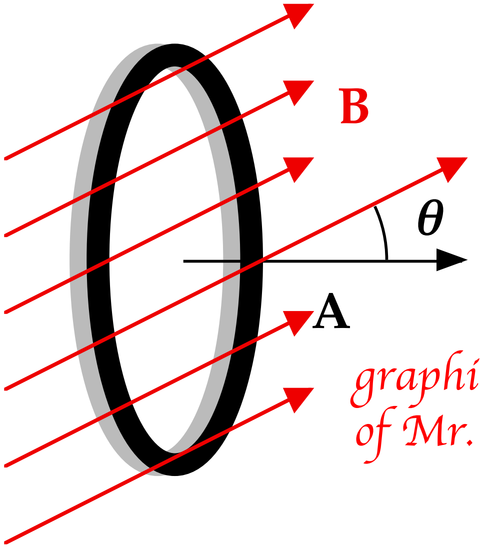
$$\Phi_B = \vec{B} \bullet \vec{A}$$
$$= BA\cos\theta$$

with

θ being the angle between the line of the area vector and the B-fld vector;

and the units being Tesla • meters^2, or *Webers*.

Preamble—Magnetic Flux



graphic courtesy of Mr. White

Shortly, we will need to use the idea of magnetic flux through a coil. We have already talked about this concept. In it's simplest form, it is defined as:

$$\Phi_B = \vec{B} \bullet \vec{A}$$
$$= BA\cos\theta$$

where:

$\vec{B}$ is the B-fld vector;

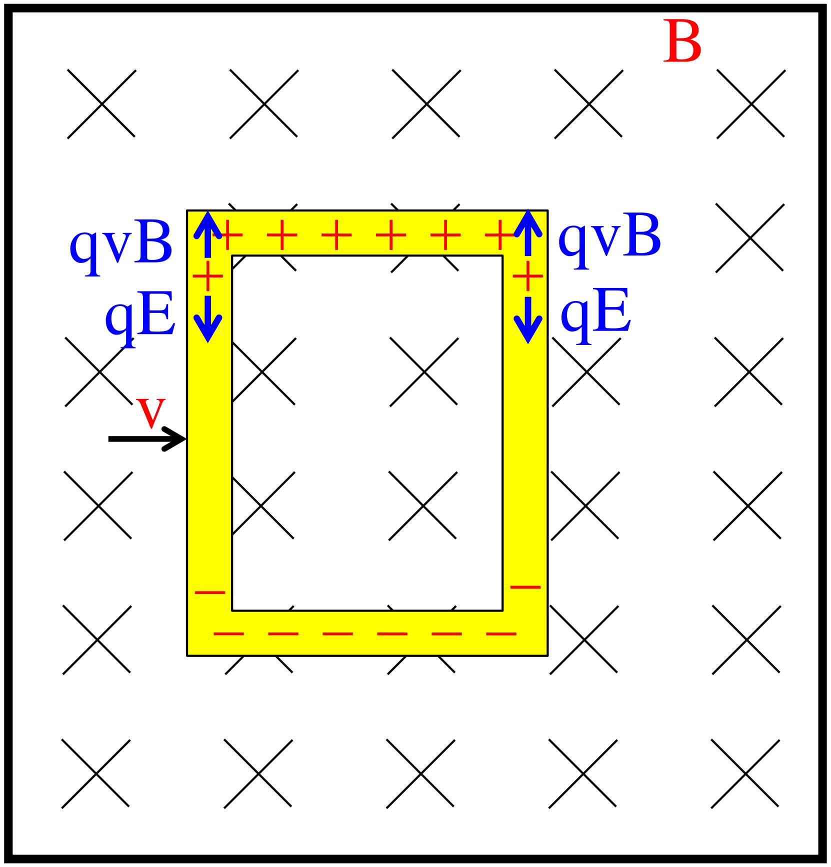
$\vec{A}$ is a vector whose direction is perpendicular to the face of the coil and whose magnitude is equal to the area of the coil's face;

θ is the angle between the line of the area vector and the B-fld vector;

and the units are Tesla • meters^2, or *Webers*.

A Conventional Approach

Consider a loop moving through the magnetic field shown in the sketch.



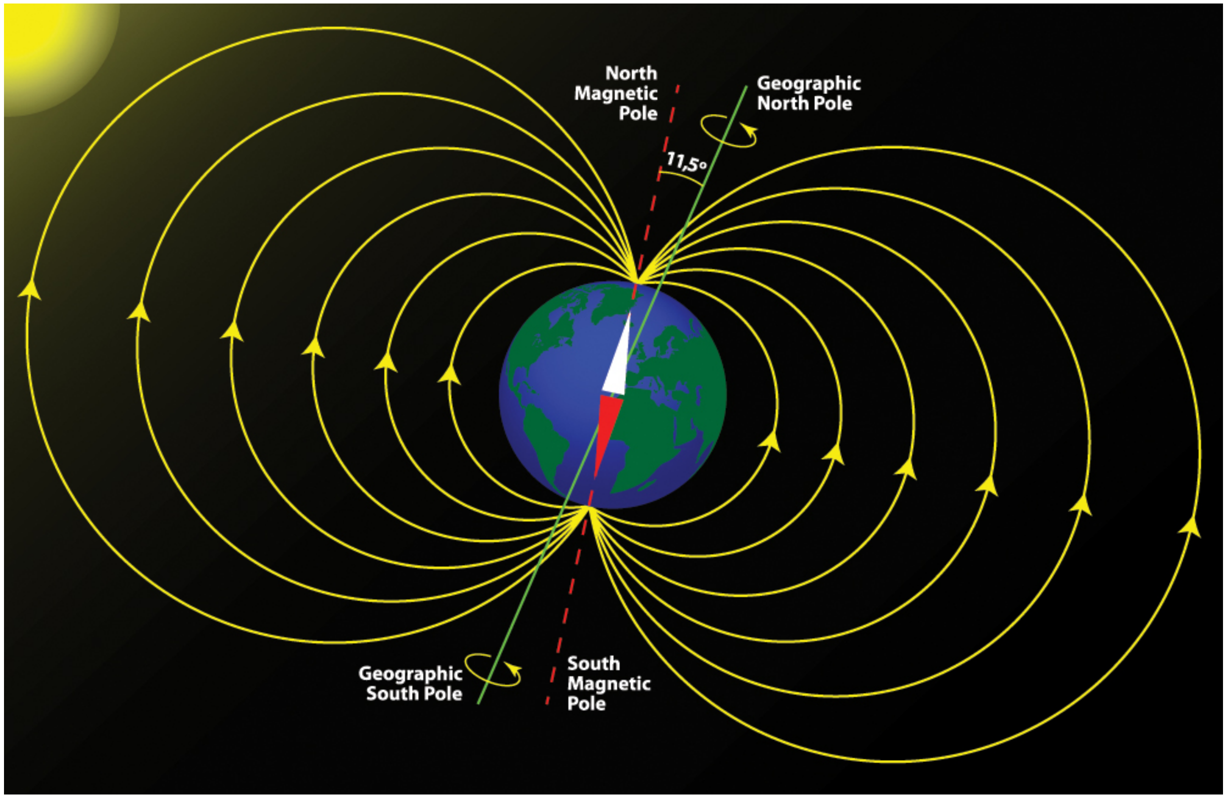
--*Initially, positive* charge carriers, assumed mobile, will interact with the *B-fld* via $\vec{F}_B = q\vec{v}x\vec{B}$ and be forced upward toward the top of the loop. Having nowhere to go, they will accumulate leaving the bottom negative.

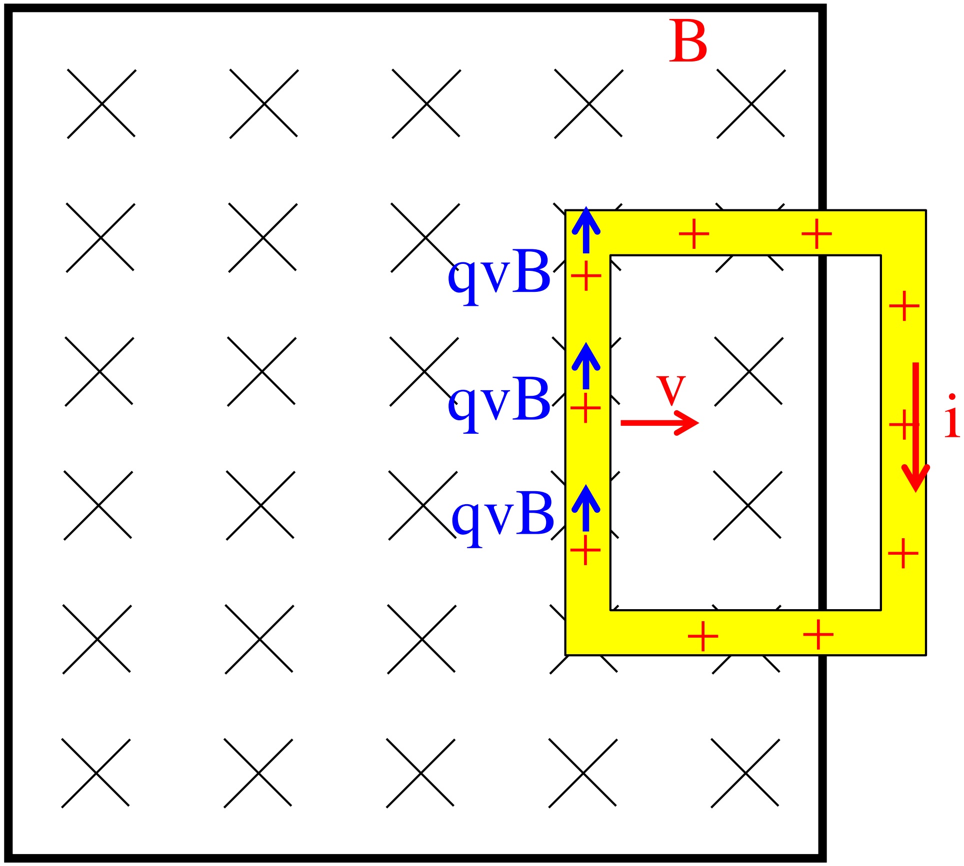
--*The electric field* they set up will produce a force that counteracts the magnetic force and the charges will come into equilibrium (i.e., $\vec{F}_e = \vec{F}_B$).

--*At that point* (and all of this will happen very quickly), we will have an electric dipole with NO CONVENTIONAL CURRENT in the loop. This is the way the loop will stay as it moves through the *B-fld*.

Exotic Aside—Example 1: *An airplane* flies from LA to Seattle and, due to its motion through the Earth's magnetic field, undergoes a motional EMF. Noting that the earth's *B-fld* points slightly downward as one flies north, which wingtip ends up positively charged, the left or the right? (courtesy of Mr. White)

Solution: The earth's *B-fld* points slightly downward as one flies north. The cross product produces a leftward EMF, which means positive charge will move leftward.





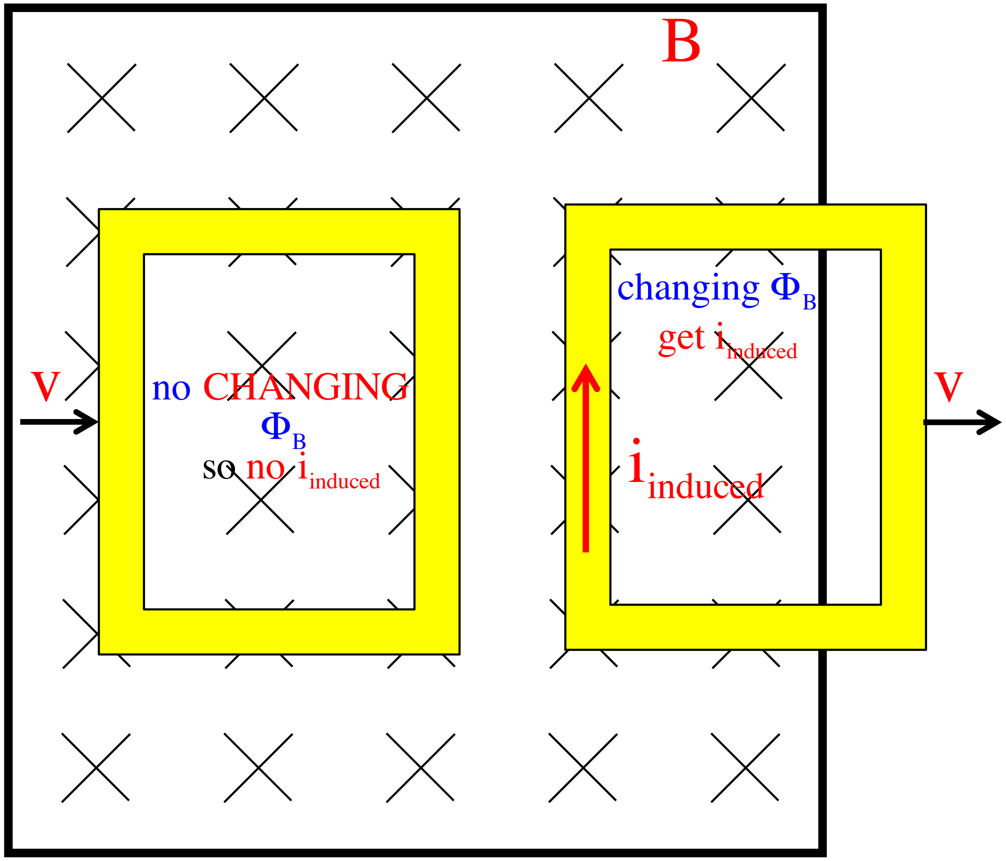
Back at the ranch: Something interesting happen when part of the coil *leaves* the field.

--*With the entire loop* in the *B-fld*, all the *positive charge carriers* felt the same magnetic force *upward*. Consequence: charge accumulated at the top of the loop.

--*But when the* right side leaves the *B-fld*, there is no longer a magnetic force upward along the exiting path. That means the positive charges forced upward in the left section *will* have some place to go—around the circuit and down the right side.

--*In other words,* what you end up with is a CONVENTIONAL CURRENT in the loop. This will persist as long as the loop is moving with part of itself in the *B-fld* while part is not.

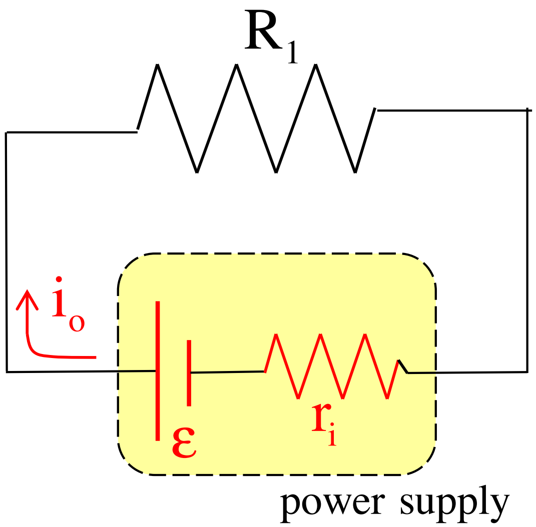
Faraday's Law



The creation of a conventional current flow as the coil leaves the constant *B-fld* has been explained using what you already know from the Classical Theory of Magnetism. Faraday viewed it differently. His approach will allow us to analyze difficult situations that are not so easily untangled with the thinking we've just presented.

Faraday, who was not interested in the dipole, noticed that you only get an induced current when there is a *changing magnetic flux* through the face of the coil. There could *be* a flux through the coil, but if it wasn't/isn't changing, no induced current.

Remember back to our discussion of real-world power supplies. Internal to a power supply is a quality that generated an electric field and that, in turn, motivated charge to move. That motivating quality was quantified in what was called an electromotive force, or EMF (symbol ε).



What Faraday deduced was that a changing magnetic flux through the face of a coil induces an EMF that creates an *E-fld* that motivates charge to move in the form of an induced current. What's more, *how fast* the EMF changes matters.

Mathematically, then, Faraday's Law is:

$$\varepsilon_{\text{induced}} = -N\frac{\Delta\Phi_B}{\Delta t} \qquad \text{(this is Faraday's Law)}$$

$$= -N\frac{\Delta\left(\vec{B}\bullet\vec{A}\right)}{\Delta t}$$

$$= -N\frac{\Delta\left(BA\cos\theta\right)}{\Delta t}$$

Note: An EMF produced by motion in a *B-fld* is called a *motional EMF.*

Example 2: *A coil* is wrapped with 200 turns of wire on the perimeter of a square frame of sides 18 cm. The total resistance of the coil is 2.0 ohms. A uniform *B-fld* is turned on perpendicular to the plane of the coil. (courtesy of Mr. White)

a.) If the field changes linearly from 0 to .5 Wb/m^2 in a time of 0.80 sec, find the magnitude of the induced EMF in the coil while the field is changing.

With everything constant, we can use the Δ version of Faraday and write:

$$\begin{aligned}\varepsilon_{\text{induced}} &= -N\frac{\Delta\Phi_B}{\Delta t}\\ &= -NA\cos 0^\circ \frac{\Delta B}{\Delta t}\\ &= -(200)(.18\text{ m})^2\cos 0^\circ \frac{(.5-0)\,\text{Wb/m}^2}{.8\text{s}}\\ &= 4.05\text{ volts}\end{aligned}$$

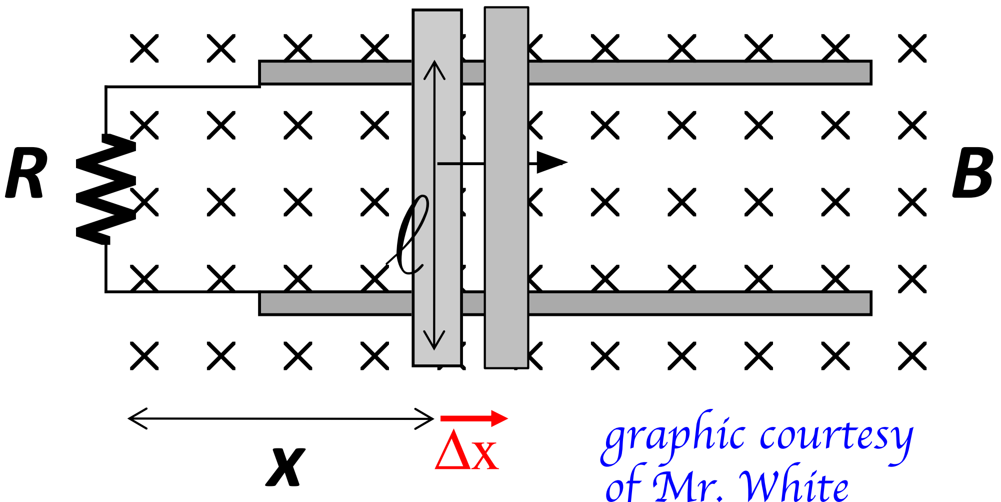
b.) Find the magnitude of the current induced in the coil while the field is changing.

Ohm's Law still work in these problems, so we can write:

$$\varepsilon_{\text{induced}} = i_{\text{ind}}R$$

$$\Rightarrow \quad i_{\text{ind}} = \frac{\varepsilon_{\text{ind}}}{R} = \frac{(4.05\text{ V})}{(2.0\ \Omega)} = 2.0\text{ A}$$

Example 3: A bar on frictionless rails is made to move with velocity *v* through a *B-fld* as shown in the sketch (you are looking down on the system).



graphic courtesy of Mr. White

a.) Derive an expression for the induced EMF in the "coil."

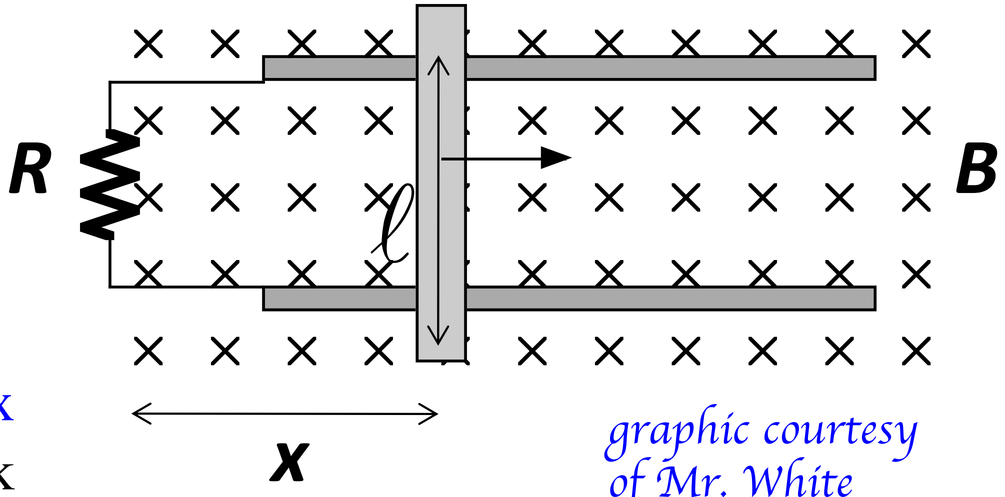
The technique here is to write out a general expression for the magnetic flux, then determine its change (which is Δx in this case). Doing so yields:

$$\begin{aligned}\Phi_B &= \vec{B} \bullet \vec{A} \\ &= BA\,\overset{1}{\cancel{\cos 0^\circ}} \\ &= B(lx)\end{aligned}$$

$$\begin{aligned}\Rightarrow \quad \varepsilon_{ind} &= -\overset{1}{\cancel{N}}\frac{\Delta\Phi_B}{\Delta t} \\ &= -\frac{d(Bl\Delta x)}{\Delta t} \\ &= -Bl\frac{\Delta x}{\Delta t} \qquad (-Blv)\end{aligned}$$

b.) In what direction is the induced current in the circuit?

There is a technique for determine the direction of an induced current in a coil due to a changing magnetic flux (it's called Lenz's Law), but we'll talk about that later. For now, simply



graphic courtesy of Mr. White

using $\vec{F}_B = q\vec{v}x\vec{B}$ on the positive charges of the bar suggest the current is upward in the bar and counterclockwise in the circuit.

c.) Current in a *B-fld* will feel a force. If all of the resistance in the circuit is wrapped up in *R*, what will be the direction and magnitude of the force on the bar due to the current's interaction with the *B-fld*?

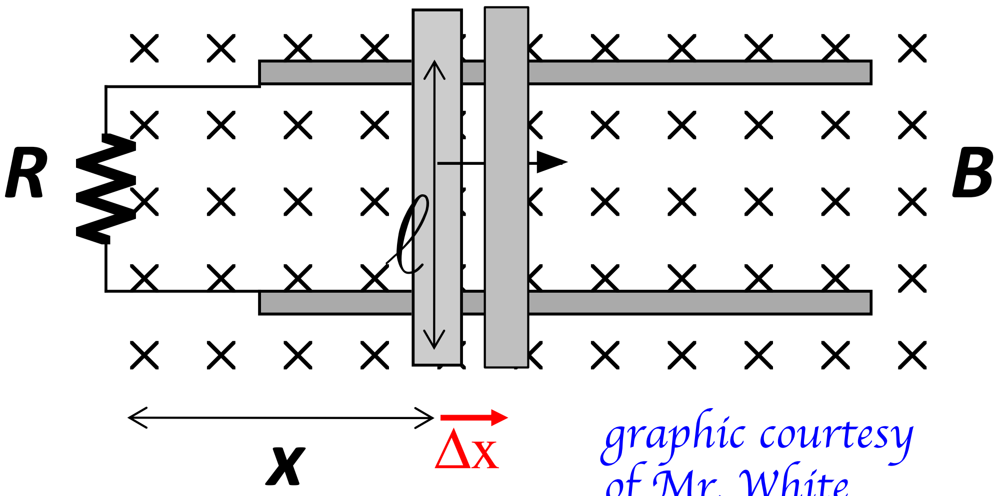
Using $\vec{F}_{wire} = i\vec{L}x\vec{B}$, the force direction on the bar is to the left.

To evaluate $\vec{F} = i\vec{L}x\vec{B}$, we need the current.

$$i_{ind} = \frac{\varepsilon_{ind}}{R} = \frac{Blv}{R}$$

$$\Rightarrow \quad \vec{F}_{wire} = i\vec{L}x\vec{B} = \left(\frac{Blv}{R}\right)lB\sin 90^\circ = \frac{B^2l^2v}{R}$$

d.) *A force* must be provided to make the bar move with constant velocity *v*. How much power must that force provide to the system?



graphic courtesy of Mr. White

$$P = \frac{W}{\Delta t}$$

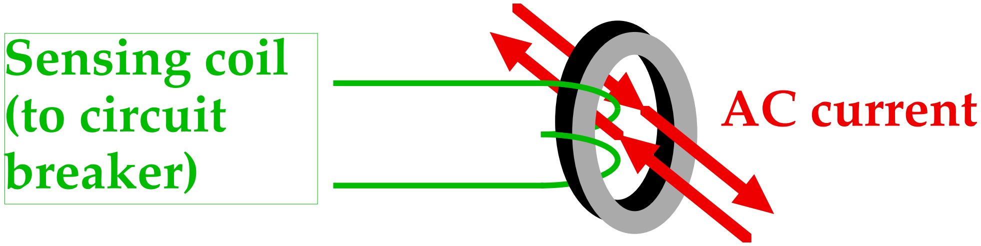
$$= \frac{\vec{F} \bullet \Delta\vec{x}}{\Delta t} = F\cancelto{v}{\frac{\Delta x}{\Delta t}}$$

$$= \left(\frac{B^2 l^2 v}{R}\right) v$$

$$= \frac{B^2 l^2 v^2}{R}$$

$$\left(= \frac{\varepsilon^2}{R} \text{ which is fitting, as this is a power relationship} \right)$$

Real Life Application (courtesy of Mr. White)

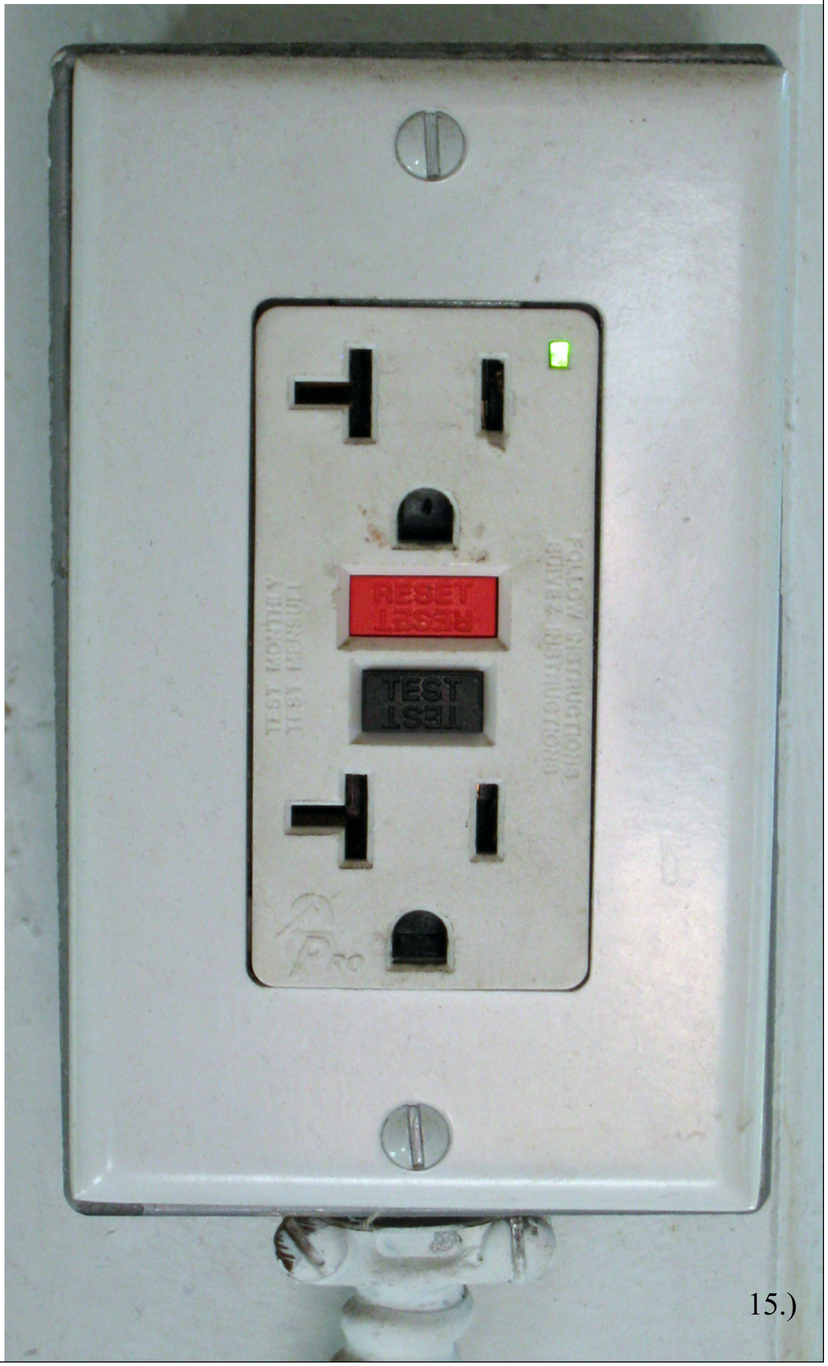


Sensing coil (to circuit breaker)

AC current

Ground Fault Interrupter

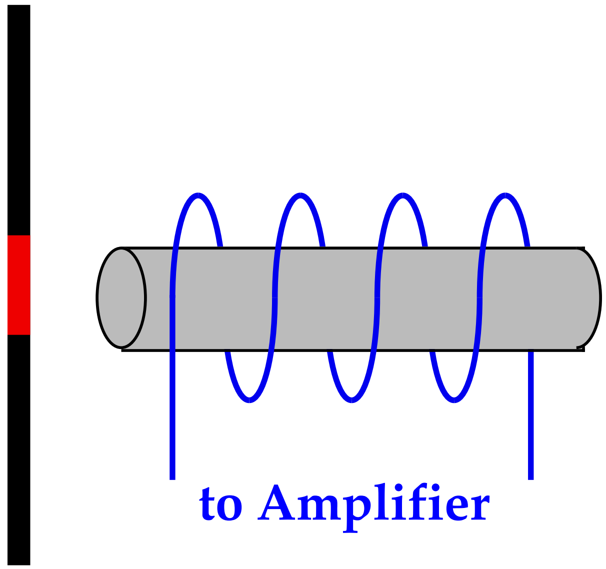
1. No net current enclose by sending coil, so no magnetic field.
2. If there's a *ground fault* (current going out but not back—this would be the case if you managed to electrocute yourself on the device by allowing current to flow to ground through you), a magnetic field is created, increasing flux, and inducing an emf that activates the circuit breaker.



Real Life Application (courtesy of Mr. White)

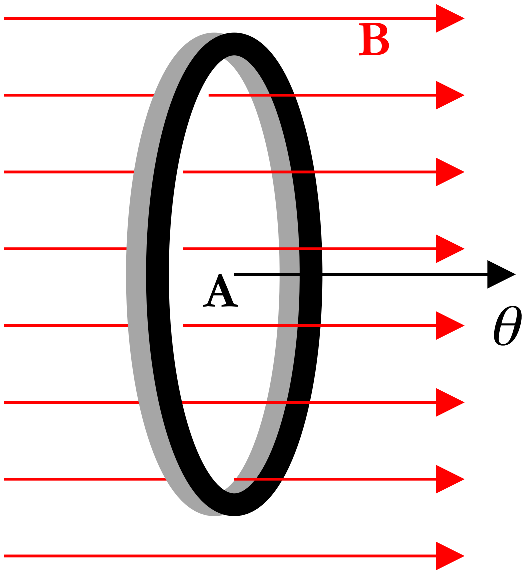
Electric Guitar Pickup

Vibrating string produces a change in magnetic flux in the coil, which is transmitted as an emf to the amplifier.



Lenz's Law

Although Faraday's Law allows us to determine the magnitude of the induced EMF set up by a changing magnetic flux through the face of a coil and, by extension, the magnitude of the induced current through the coil, it says nothing about the *direction* of the induced current set up by the EMF. Lenz's Law is designed to fill in that gap.

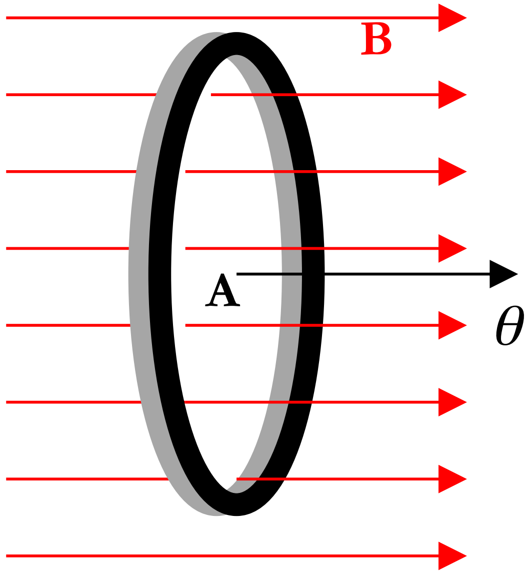


graphic courtesy of Mr. White

Lenz's Law maintains that an induced EMF through a coil (or loop) will produce an induced current that will create its own induced magnetic flux, and that that induced magnetic flux will *oppose the change* of magnetic flux through the loop that started the process off in the first place.

Confused? That's the statement of Lenz's Law in the raw. Its message can be more economically unpacked with three easy steps.

The easy way:



graphic courtesy of Mr. White

1.) Identify the direction of the external *B-fld*.

2.) Identify whether the external magnetic flux is increasing or decreasing.

3a.) If the flux is INCREASING, the coil's induced current will set up an induced magnetic field through the coil's face that is OPPOSITE the direction of the external *B-fld*. Use the right-hand rule to determine the current direction that does this. (See note below.)

3b.) If the flux is DECREASING, the coil's induced current will set up an induced magnetic field through the coil's face that is IN THE SAME DIRECTION AS the direction of the external *B-fld*. Use the right-hand rule to determine the current's direction.

Note: Remember, the way you relate the *direction* of the magnetic field set up in a coil by a *current* through the coil is by laying your right hand on the coil with your fingers in the direction of the current. Your thumb will point in the direction of the *B-fld* down the axis. You will simply be using this technique backwards here (starting with the *B-fld* and figuring the current that produced it).

Try it on the figure to the right. Assume the field is diminishing.

1.) Identify the direction of the external *B-fld*.

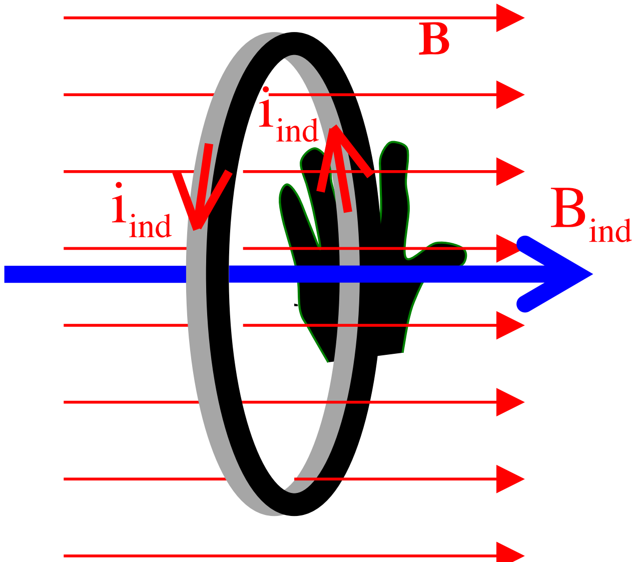
to the right

2.) Identify whether the external magnetic flux is increasing or decreasing.

it's decreasing

3.) As the flux is DECREASING, the coil's induced current will set up an induced magnetic field through the coil's face that is IN THE SAME DIRECTION as the external *B-fld* (as determined in #1).

so the induced B-fld is to the right

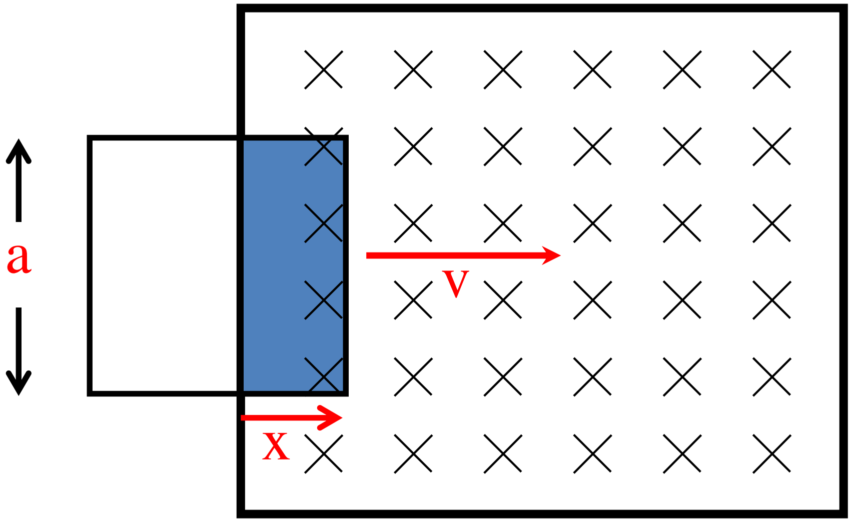


graphic, heavily modified, courtesy of Mr. White

How do you have to wrap my fingers (right hand) on the coil to get an induced *B-fld* to the right?

Wrapped them counterclockwise, so that's the direction of the current.

So Back to Motional EMFs



Example 4: A square coil of resistance *R* and sides of length *a* enters a region in which there is a constant *B-fld*. It is moving with constant velocity *v* as shown in the sketch:

a.) Is there a magnetic flux through the coil?

yes, magnetic field lines are piercing the face of the coil

b.) Is there an induced EMF set up in the coil (justify)? If so, what is its magnitude?

yes, the magnetic flux is CHANGING through the face of the coil

What was the technique to determine the EMF? Define the magnetic flux, then determine its change! With N = 1:

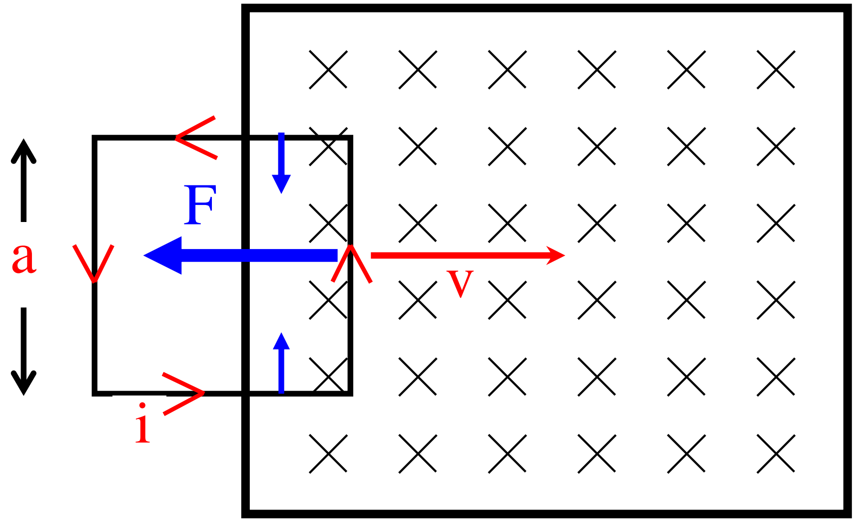
$$\varepsilon_{ind} = -N\frac{\Delta\Phi_B}{\Delta t}$$

$$= -\frac{(Ba\Delta x)}{\Delta t} \quad \ldots \text{where } \frac{\Delta x}{\Delta t} = v$$

$$= -Bav$$

c.) What is the induced current in the coil?

$$i_{ind} = \frac{\varepsilon_{ind}}{R} = \frac{Bav}{R}$$



d.) What is the direction of the current?

Lenz's Law:

--external *B-fld* into the page;

--magnetic flux increasing,

--so induced *B-fld OUT OF PAGE* (opposite external field). Current has to flow counterclockwise to achieve that.

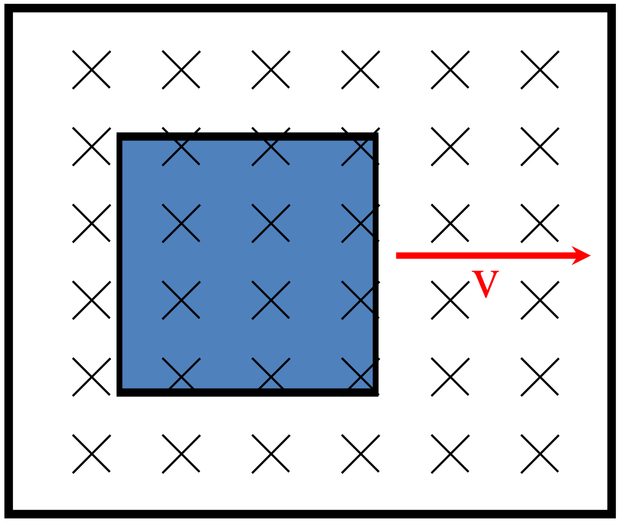
e.) The induced current will interact with the external *B-fld* and feel a force. In what direction will be that net force?

The magnitude would be the magnitude of $\vec{F}_{wire} = i\vec{L}x\vec{B}$, which we could figure out, but all that was asked for was the direction, which is the direction of that cross product. The force on the two horizontal wires will cancel, but the force on the vertical wire in the *B-fld* will be to the left, as shown on the sketch.

The coil proceeds into the *B-fld*, fully immersing itself. At that point:

f.) Is there a magnetic flux through the coil?

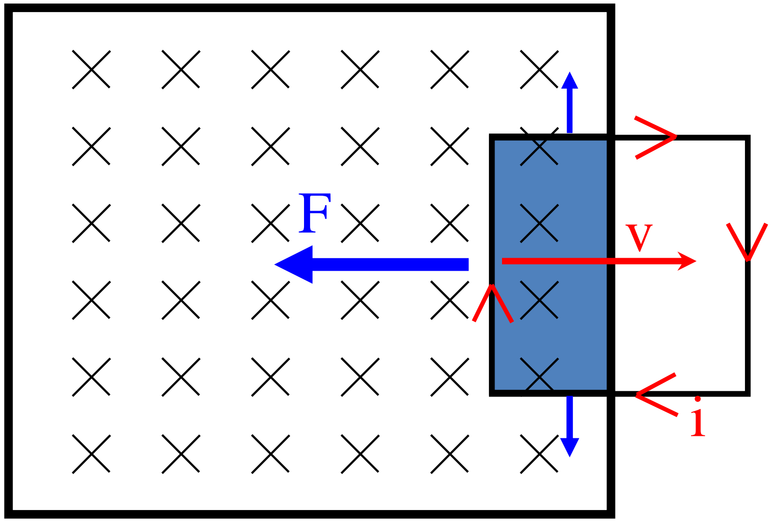
yes, magnetic field lines are piercing the face of the coil



g.) Is there an induced EMF set up in the coil (justify)? If so, what is its magnitude?

Nope, the magnetic flux is NOT CHANGING through the face of the coil, so there is no induced EMF set up in the coil.

And that means there's *no induced current* and *no magnetic force* acting to fight the motion of the coil as it moves through the field (there will be that dipole, but it won't retard the motion).



The coil proceeds out of the *B-fld*, leaving it with time. At the point shown:

h.) Is there a magnetic flux through the coil?

yes, magnetic field lines are piercing the face of the coil

i.) Is there an induced EMF set up in the coil (justify)? If so, what is its magnitude?

yes, the magnetic flux is CHANGING through the face of the coil. We won't do the calculation—it will be similar to what we did earlier—but we could.

j.) What is the direction of the induced current?

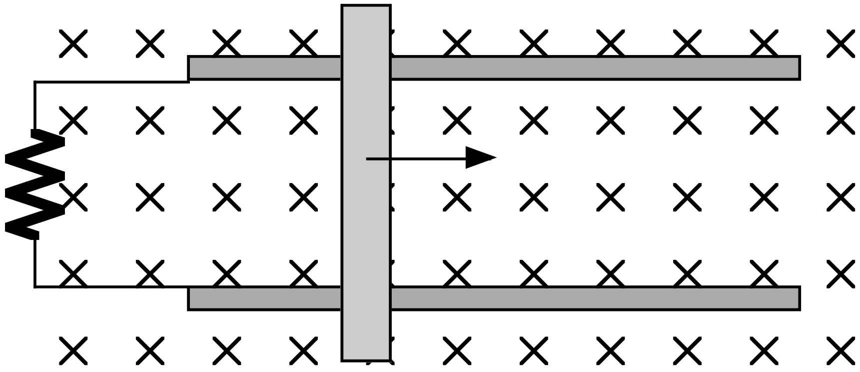
Lenz's Law maintains clockwise (go through the steps for the practice).

e.) The direction of the induced force on the coil?

$\vec{F}_{wire} = i\vec{L}x\vec{B}$ Says the vertical wire will feel a force *to the left* (again—do it!).

Huge observation: Induced currents will ALWAYS generate forces that fight what you are trying to do. Try to move the coil OUT OF THE FIELD—the induced force will fight you. Try to move the coil INTO THE FIELD—the induced force will fight you . . . they always *fight the change*.

Example 5: (courtesy of Mr. White) What direction is the current flow in this loop?

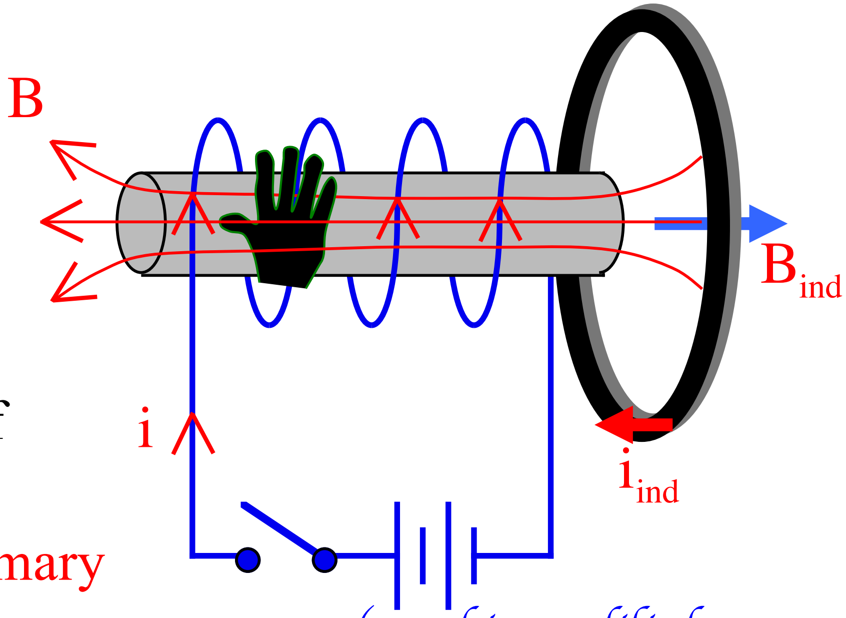


Procedure:

1. Determine direction of magnetic field
2. Determine whether flux in the area is increasing or decreasing in the direction of magnetic field
3. Use second RHR, with fingers pointing in opposite direction, to determine direction of current flow.

Example 6: (courtesy of Mr. White)

For the circuit shown:



(graphic, modified, courtesy of Mr. White)

a.) Is there an induced current in the secondary coil when the switch is thrown? If so, in what direction will the current be?

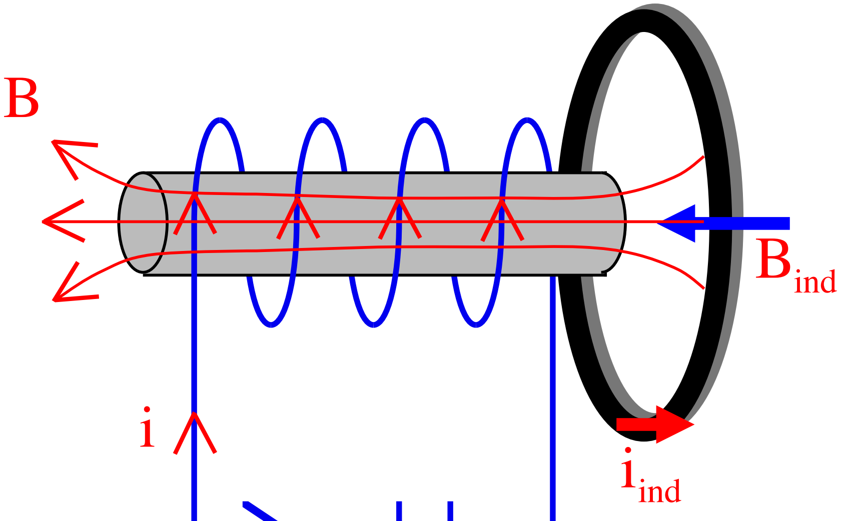
Some very funky stuff happens in the primary coil when the switch is thrown, but what happens in the secondary is straightforward.

The battery-driven current in the primary coil generates a *B-fld* down the axis of the primary coil to the left. As that *B-fld* is increasing, the induced *B-fld* due to induced current in the secondary coil will be OPPOSITE that direction, or to the right. The r.h.r. predicts an induced current in the secondary coil that is clockwise, as viewed from our perspective.

b.) Is there an induced current in the secondary coil after the switch has been closed for a long time? If so, in what direction will the current be?

Nope—once the battery-driven current in the primary coil gets to steady-state, the magnetic flux becomes constant and the induced EMF ceases.

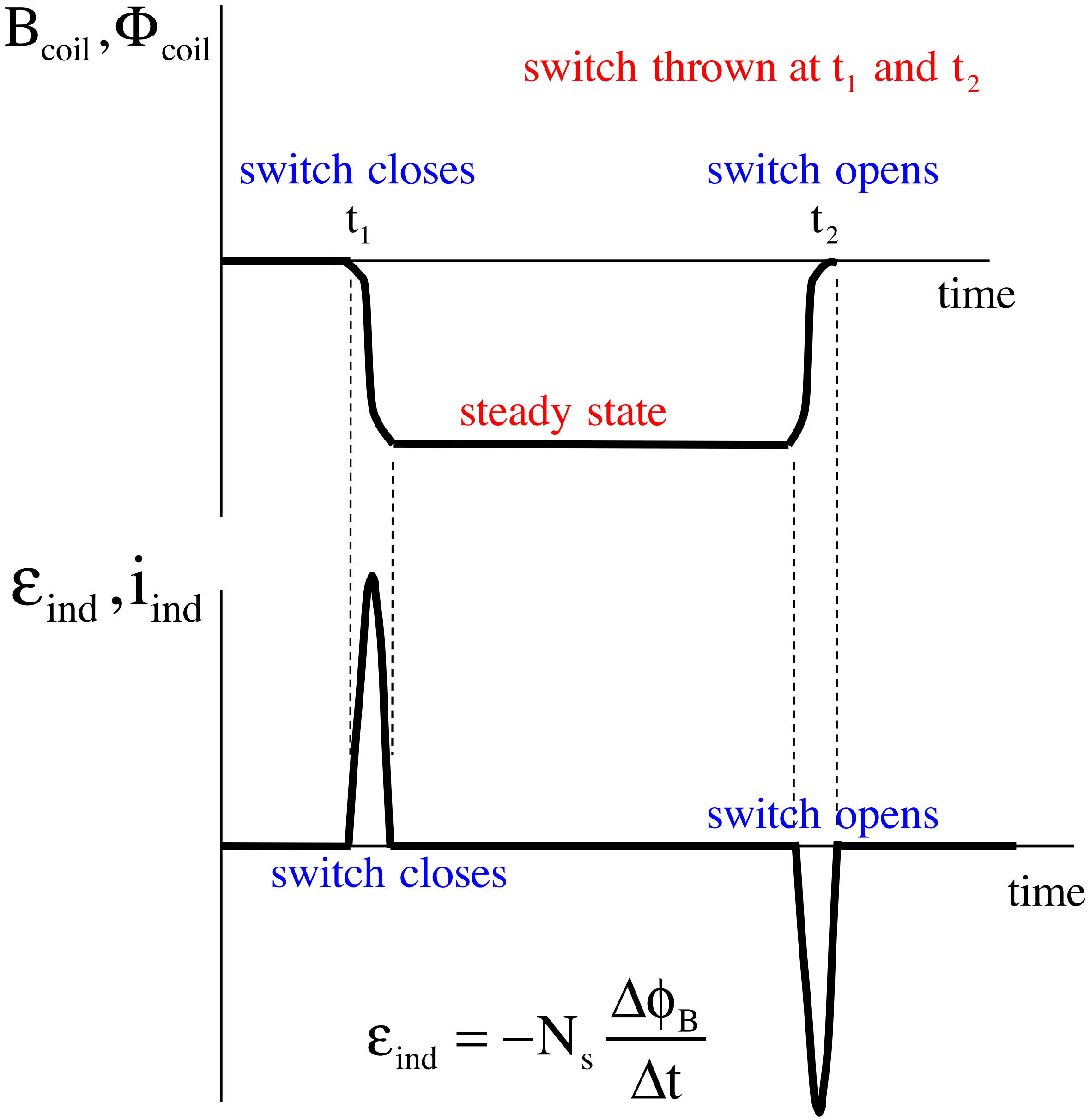
c.) Is there an induced current in the secondary coil when the switch is opened after being closed for a long time? If so, in what direction will the current be?



(graphic, modified, courtesy of Mr. White)

The direction of the coil's B-fld down the coil's axis won't changed, but now it will diminish to zero. That means the induced EMF in the secondary coil will produce an induced *B-fld* that is in the SAME DIRECTION AS the external field, or to the left. That will require a counterclockwise induced current.

d.) What will the graph of the current in the second coil look like as a function of time?



$$\varepsilon_{ind} = -N_s \frac{\Delta\phi_B}{\Delta t}$$

Example 7: (modified from Mr. White)

a.) In what direction will the induced current flow in the loop of wire?

--The direction of the *external B-fld* is?

downward

--Is the flux increasing or decreasing?

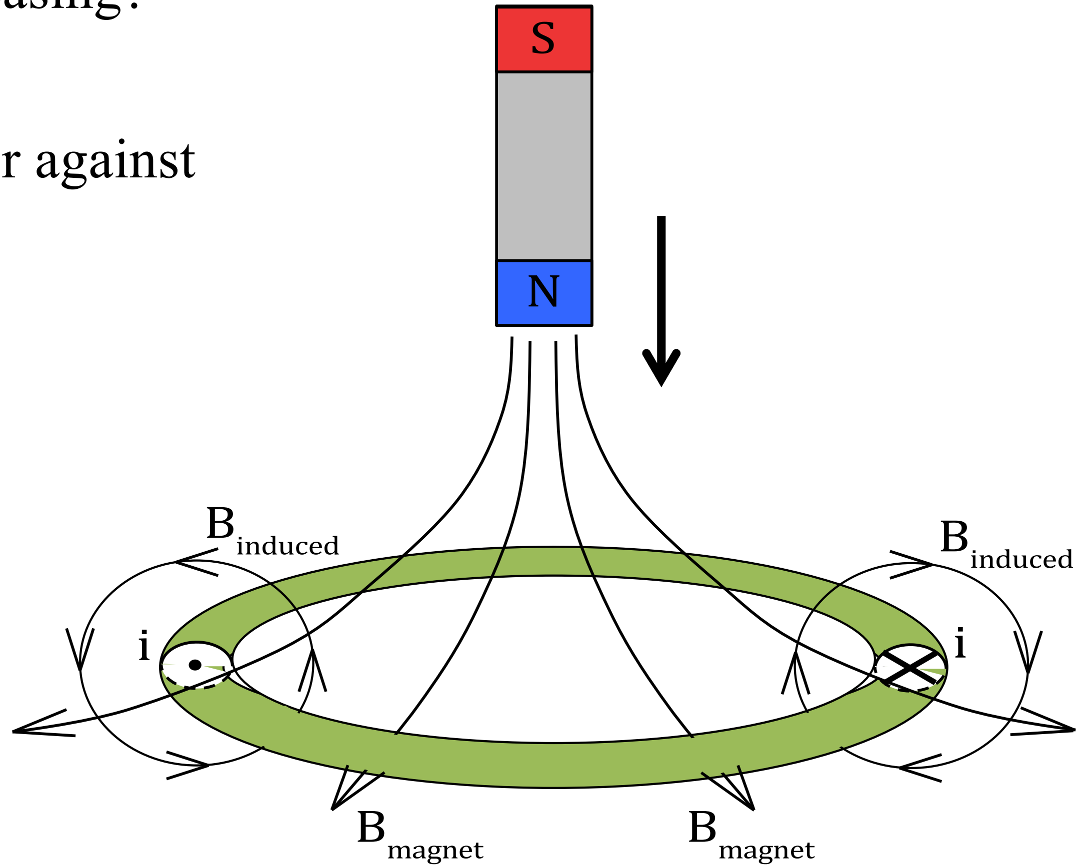
it's increasing

--So the *induced B-fld* is with or against the external field?

against

--Which means the induced current must be

using the modified right-hand trick to determine the direction of a current-carrying coil's B-fld down its axis, you get *into the page* on the right . . .

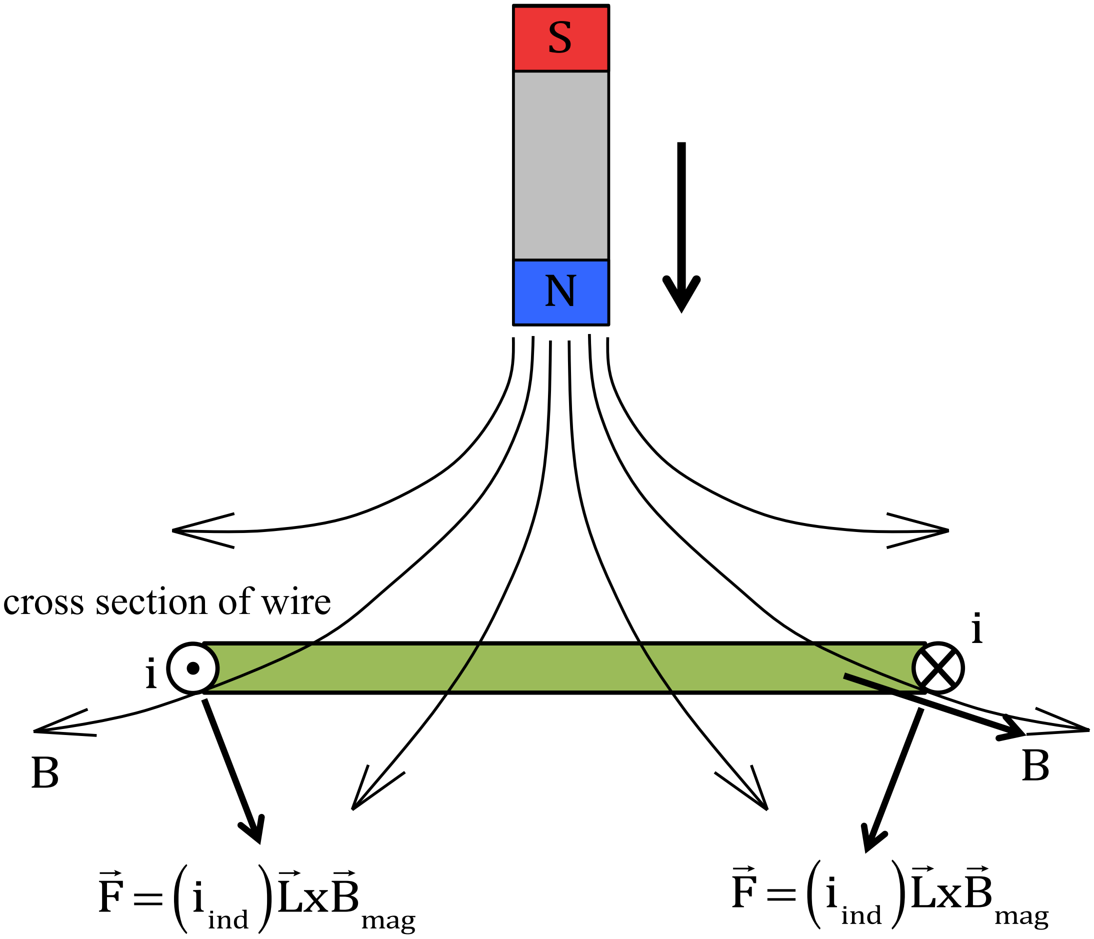


b.) Will there be a force on the coil, and in what direction will it be if there is one? How about the magnet?

Yes, there will be. There are two ways to see the direction.

The hard way:

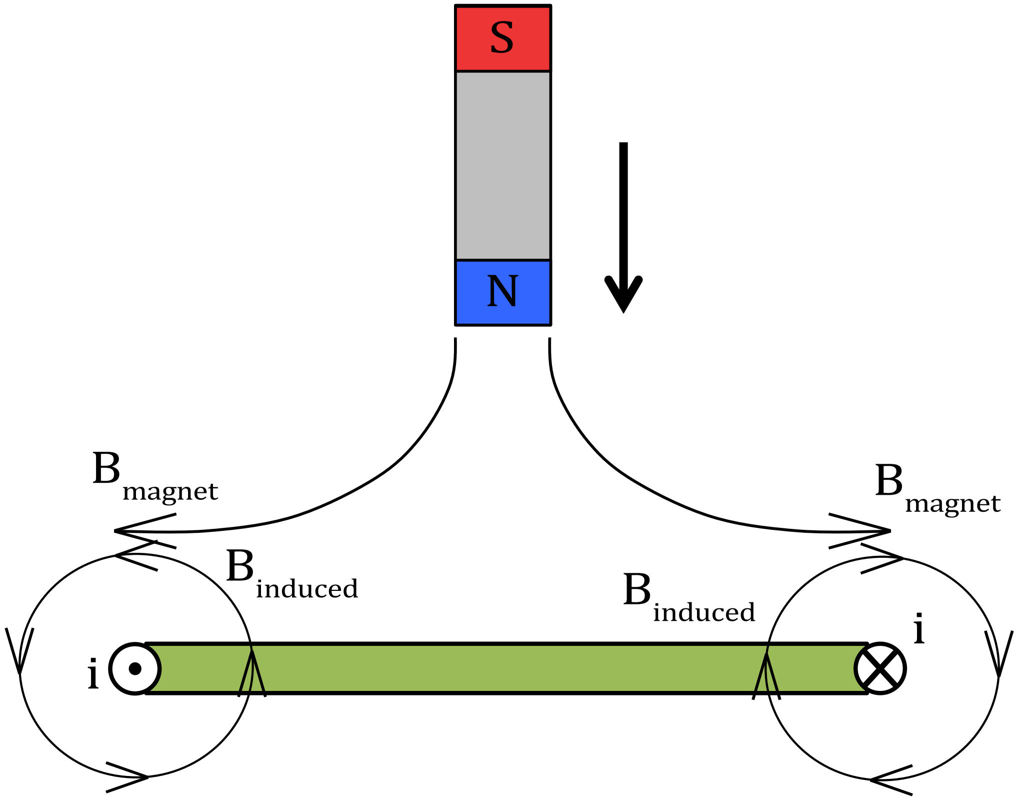
The magnet is producing an external B-fld. There is a current in the wire (it's induced, but it's nevertheless there). A current in a section of wire will feel a force equal to $\vec{F} = \left(i_{ind}\right)\vec{L}x\vec{B}_{mag}$.



Consider the section of wire at the far right side of the coil in the plane of the page (i.e., where the ⊗ is. At that point, the "i" is into the page and B is as shown. The cross product yields a force as shown. A similar force is shown on the other side of the coil. Notice that the horizontal components will cancel leaving only downward vertical components. Translation: the coil will feel a force of repulsion that is DOWNWARD, while the magnet will feel an equal and opposite force UPWARD (Newton's third law).

Easy way #1:

When the region between two B-fld producing structures has magnetic field lines that are parallel to one another (or parallelish), the structures will be acting like *like-poles* and you will get repulsion. That's the case in this situation, so the coil will be repulsed by the magnet and feel a force DOWNWARD.



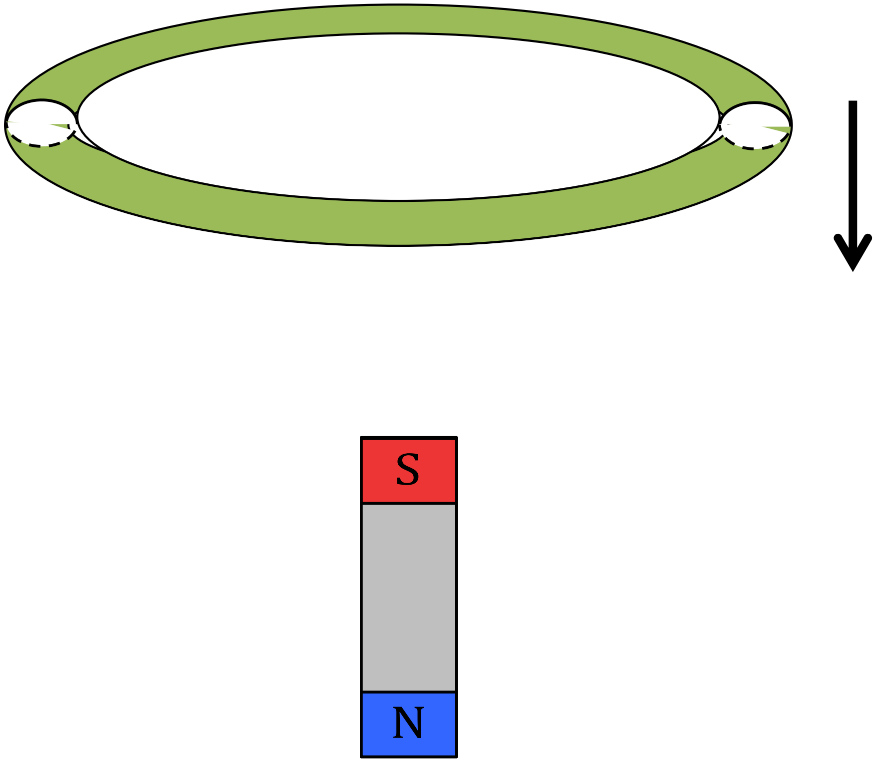
Easy way #2:

IN ALL CASES, the induced magnetic field and current will set itself up in such a way as to OPPOSE WHATEVER CHANGE is occurring. In this case, you are trying to shove the magnet toward the coil (or the coil toward the magnet), so the induced force will fight you by generating a repulsive force between the entities . . . and the force on the coil will be DOWNWARD.

Example 8: (modified from Mr. White)

a.) In what direction will the induced current flow in the loop of wire?

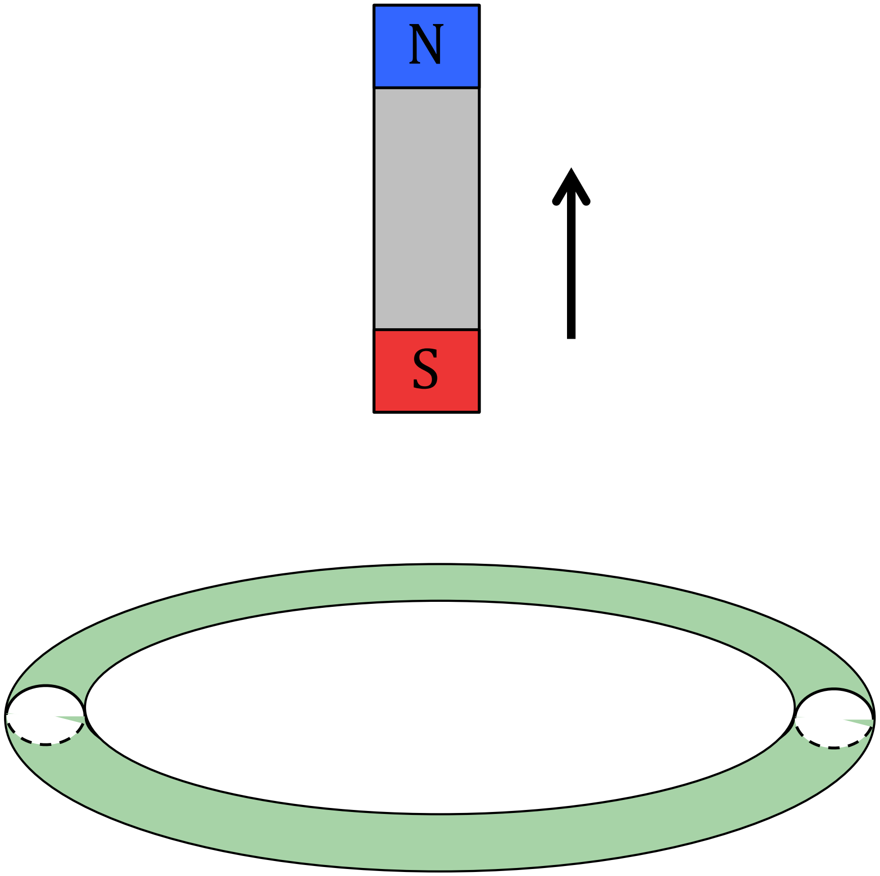
b.) Will there be a force on the coil, and in what direction will it be if there is one? How about the magnet?



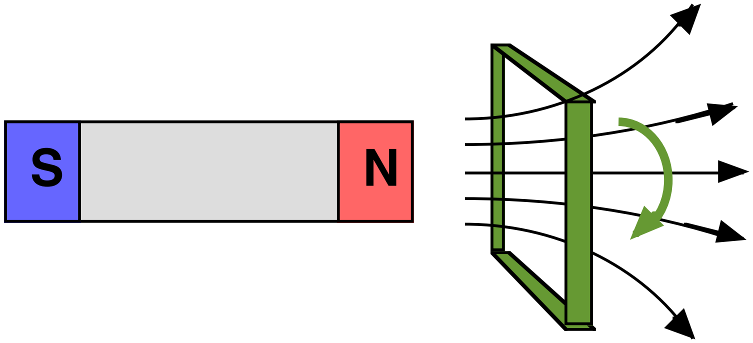
Example 9: (modified from Mr. White)

a.) In what direction will the induced current flow in the loop of wire?

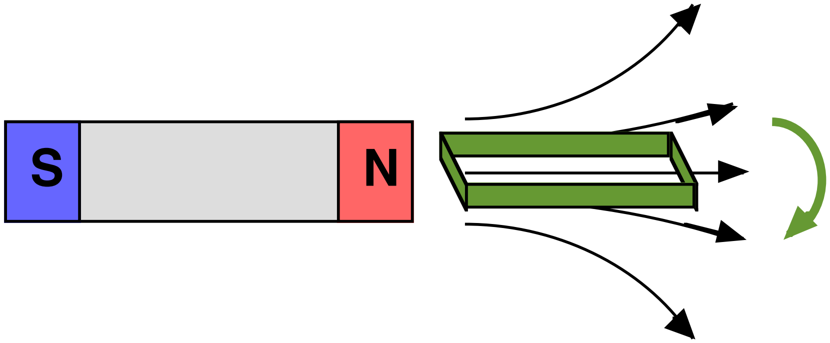
b.) Will there be a force on the coil, and in what direction will it be if there is one? How about the magnet?



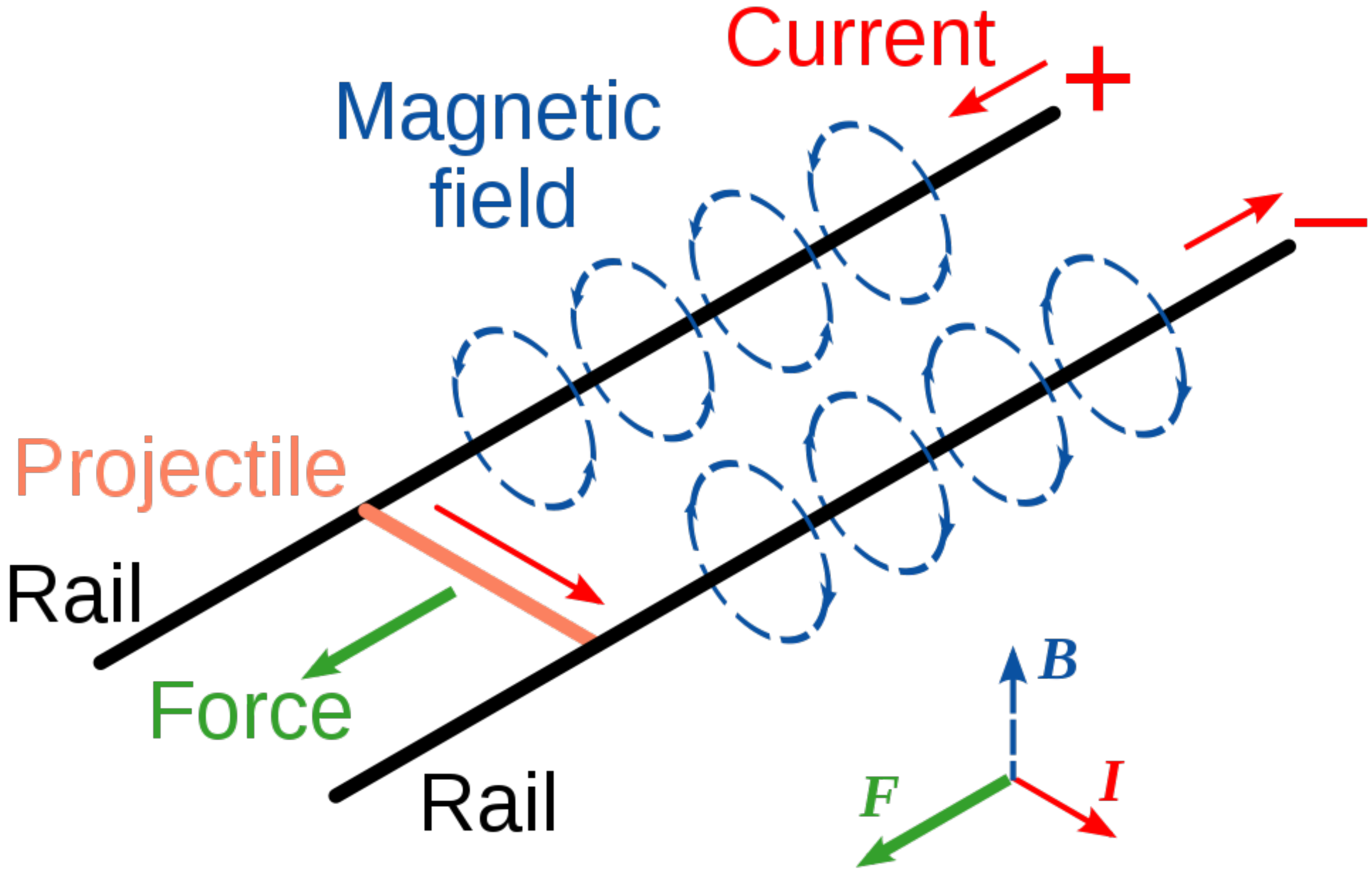
Example 10: (courtesy of Mr. White) In what direction will the induced current flow in a loop of wire shown?



Example 11: (courtesy of Mr. White) In what direction will the induced current flow in a loop of wire shown?



Example 13: (courtesy of Mr. White) Given the schematic diagram, explain how the "rail gun" works to accelerate a projectile. (Note that this is how they propel the Superman ride at Magic Mountain.)



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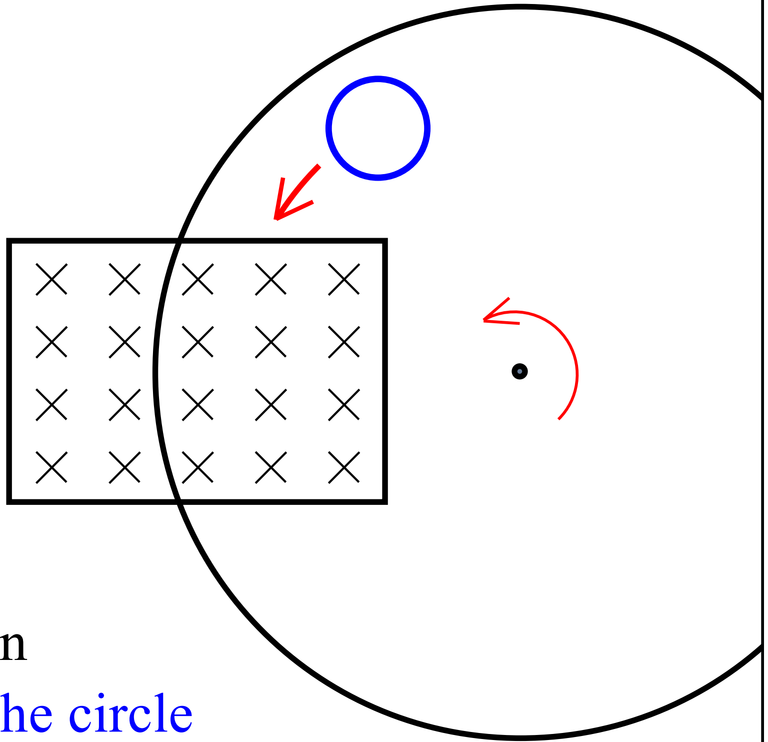
https://www.youtube.com/watch?v=eObepuHvYAw

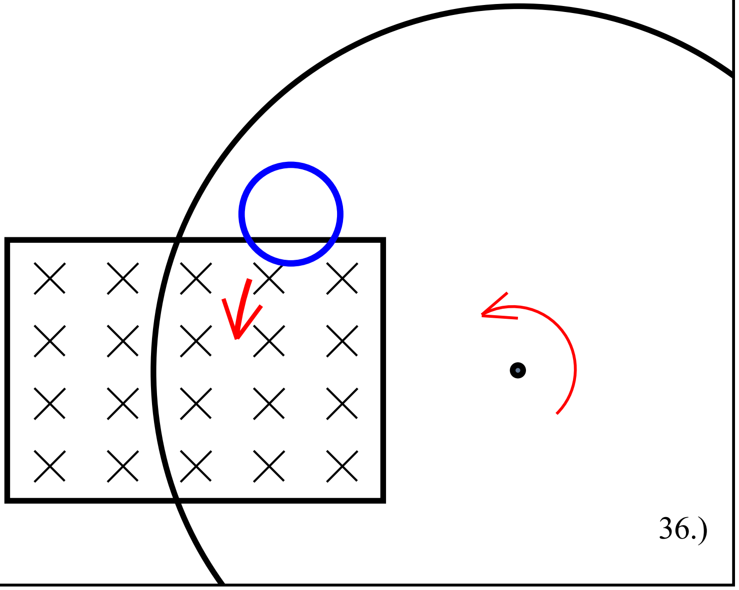
Eddy Currents



Aluminum is NOT magnetizable (it's a metal, so you can get a current in it, but it doesn't have magnetic domain like iron). So consider an aluminum disk rotating about its central axis that has a portion of its surface continuously passing through a *B-fld* directed *into the page*.

What is going to happen in this situation?

This is easiest to see by considering what will happen underneath a circle drawn on the disk if we think of the circle as being a coil of wire, and watch it over time.



--While away from the magnetic field, there will be no magnetic flux through the coil, so there will be no induced current in the coil.

--As it enters the magnetic field, there will be a change of magnetic flux and an induced current.

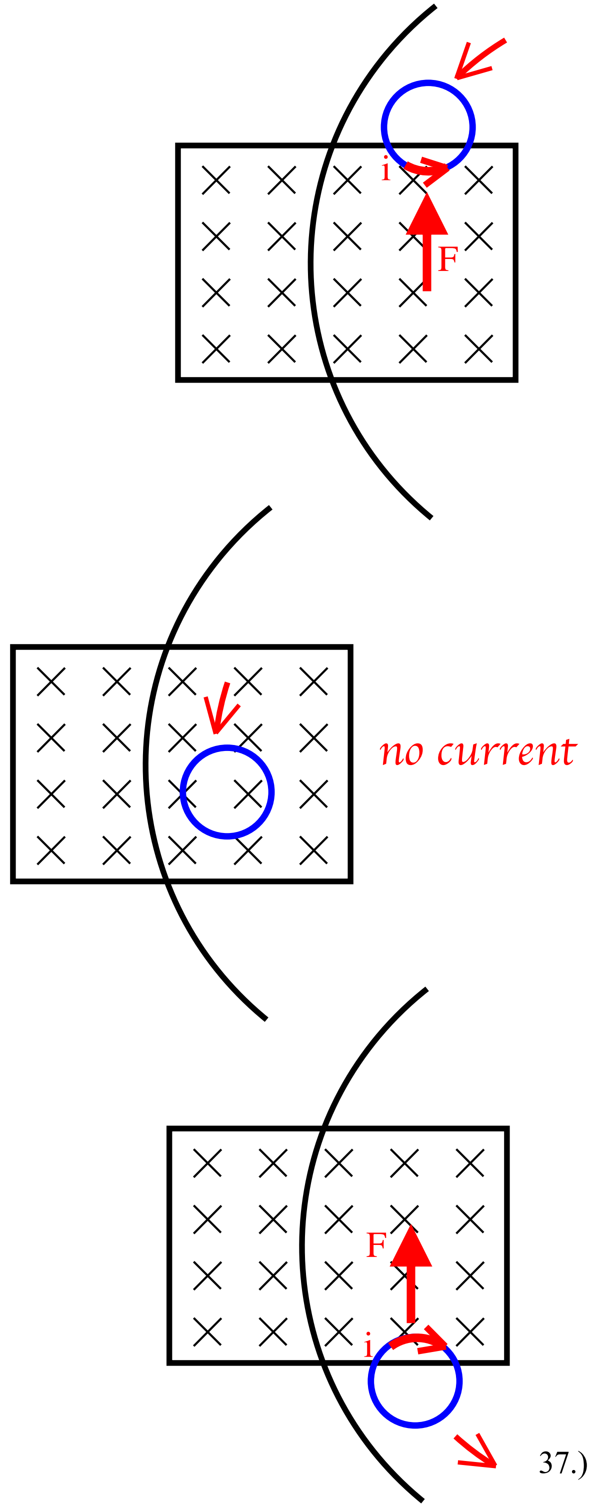
--The flux is increasing, so the induced current will be counterclockwise (producing an induced *B-fld* opposite the external field and out of the page).

> *--The induced current* will interact with the external *B-fld* producing a force via $\vec{F}_{wire} = i\vec{L}x\vec{B}$ that is *upward,* fighting the entrance of the "coil" into the external *B-fld.*

--Once completely into the *B-fld,* there will no longer be a changing magnetic flux, and the induced current and induced force will go away.

--As it begins to exit the *B-fld,* there will again be a changing magnetic flux and the induced current and induced force, with this induced force again being directed upward (check it).

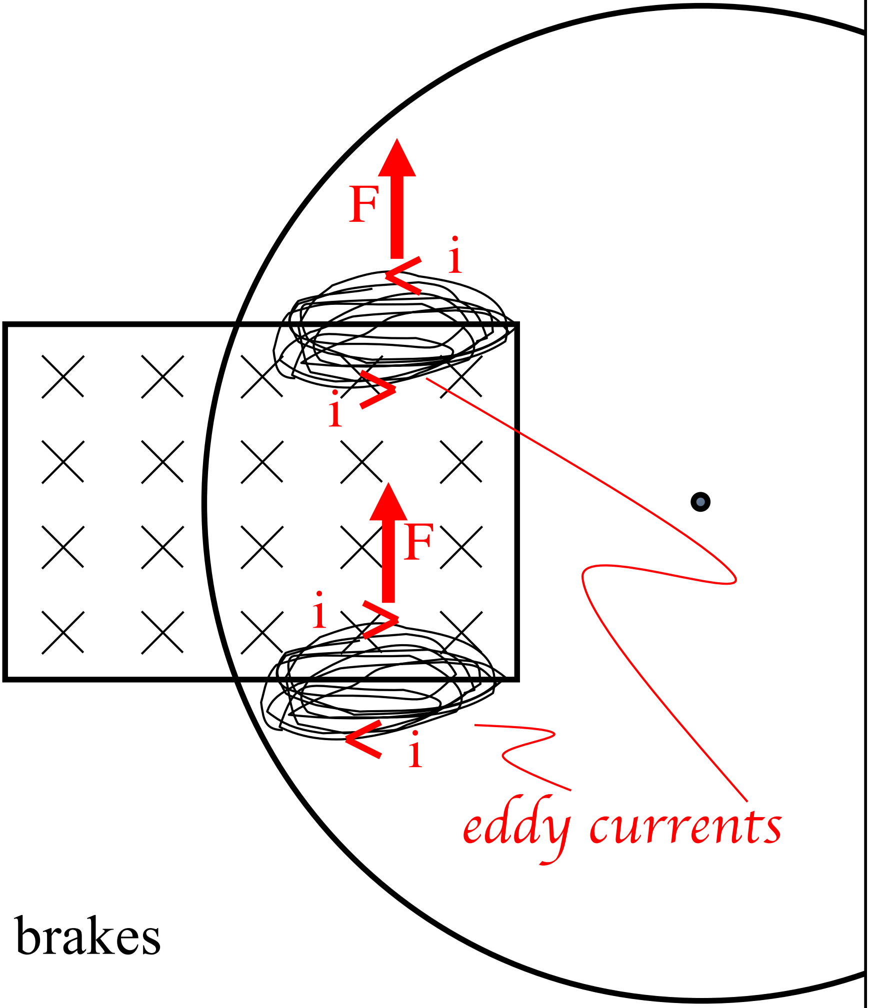
--And once out again and on its way, no changing flux and no force.



What's really happening is not a single loop of wire moving into, then out of a *B-fld* but, rather, a continuous piece of metal moving through our *B-fld*.

The consequence is a permanent swirl of charge, called an *eddy current*, at the boundary of the *B-fld*.

These eddy currents constantly create a retarding force on the disk, hence the *eddy current brake*.



Large, heavy objects like trains use eddy current brakes for braking.

Example 14: What will happen when a magnet is dropped down an aluminum tube?

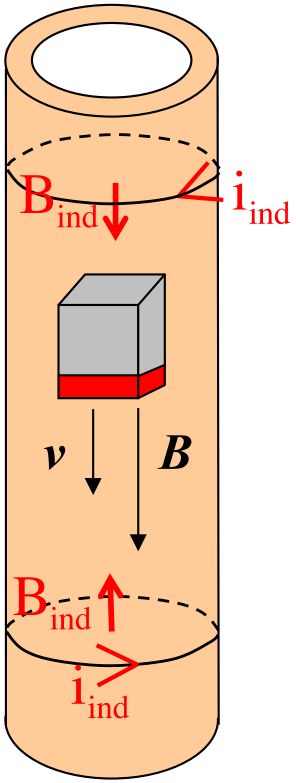
--Consider a loop below the position of the magnet at a given instant. There is an increasing magnetic flux through it, so the induced current and its associated induced *B-fld* will set itself up how?

> *--Notice* we find *two north poles* juxtaposed against one another due to the induced current. This will produce a magnetic force UPWARD on the magnet.

--Now consider a loop above the magnet at a given instant. There is an decreasing magnetic flux through it, so the induced current and its associated induced *B-fld* will set itself up how?

> *--Notice* we find one *north poles* and one *south pole* juxtaposed against one another. This will produce a magnetic force UPWARD on the magnet.

--Bottom line: It should take a *long time* for the magnet to free fall to the bottom of the tube!



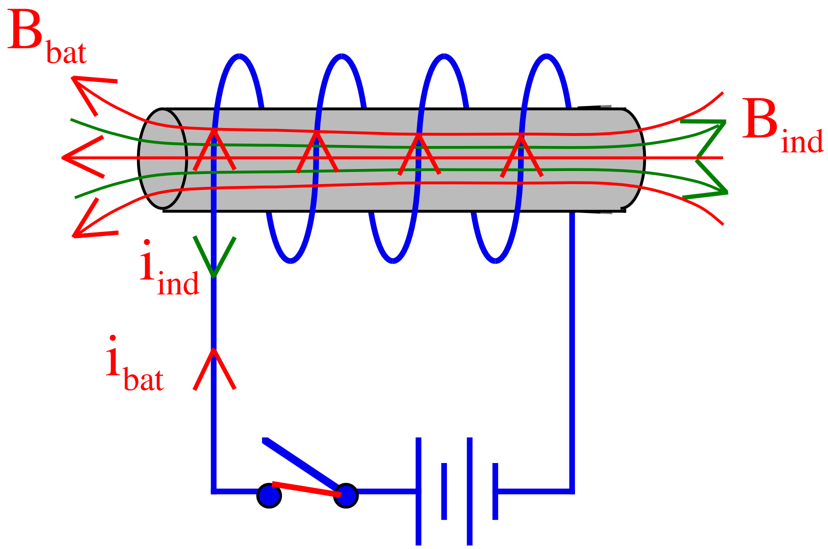
(graphic, with considerable modification, courtesy of Mr. White)

Maxwell's Equations

Around 1862, James Clerk Maxwell compiled the four equations, known as Maxwell's equations, that govern the world of electricity and magnetism. The equations come in two forms, the integral form you are familiar with and the derivative form used for boundary value problems. Both are shown below:

	Integral form	*Differential form*
Gauss's Law	$\Phi_E = \oint_S \vec{E} \bullet d\vec{A} = \frac{q_{enc}}{\varepsilon_o}$	$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_o}$ where ρ is charge per unit volume
Gauss's Law for Magnetism	$\Phi_B = \oint_S \vec{B} \bullet d\vec{A} = 0$	$\vec{\nabla} \bullet \vec{B} = 0$
Faraday's Law	$\oint \vec{E} \bullet d\vec{l} = -\frac{d\Phi_B}{dt}$	$\vec{\nabla} x \vec{E} = -\frac{\partial B}{\partial t}$
Ampere's Law	$\oint \vec{B} \bullet d\vec{l} = \mu_o i_{enc} + \mu_o \left(\varepsilon_o \frac{d\Phi_E}{dt} \right)$	$\vec{\nabla} x \vec{B} = \mu_o \left(J + \varepsilon_o \frac{\partial \vec{E}}{\partial t} \right)$ where J is current per unit area

Inductance



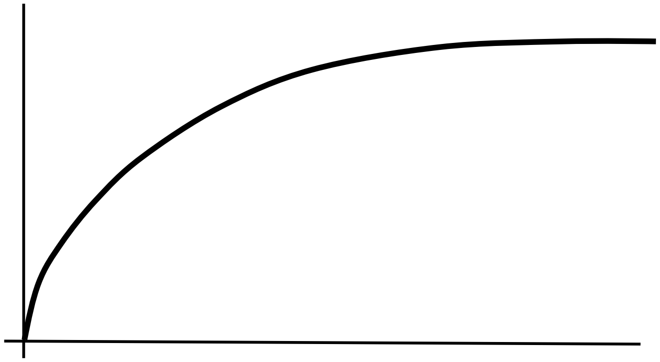
Example 15: When this set-up (somewhat modified) was first presented, it was pointed out that, "Some very funky stuff happens in the primary coil when the switch is thrown." What is that funkiness?

--*As the current* increases from zero to something, the *B-fld* down the coil's axis increases.

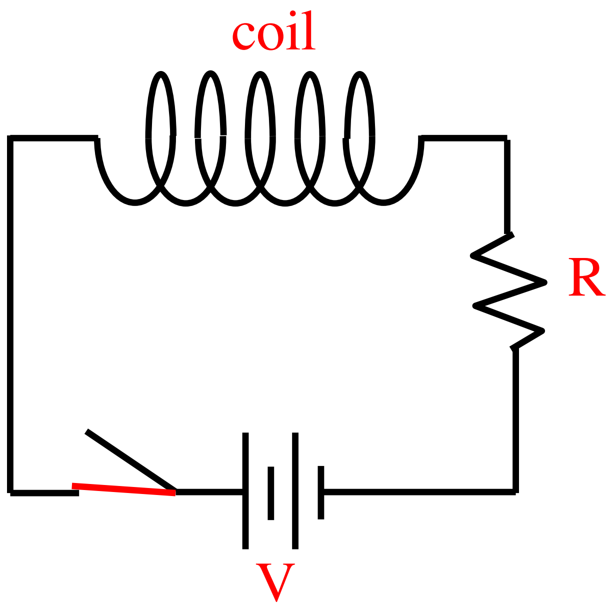
--*That increase* of *B-fld* in turn produces an increasing magnetic flux.

--*The increasing flux* produces a back EMF that tries to force current to flow *opposite* the direction of the current generated by the battery (that is, a current that *fights the change*) and a *B-fld* opposing the battery's *B-fld*.

--*The net effect* is that current in the coil is initially stymied, increasing only slowly with time.



The circuit symbol for a coil is shown in the circuit to the right. We need some way to quantify how "big" the coil is.



Using Faraday's Law, we could identify the induced EMF generated in the coil due to the throwing of the switch. That would be:

$$\varepsilon_{\text{ind}} = -N\frac{d\Phi_B}{dt}$$

Problem is, a coil in an electrical circuit is often encased in ceramic, so there is no easy way to measure the *time rate of change of magnetic flux* down its axis.

What was observed was that it is the change of current that actually induces the EMF, which means we *could* write: $\varepsilon_{\text{ind}} \alpha -\frac{di}{dt}$

To make this into an equality, we need a proportionality constant. In this case, the constant is called the *inductance* of the coil. Its symbol is *L* and its units are *henrys*. With that, Faraday's Law becomes:

$$\varepsilon_{\text{ind}} = -L\frac{di}{dt}$$

Observations

Notice that all of the parameters that characterize the size of the circuit elements have been proportionality constants linking two characteristics of an element.

The proportionality constant between the voltage across a resistor and the current through a resistor is the resistance *R* of a resistor.

$$V_{\text{across}} = R\left(i_{\text{thru}}\right)$$

The proportionality constant between the voltage across a capacitor and the charge on one plate is the *capacitance C* of a capacitor.

$$Q_{\text{on one plate}} = C\left(V_{\text{across plates}}\right)$$

The proportionality constant between the induced EMF of a coil and the rate as which current changes in the coil is the *inductance L* of an inductor.

$$\varepsilon_{\text{ind}} = -L\frac{di}{dt}$$

Additional Observations

Coils are called by several names. The most common are:

Coils: Mundane, but to the point;

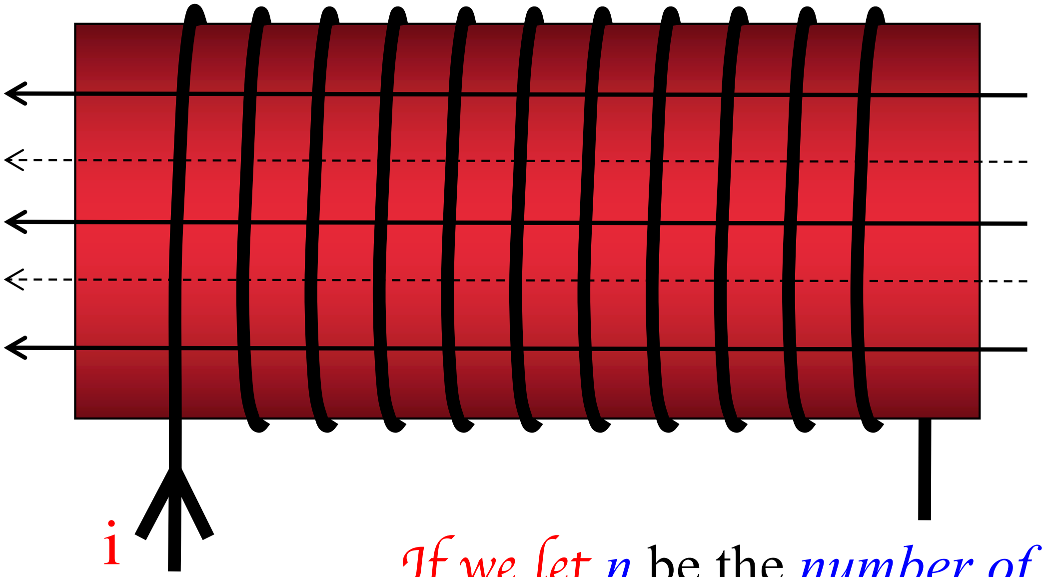
Solenoids: This term was used earlier;

Inductors: This is what coils are called when being used as a circuit element;

Chokes: This is slang used by electronics nerds in the 1950's. You get extra points if you use terms like this in the presence of old people.

Minor Point: Because inductors are made of wire, inductors have resistor-like resistance associated with them. That means that when an inductor is placed in a circuit, you will often be told both its inductance L and it's resistance r_L.

Example 16: Derive an expression for the inductance of a solenoid of length *l,* radius *r* and total number of turns *N.*



$$\Phi_B = B_{coil} A \cos 0^0$$
$$= B_{coil} A$$
$$= (\mu_o n i)(\pi r^2)$$
$$= \left(\mu_o \frac{N}{l} i\right)(\pi r^2)$$

$$L\frac{di}{dt} = N\frac{d\Phi_B}{dt}$$

$$\Rightarrow \quad L\int_0^i \frac{di}{dt}dt = N\int_0^{\Phi_B}\frac{d\Phi_B}{dt}dt$$

$$\Rightarrow \quad Li = N\Phi_B$$

$$\Rightarrow \quad Li = N\left(\mu_o \frac{N}{l} i\right)(\pi r^2)$$

$$\Rightarrow \quad L = \left(\mu_o \frac{N^2}{l}\right)(\pi r^2)$$

If we let n be the *number of winds per unit length* (i.e., *N/l*), and noting the volume of the coil is $\pi r^2 l$, we can write:

$$L = \frac{\mu_o N^2 \pi r^2}{l}$$

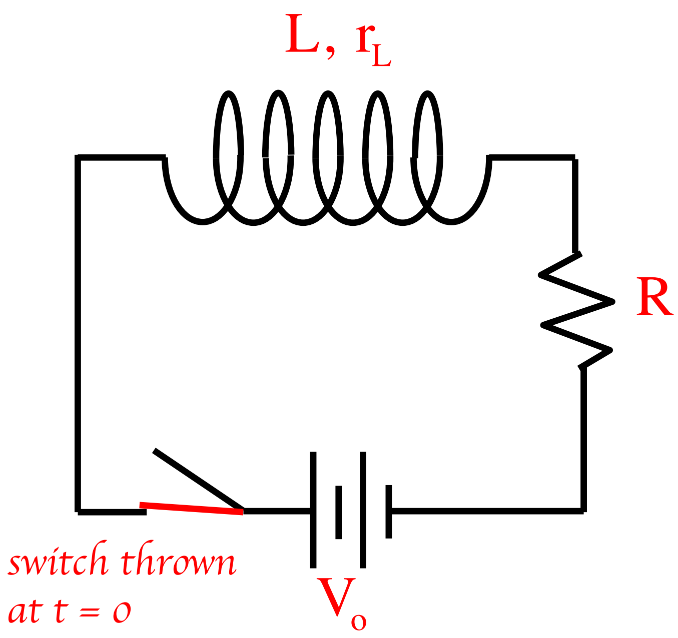
$$= \frac{\mu_o (n^2 l^2)\pi r^2}{l} = \frac{\mu_o (n^2 \cancel{l})\overset{\text{volume}}{\cancel{(\pi r^2 l)}}}{\cancel{l}}$$

$$= \mu_o n^2 V$$

Example 17: For the circuit to the right:

a.) What is the initial current in the circuit just after the switch is closed?

The back-EMF in the coil generated by the attempt of the battery to increase current in the coil (hence increase the *B-fld down the coil's axis*) will fight the build-up of current in the system, so the initial current will be *ZERO*!



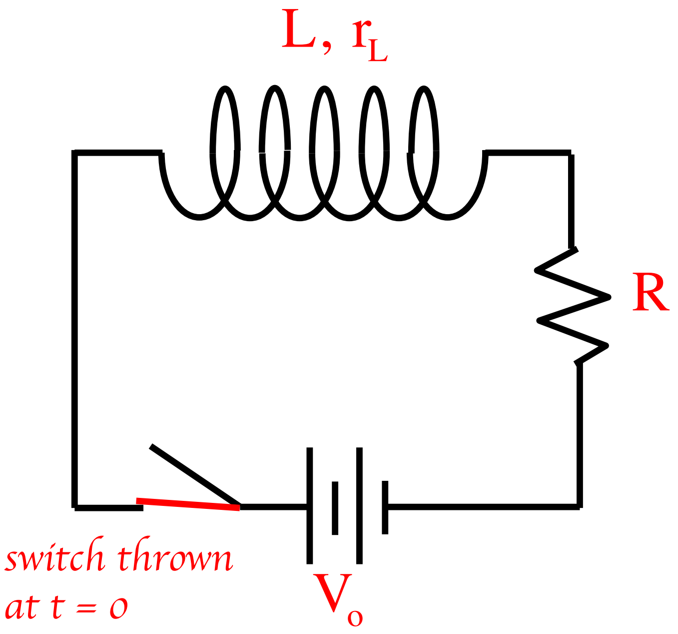
b.) What is the current in the circuit after a long period of time?

Once the current goes steady-state, the inductor will no longer experience a CHANGING magnetic flux (there *will* be a magnetic flux, but it won't be changing), and all of the voltage drops in the circuit will be due to resistor-like resistance. In other words:

$$i_{max} r_L + i_{max} R = V_o$$

$$\Rightarrow \quad i_{max} = \frac{V_o}{(r_L + R)}$$

$$= \frac{V_o}{R_{net}}$$



c.) Write a differential equation that characterizes the *current as a function of time*.

This is a Kirchoff's Law problem. Summing the voltage changes around a closed path yields:

$$-L\frac{di}{dt} - ir_L - iR + V_o = 0$$

d.) Solve the differential equation for the current as a function of time.

This is exactly like the differential equations we've solved before, complete with the natural log function and exponential solution (see next page for the gruesome details).

$$-L\frac{di}{dt} - ir_L - iR + V_o = 0 \quad \Rightarrow \quad L\frac{di}{dt} + i\left(r_L + R\right) = V_o$$

$$\Rightarrow \quad \frac{L}{R_{net}}\frac{di}{dt} + i = \frac{V_o}{R_{net}} \quad \Rightarrow \quad \frac{L}{R_{net}}\frac{di}{dt} + i = i_{max}$$

$$\Rightarrow \quad -\frac{L}{R_{net}}\frac{di}{dt} = i - i_{max} \Rightarrow \quad \int_{i=0}^{i}\frac{di}{\left(i - i_{max}\right)} = -\frac{R_{net}}{L}\int_{t=0}^{t} dt$$

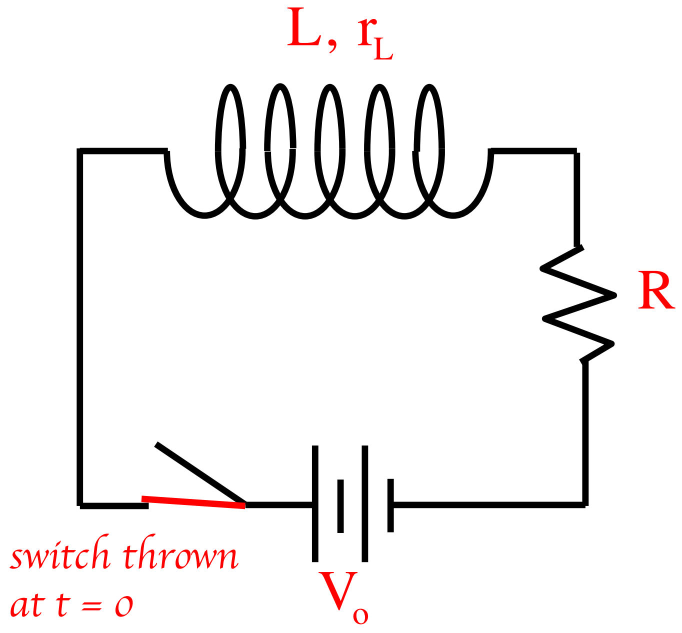
$$\Rightarrow \quad \ln\left|i - i_{max}\right|\Big|_{i=0}^{i} = -\left(\frac{R_{net}}{L}\right)t \quad \Rightarrow \quad \ln\left|i - i_{max}\right| - \ln\left|-i_{max}\right| = -\left(\frac{R_{net}}{L}\right)t$$

$$\Rightarrow \quad \ln\left(i_{max} - i\right) - \ln\left(i_{max}\right) = -\left(\frac{R_{net}}{L}\right)t \quad \Rightarrow \quad \ln\left(\frac{i_{max} - i}{i_{max}}\right) = -\left(\frac{R_{net}}{L}\right)t$$

$$\Rightarrow \quad e^{\ln\left(\frac{i_{max} - i}{i_{max}}\right)} = e^{-\left(\frac{R_{net}}{L}\right)t} \quad \Rightarrow \quad \left(\frac{i_{max} - i}{i_{max}}\right) = e^{-\left(\frac{R_{net}}{L}\right)t}$$

$$\Rightarrow \quad i_{max} - i = i_{max}e^{-\left(\frac{R_{net}}{L}\right)t}$$

$$\Rightarrow \quad i(t) = i_{max}\left(1 - e^{-\left(R_{net}/L\right)t}\right)$$



e.) *What is the circuit's* time constant?

As was done with capacitors, the time constant is the inverse of the argument of the exponential in the current function. That means that for our inductor:

$$\tau = L/R_{net}$$

Again, as before, after one time constant the current will be:

$$i(t) = i_{max}\left(1 - e^{-\left(R_{net}/L\right)\left(L/R_{net}\right)}\right)$$

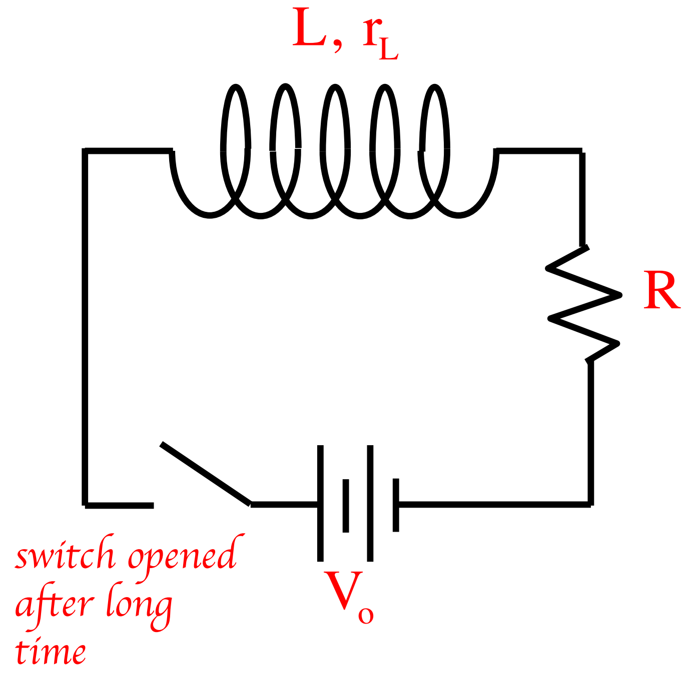
$$= i_{max}\left(1 - e^{-1}\right)$$

$$= i_{max}\left(1 - \frac{1}{2.7}\right)$$

$$= .63i_{max}$$

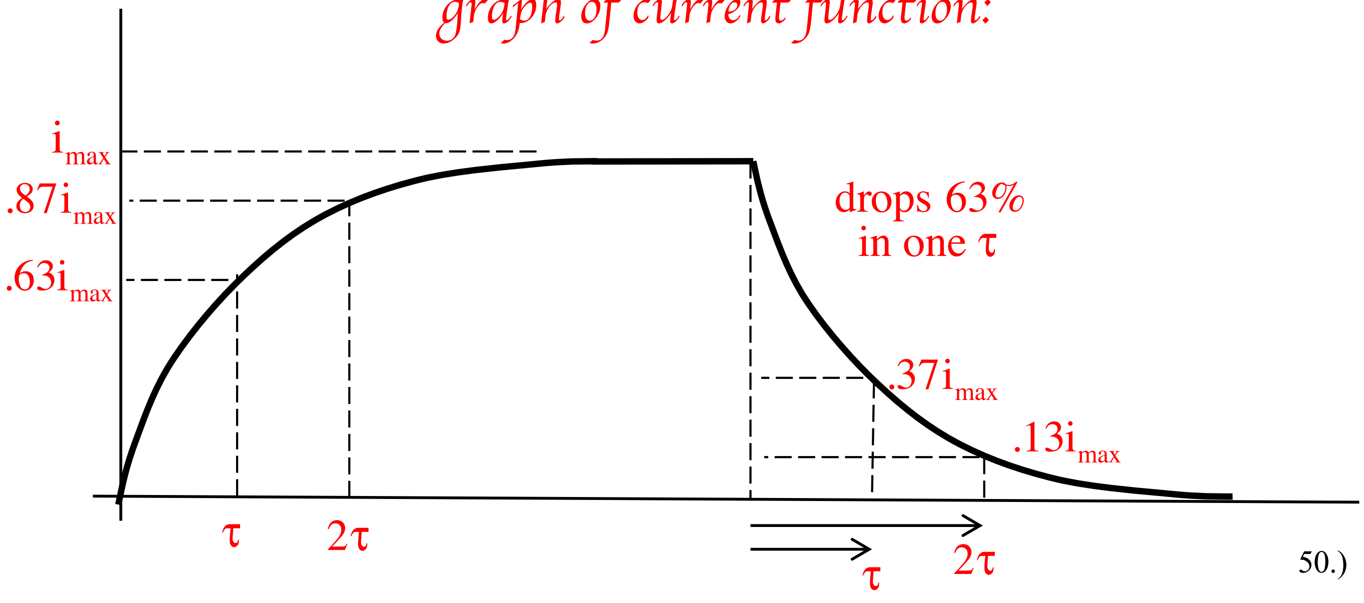
f.) If current has been flowing for a long time, what happens when you open the switch?

An attempted drop in battery-current will instigate an attempted drop in *B-fld* down the axis of the coil. That will induce an EMF that *fights the change*, which in this case means it will force current to flow *even longer than it normally would*. Due to the symmetry of the situation, it will take *one time constant* for the current to drop 63% of its maximum.



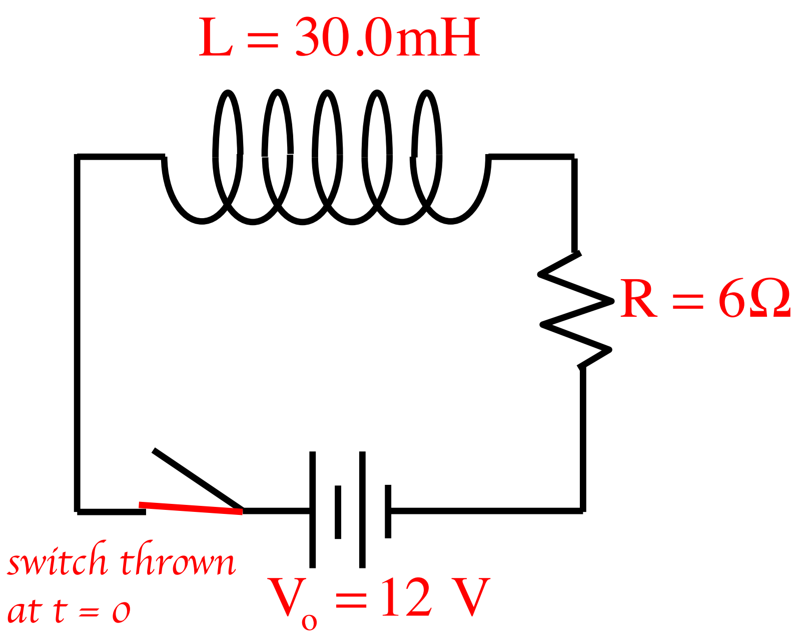
g.) The switch is closed, then after a long time it is opened. What will a graph of the *current vs time* look like for the system?

graph of current function:



Example 18: (problem courtesy of Mr. White)

For the circuit to the right:



a.) What is the time constant for the circuit?

$$\tau = \frac{L}{R_{net}}$$

$$= \frac{30x10^{-3}H}{6\Omega}$$

$$= 5x10^{-3}s$$

b.) What is the current just after the switch is thrown? *zero*

c.) What is the current

at *t = 2.0 milli-seconds*?

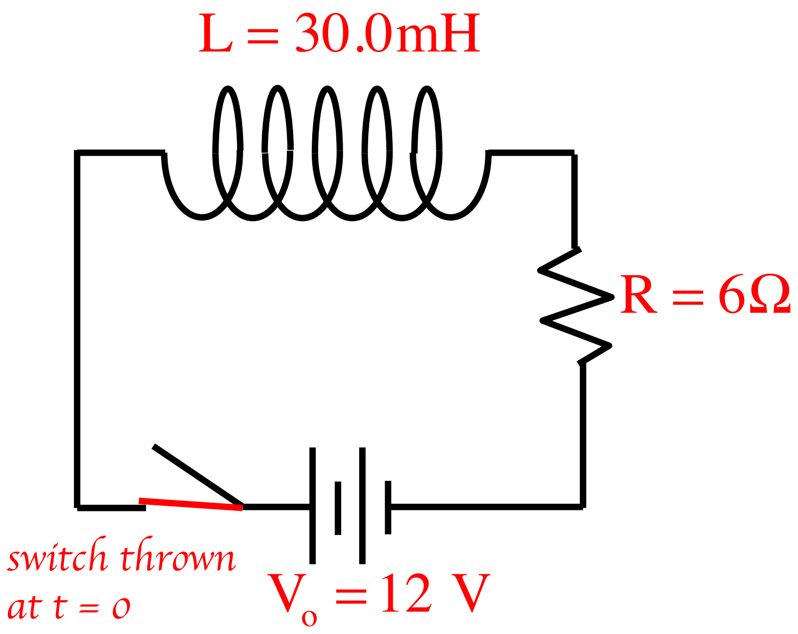
$$i(t) = i_{max}\left(1 - e^{-\left(R_{net}/L\right)t}\right)$$

$$= \frac{V_o}{R_{net}}\left(1 - e^{-t/\tau}\right)$$

$$= \frac{(12\ V)}{(6\ \Omega)}\left(1 - e^{-\left(2x10^{-3}\right)/\left(5x10^{-3}\right)}\right)$$

$$= .66\ A$$

d.) What is the current after one time constant?

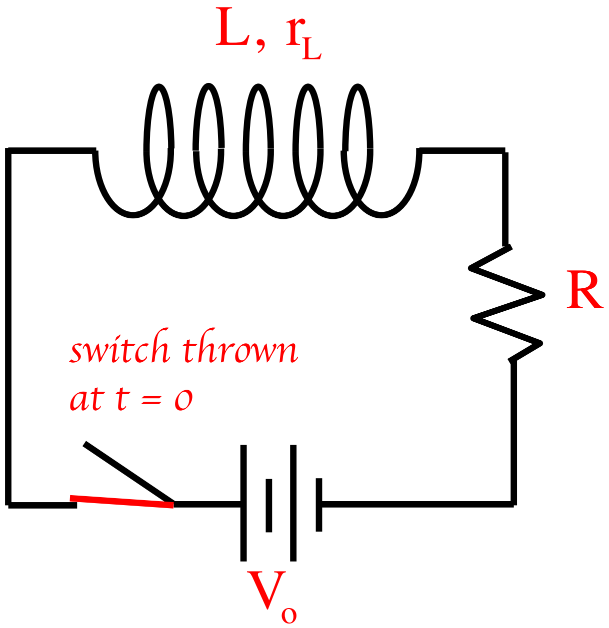


$$
\begin{aligned}
i(t=\tau) &= .63 i_{max} \\
&= .63\left(\frac{V_o}{R_{net}}\right) \\
&= .63\frac{(12\ \text{V})}{(6\ \Omega)} \\
&= 1.26\ \text{A}
\end{aligned}
$$

e.) What is the current a long time after the switch is thrown?

$$
\begin{aligned}
i_{max} &= \left(\frac{V_o}{R_{net}}\right) \\
&= \frac{(12\ \text{V})}{(6\ \Omega)} \\
&= 2.0\ \text{A}
\end{aligned}
$$

Energy in an Inductor



Think back. A *capacitor* stores its energy in the *electric field* that exists between its plates.

Not surprisingly, an inductor stores *its* energy in the *magnetic field* that exists down its axis.

To see *how much* energy is stored in an inductor, consider the circuit shown to the right.

Kirchoff's Law yields: $V_o - iR_{net} - L\frac{di}{dt} = 0$

Multiplying by the current yields: $iV_o - i^2R_{net} - Li\frac{di}{dt} = 0$

Notice:

-- iV_o is the power provided by the battery;

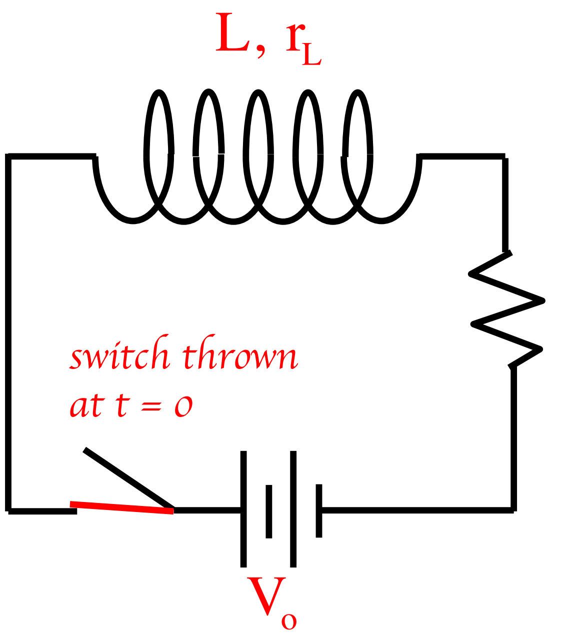
-- i^2R_{net} is the power dissipated by the resistors in the circuit;

-- $Li\,{}^{di}\!/_{dt}$ *must be* the power (i.e., the work the inductor does per unit time) associated with the inductor while in the circuit.

Knowing the power rating of an inductor, we can write:

$$P = \frac{dW}{dt} = \frac{-dU}{dt}$$

$$= -Li \, {}^{di}\!/_{dt}$$



Note: The *negative sign* in the second line is due to the fact that the power *stored* in an inductor (versus the power *dissipated* by an inductor) will (using Faraday's Law) be:

$$i\varepsilon = i\left(-L\frac{di}{dt}\right)$$

Continuing:

$$\frac{-dU}{dt} = -Li \, {}^{di}\!/_{dt}$$

$$\Rightarrow \quad dU = (Li)\,di$$

$$\Rightarrow \quad \int dU = L\int_{i=0}^{i} i\,di$$

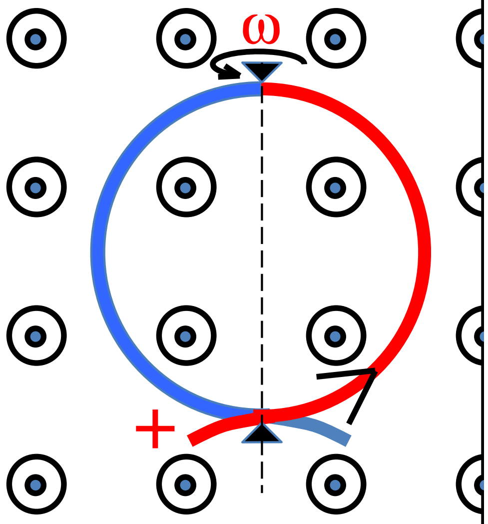
$$\Rightarrow \quad U_L = \frac{1}{2}Li^2$$

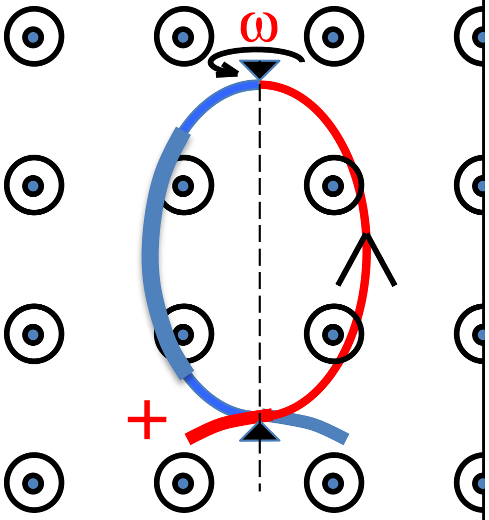
The Production of AC

As one more nod to the use of Lenz's Law, consider a coil that is spinning with constant angular velocity ω in a constant *B-fld* coming *out of the page*. To make the *keeping track of things* easier, I've made one side of the coil blue and one side red, and I'm making the side closest to you bigger.

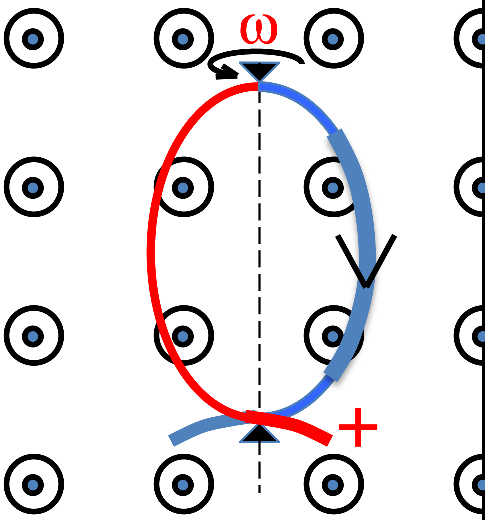
We want to keep track of the induced current (or, more to point, which terminal is positive and which negative) as the coil rotates.

--At the point shown in the rotation, the loop's external magnetic flux is entering a period of diminishing. The induced *B-fld* is *out of the page*. This happens due to an induced current that is c.c. That means the red end will act like a high voltage (+) source.

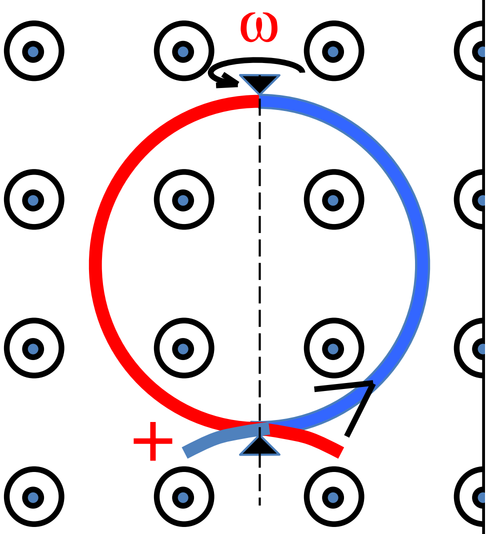




--As the loop continues to rotate (now the blue part is closest to you), the loop's external magnetic flux continues to diminish so the induced current continues c.c. The red end continues to act like a high voltage (+) source.

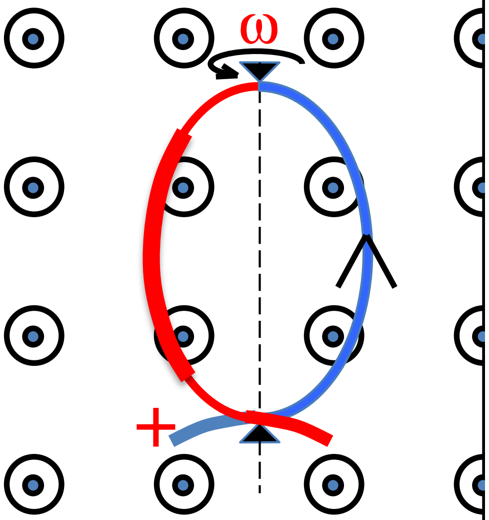


--Once the loop has passed the quarter-turn point, it's external magnetic flux begins to increase. The induced *B-fld* must be *into the page,* which means the induced current must now be *clockwise*. To effect that current, though, the red end CONTINUES to act like a high voltage (+) source.



--The loop reaches the 180° point (halfway through the rotation), the external magnetic flux begins to enter a period of diminishing, so the induced *B-fld* orients *into the page* as the induced current goes c.c. Now the blue end acts like a high voltage (+) source.

--This is a BIG DEAL! The end-polarities have switched.



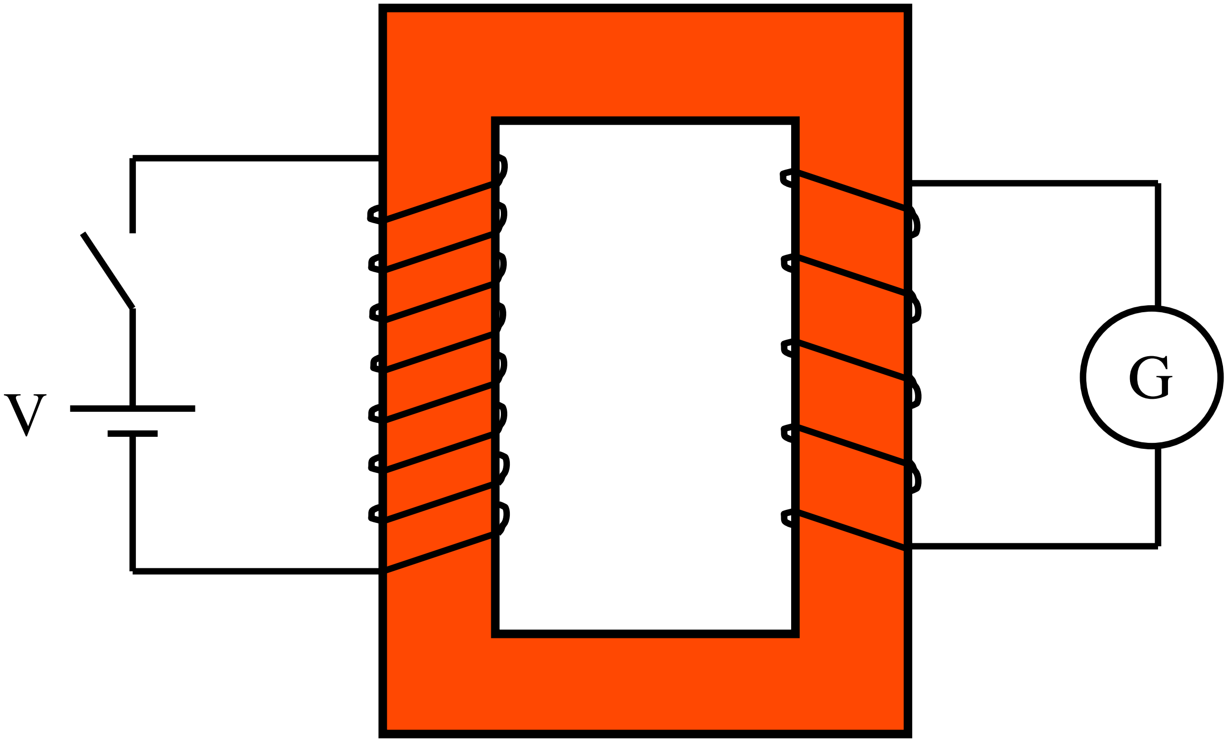
--The loop continues with the motion, mimicking the action of the first half cycle.

--This continues as every half-cycle, the polarity across the ends changes. We have just produced *alternating current: AC*!

AP Ideas (Induction) with Non-AP Math
The Transformer

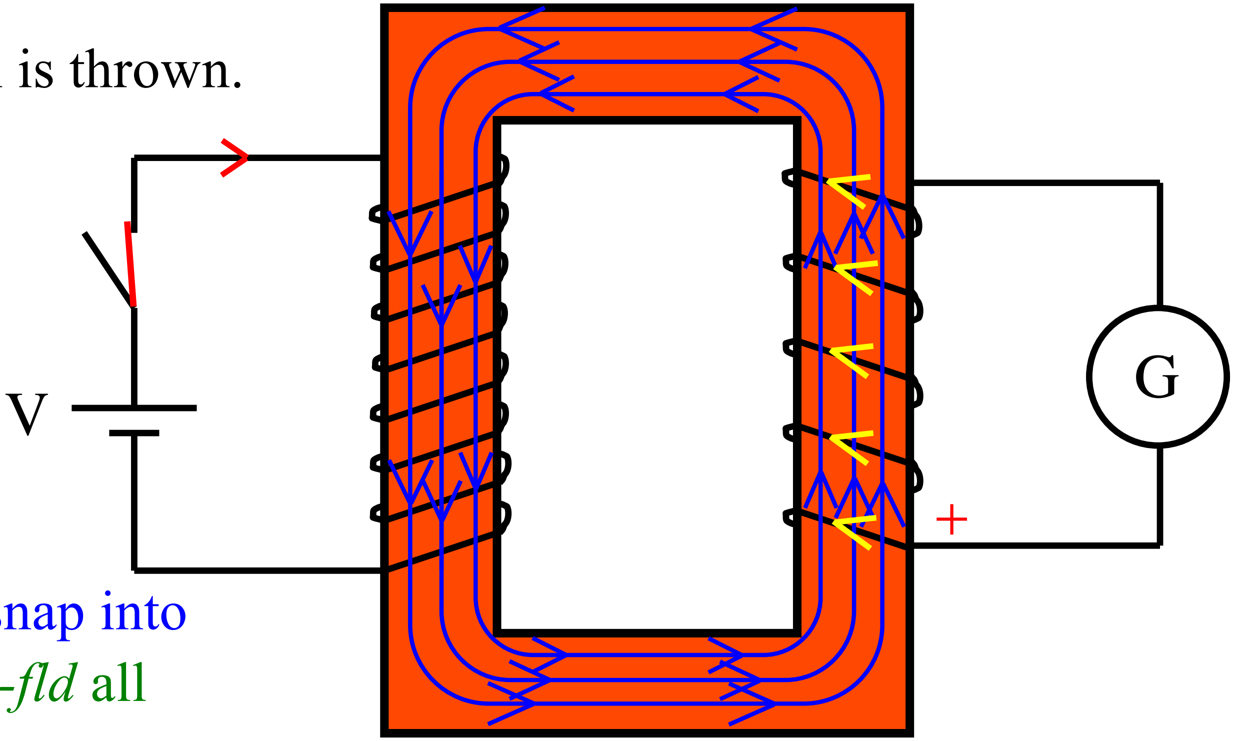
To the right you see an iron yoke about which is wrapped two independent coils that are insulated from one another (see sketch). To whit:

--*The left circuit,* called *the primary* because it includes a power supply in it, has some number of winds N_p in its coil.



--*The right circuit,* called *the secondary* because it includes a device you are transferring power to, has some number of winds N_s in its coil.

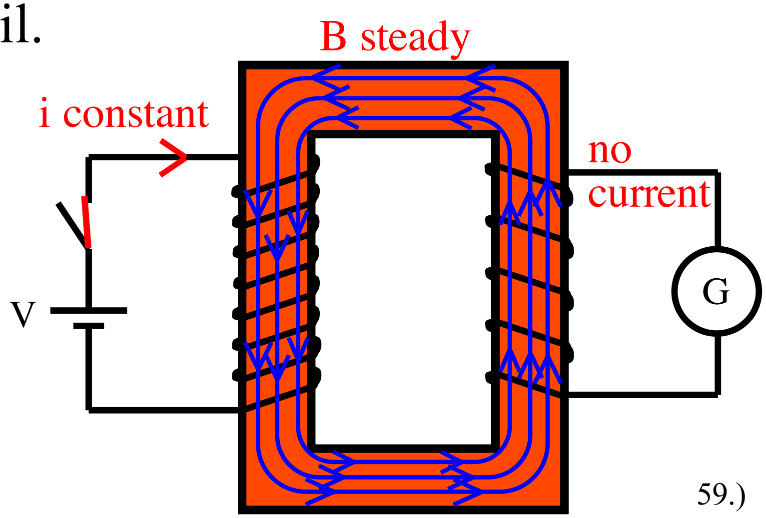
At some time, the switch is thrown.



- *--Current in the primary* coil experiences a back EMF, but it slowly builds generating a slowly escalating *B-fld* down its axis.

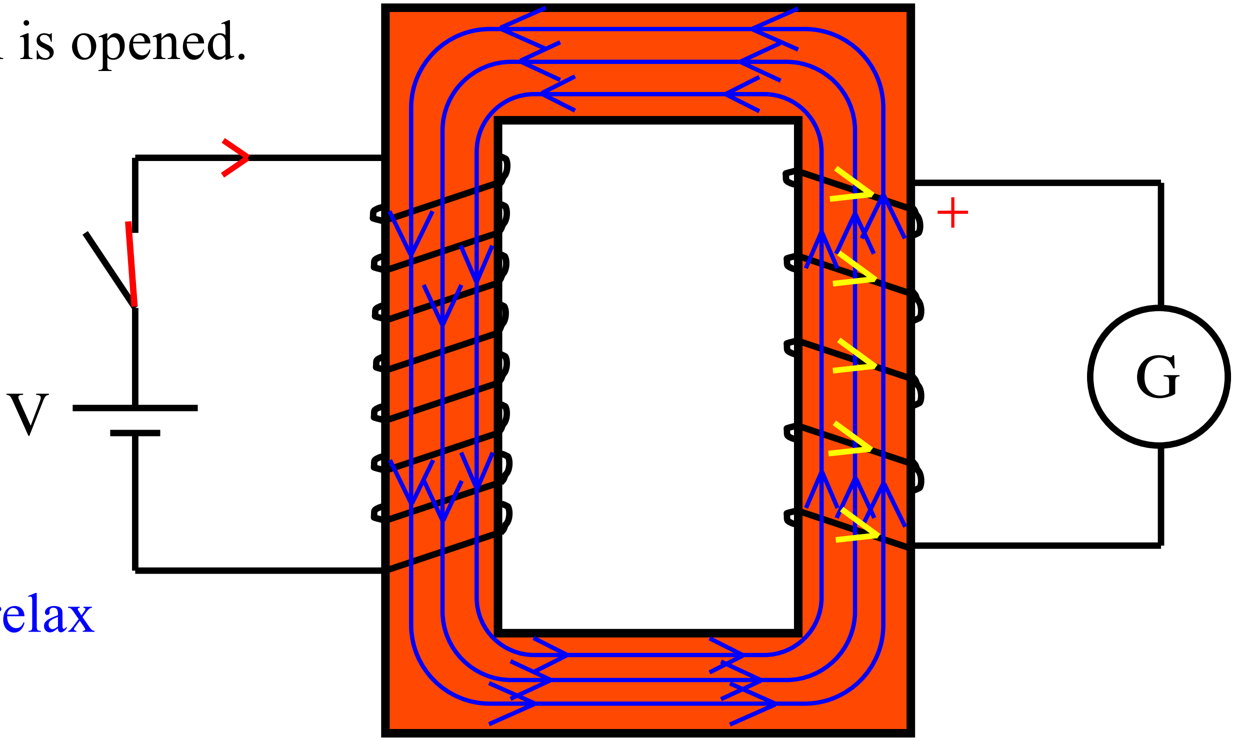
- *--The domains* in the yoke snap into alignment telescoping that *B-fld* all around the yoke.

- *--As the B-fld* increases, an induced EMF is set up in the secondary coil producing an induced current in the secondary coil.



- *--The induced current* in the secondary coil CONTINUES until the current in the primary has reached steady state whereupon the *B-fld* in the yoke ceases to change and there is no longer a *changing magnetic flux* through the secondary coil.

At some time, the switch is opened.

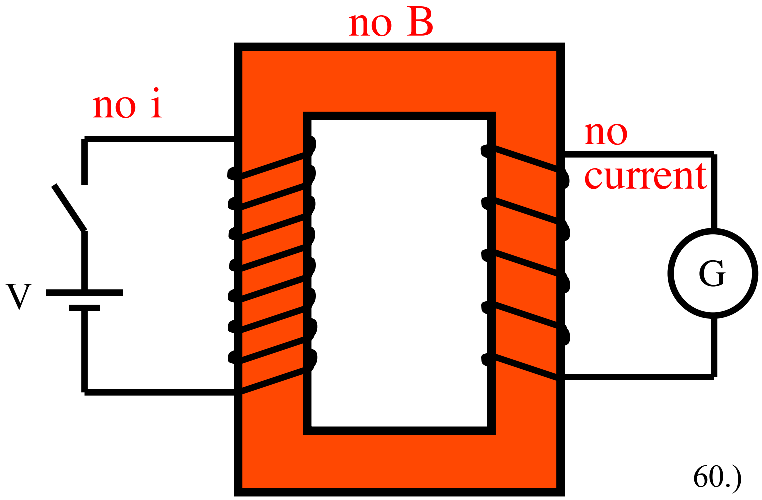


--*As the current* in the primary tries to drop, it (again) experiences an EMF which slows the diminishing as the *B-fld* down the axis erases.

--*The domains* in the yoke relax with the *B-fld* diminishing.

--*As the B-fld* decreases, an induced EMF is set up in the secondary coil opposite the original producing an induced current in the secondary coil.

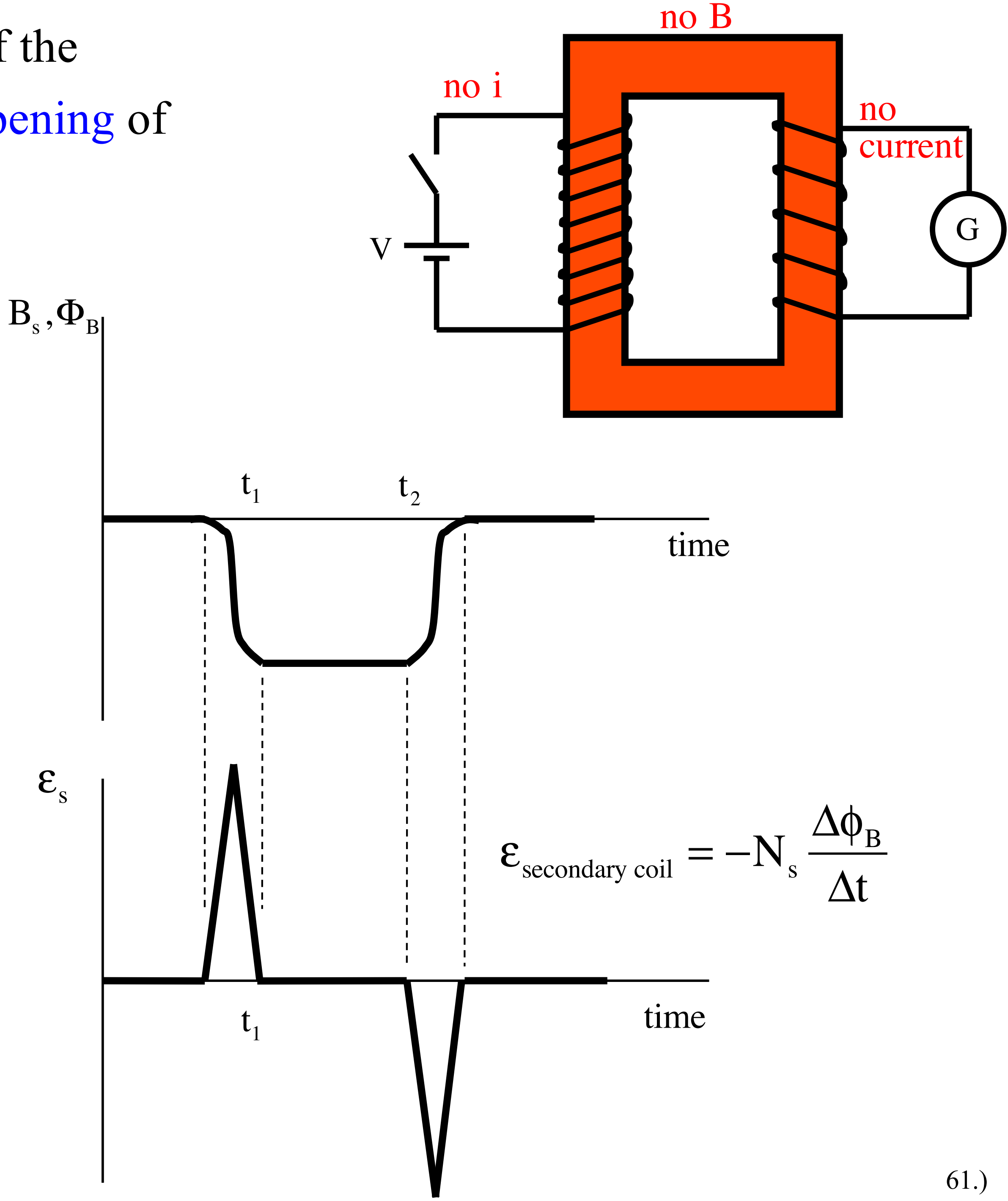
--*The induced current* in the secondary coil WILL CEASE once the current in the primary has dropped to zero, as the *B-fld* in the yoke will cease to change at that point and there will no longer be a *changing magnetic flux* through the secondary coil.



A rudimentary graph of the proceedings from the closing to opening of the switch:

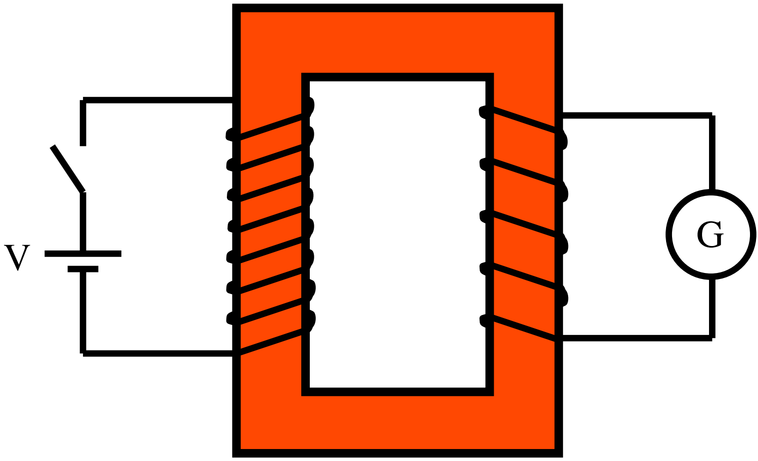
--The first graph animates the *B-fld* and magnetic flux down the axis of the primary coil;

--The second graph animates the EMF induced in the secondary coil when current in the primary changes.



$$\varepsilon_{\text{secondary coil}} = -N_s \frac{\Delta\phi_B}{\Delta t}$$

What does the math suggest?



--Transformers transfer power from the primary coil to the secondary coil via a *changing magnetic flux* generated via a shared *B-fld.* That means we could write:

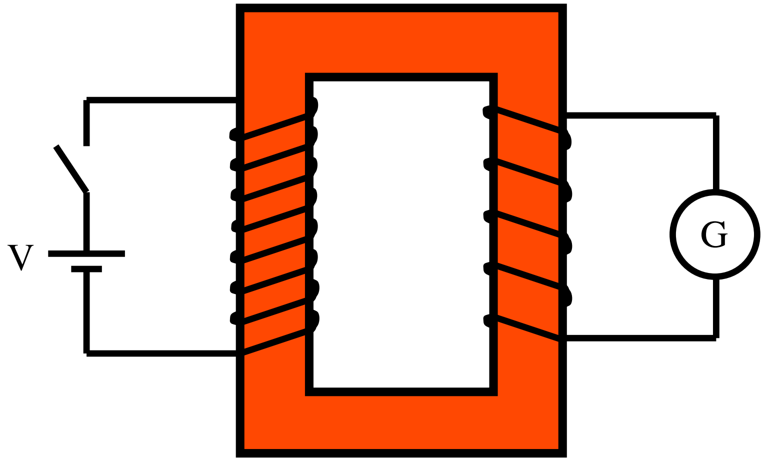
for the primary coil: $$\varepsilon_{\text{primary}} = N_p \frac{\Delta\phi_B}{\Delta t}$$

and for the secondary coil: $$\varepsilon_{\text{secondary}} = N_s \frac{\Delta\phi_B}{\Delta t}$$

Taking the radio yields:

$$\frac{\varepsilon_{\text{secondary}}}{\varepsilon_{\text{primary}}} = \frac{N_s\left(\cancel{\frac{\Delta\phi_B}{\Delta t}}\right)}{N_p\left(\cancel{\frac{\Delta\phi_B}{\Delta t}}\right)}$$

$$\Rightarrow \quad \frac{\varepsilon_s}{\varepsilon_p} = \frac{N_s}{N_p}$$



This ratio suggests that if you have more winds in the secondary coil, you will end up with a larger EMF in the secondary coil . . . You will have *stepped the voltage up,* so to speak.

> *This kind of transformer* is called a *step-up* transformer. It's characteristic is that:

$$N_s > N_p$$

You never get something for nothing, though. What is being transferred is *power*, and assuming ALL the power is transferred with no loss, we could write:

$$P_{seconary} = P_{primary}$$

$$\varepsilon_s \quad i_s = \varepsilon_p \quad i_p$$

$$\left(N_s \cancel{\frac{\Delta\phi_B}{\Delta t}}\right) i_s = \left(N_p \cancel{\frac{\Delta\phi_B}{\Delta t}}\right) i_p$$

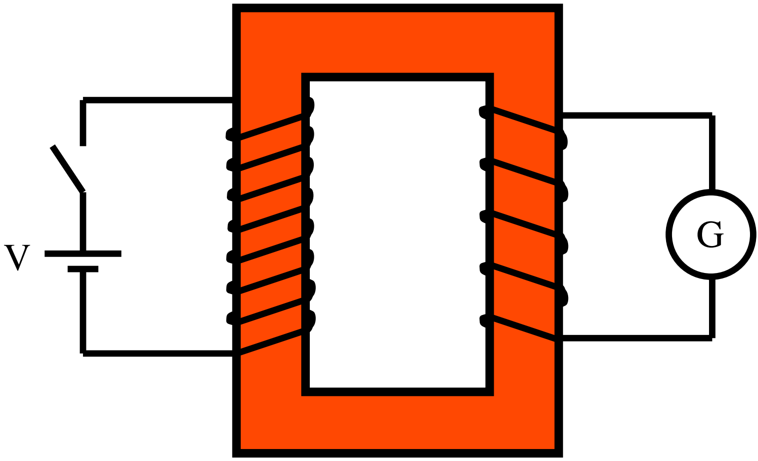
$$\Rightarrow \left(N_s\right) i_s = \left(N_p\right) i_p$$

$$\Rightarrow \frac{N_s}{N_p} = \frac{i_p}{i_s}$$

Translation: If the voltage goes up in a step-up transformer, the current provided to the secondary coil must go *down*.

NOTE: Using a transformer in a DC setting is nonsensical. The only time you get action in the secondary is when you change something in the primary. But using it with an AC source and ahhhhh, that's when you get poetry.

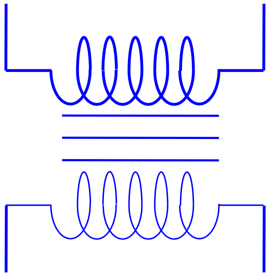
Summary:



--A transformer allows you to transfer power from one part of an electrical circuit to another without electrically connecting the parts. It does it by utilizing two coils that are not electrically connected, but that share a common magnetic field.

--Manipulating the *winds ratio* allows you to step-up the voltage or step-down the voltage from a source. This means that almost every electrical device you use has a transformer in it. (Example: the motherboard of a computer requires between 2 and 5 volts, but an AC wall socket is rated at 120 volts. The first thing your power cord runs into when it enters a computer is a transformer that steps the voltage down to a useable rating.)

--And FYI, the symbol for a transformer in a circuit is shown to the right (it's supposed to signify two coil that aren't connected to one another, coupled by a magnetic field signified by the three lines between the coils):



Example 19: You are off to Europe where the wall socket voltage is 240 volts. You want to take a hair dryer (bad idea as all the hotels will have them for free, buy you're stubborn and want your favorite dryer).

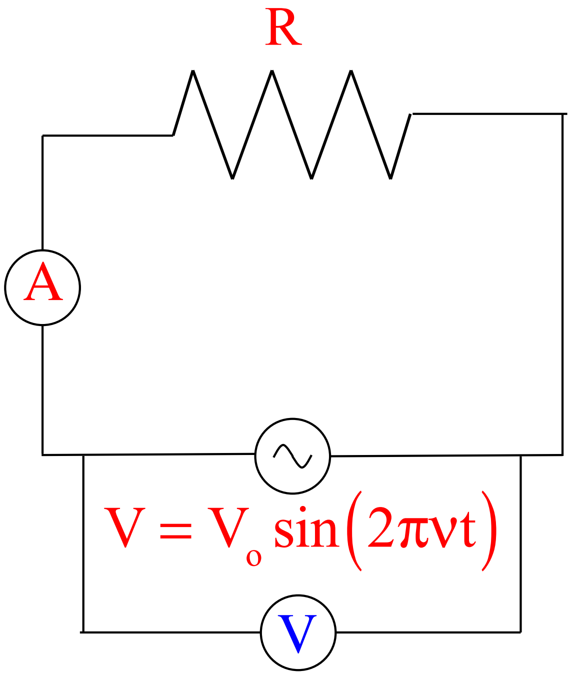
a.) What kind of transformer are you going to want to take on the trip?

U.S. wall sockets are 120 volts, so you want a *step-down* transformer.

Huge minor point: What do we mean when we say the wall socket voltage is 120 volts AC?

An AC voltages across the terminals of a power supply means the voltage difference across the terminals varies in time and, in fact, actually changes polarity periodically. This is usually characterized as sine function:

$$V = V_o \sin(2\pi\nu t)$$



This motivates charge carriers in the circuit to jiggle back and forth in response to the alternating electric field set up by the alternating voltage across the terminals. This, in turn, means that the idea of a current (number of charge carriers passing a point per unit time) kind of loses its meaning. So what does the ammeter and voltmeter in an AC circuit read?

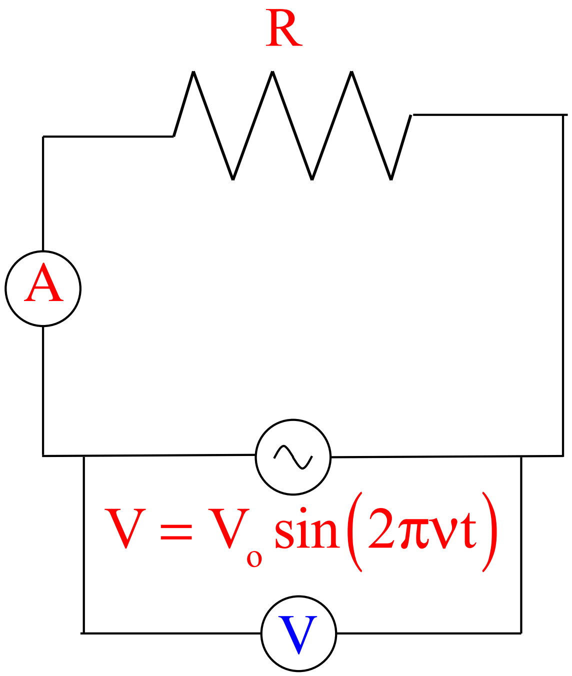
Without going into the math, the short answer is:

The AC ammeters in the circuit shown tells you how much DC current would be required to provide the same amount of power to the circuit as the AC source is providing.

It gives, in other words, the DC-equivalent current for the circuit.

Called the “root, mean, squared” value of the current (RMS, denoting how the value is derived using the current squared), this value equal to:

$$i_{RMS} = .707 i_o$$



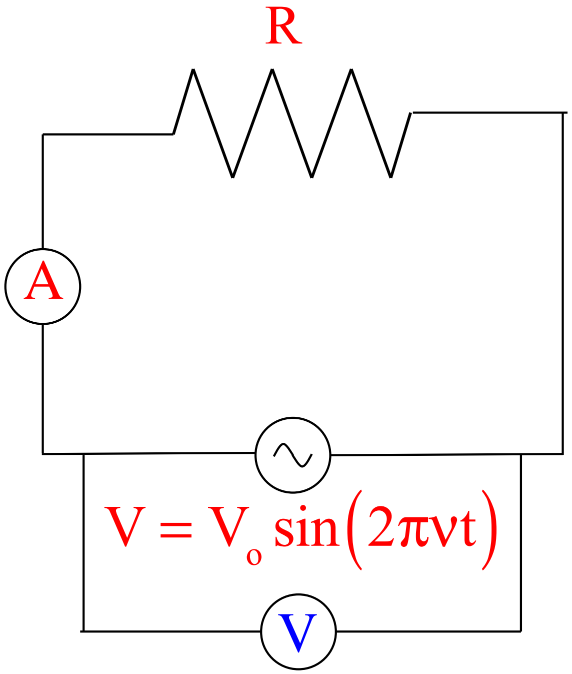
A similar approach is used for voltage values, with

$$V_{RMS} = .707V_o$$

So when you plug a voltmeter into a wall socket and it reads 120 volts, then, that AC meter is giving you the RMS value for that source.

As a bit more minutia, that means the maximum voltage difference across the terminals of the wall socket is:

$$V_o = \frac{V_{RMS}}{.707} = \frac{120V}{.707} = 169.7V$$



And as a wall socket's frequency is 60 Hz, that means the voltage function for a wall socket is:

$$\begin{aligned} V(t) &= V_o \sin(2\pi\nu t) \\ &= (169.7V)\sin(2\pi(60)t) \\ &= 170\sin(377t) \end{aligned}$$

Example 19: (So back at the ranch) You are off to Europe where the wall socket voltage is 240 volts. You want to take a hair dryer (bad idea as all the hotels will have them for free, buy you're stubborn and want your favorite dryer).

b.) What kind of winds ratio will your transformer sport?

$$\frac{\varepsilon_s}{\varepsilon_p} = \frac{120}{240} = \frac{N_s}{N_p} = \frac{1}{2}$$

c.) If your dryer runs on 3 amps, how much current will be drawn from the French grid?

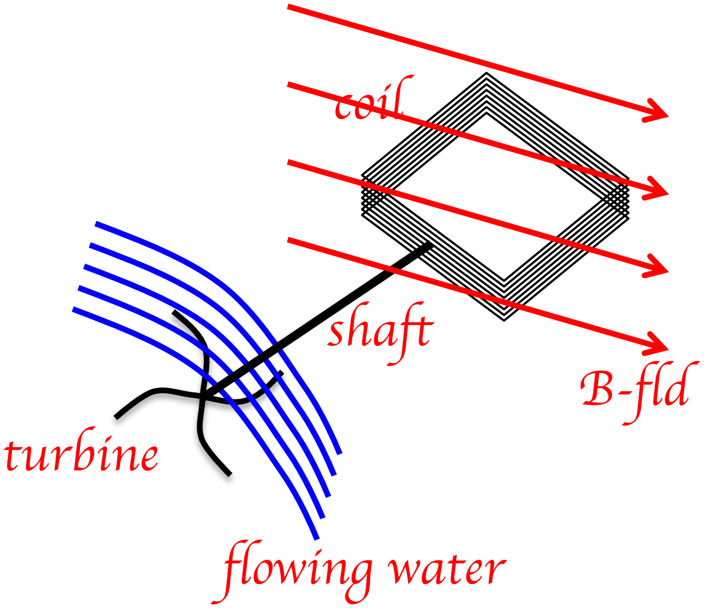
you'll require 3 amps in the secondary, so:

$$\frac{N_s}{N_p} = \frac{\varepsilon_s}{\varepsilon_p} = \frac{i_p}{i_s}$$

$$\Rightarrow \quad \frac{N_s}{N_p} = \frac{1}{2} = \frac{i_p}{i_s} = \frac{i_p}{(3A)}$$

$$\Rightarrow \quad i_p = 1.5 \text{ A}$$

Example 20: How is AC power produced in hydroelectric power plants, and how is it transferred from the dam to the city?



Water is run over the blades of a turbine whose shaft is attached to a coil that is bathed in a magnetic field. The coil is constrained to rotate at a fixed angular frequency ω (for the U.S., its 60 Hz; for Europe, it's 50 Hz), so the EMF generated is:

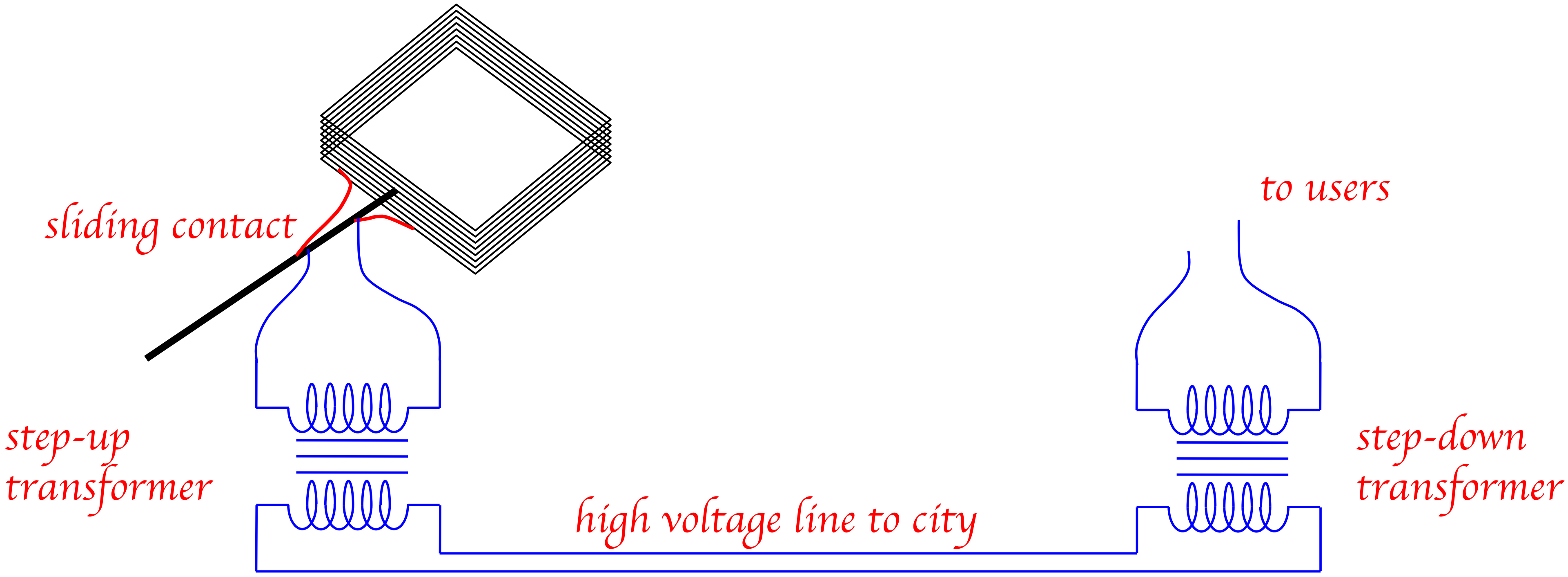
$$\begin{aligned}\varepsilon &= -N\frac{d\Phi_B}{dt} \\ &= -N\frac{d\left(BA\cos(\omega t)\right)}{dt} \\ &= -NBA\omega\sin(\omega t)\end{aligned}$$

It produces, in other words, periodic, alternating current that can be described with a sine function.

The problem with energy transfer is energy loss to heat. Heat comes from high current, so the trick is to lower the current for transfer to the city.

The coil's ends are connected via sliding contacts to terminals that are, themselves, connected to a *step-up transformer*. This steps the voltage up (consequence: 50,000 volt high tension lines) and drops the current down close to zero.

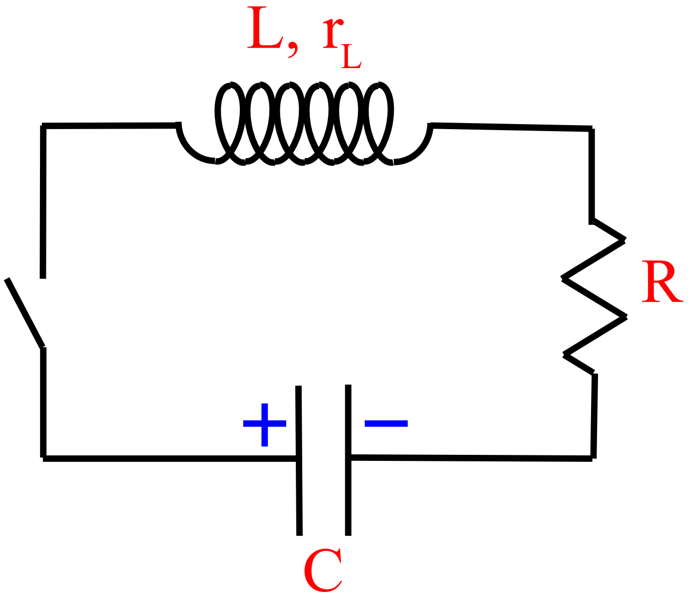
As there are no toasters that can handle 50,000 volts, a step-down transformer is located in the city to drop the voltage down to 120 volts or 240 volts with appropriate rise in possible current.



RLC Circuits

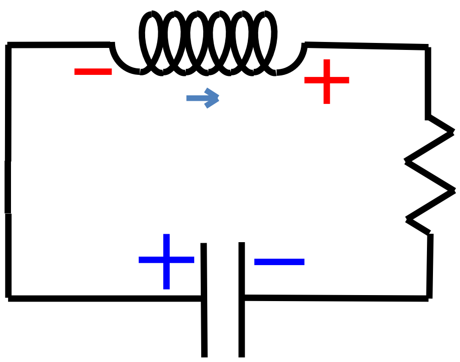
You know how capacitors act in a circuit, and you know how inductors act in a circuit, so how do the two act together in a circuit?

In the circuit to the right, assume the capacitor is initially charged with polarity as shown. What will happen when the switch it thrown?



Initially, the cap tries to discharge. Problem is, the inductor responds to the increase in current by producing a back-EMF to fight the change of magnetic flux through its cross-section. The current in the circuit increases, but slowly.

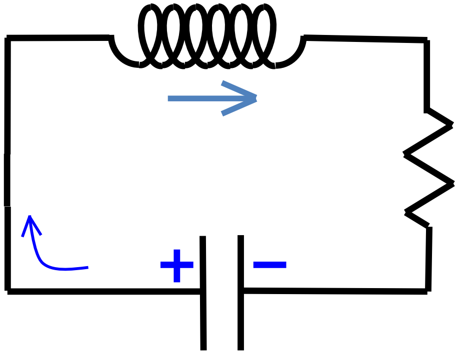
current grows slowly

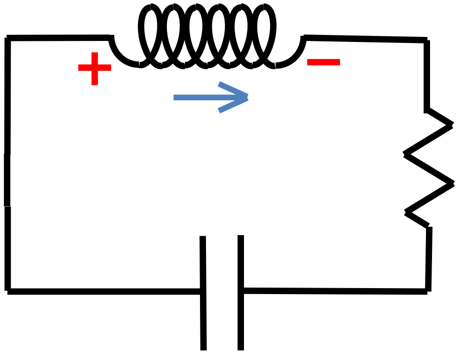


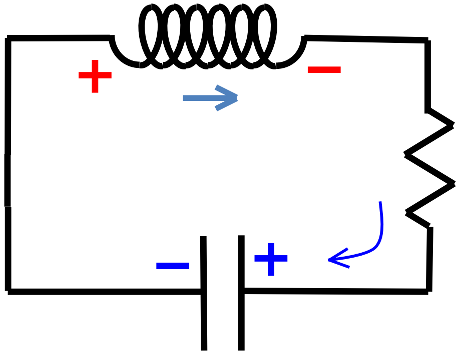
As the cap continues to discharge, the current approaches steady state and the induced EMF disappears. The current continuing to flow.

What's interesting is that as the cap's charge diminishes, the current from the cap begins to fall off. The creates another *changing magnetic flux* through the coil, which produces an EMF that fights the decreasing current. EVEN AFTER ALL THE CHARGE ON THE CAP IS EXHAUSTED, CURRENT CONTINUES TO FLOW DUE TO THIS INDUCED EMF!!!

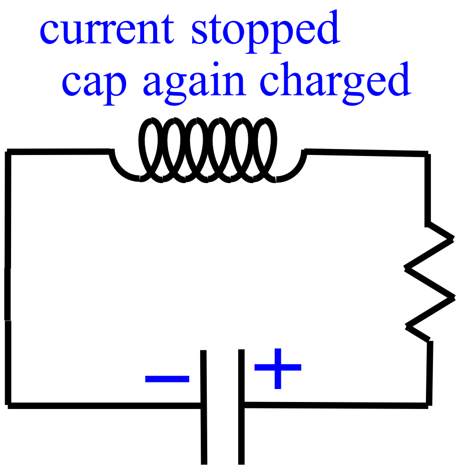
And as it does, that current begins to charge the capacitor up *with its polarity reversed . . .*


current steady state

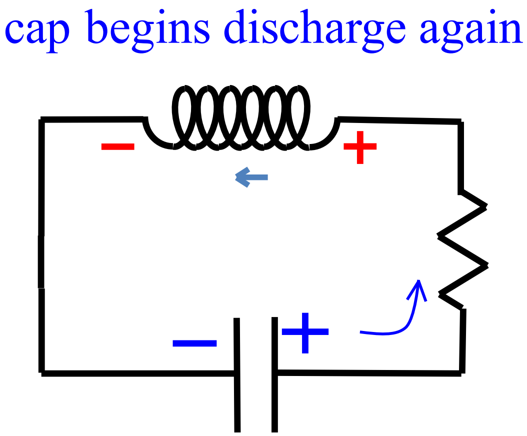

cap discharged, but induced EMF forces current to continue


cap recharges

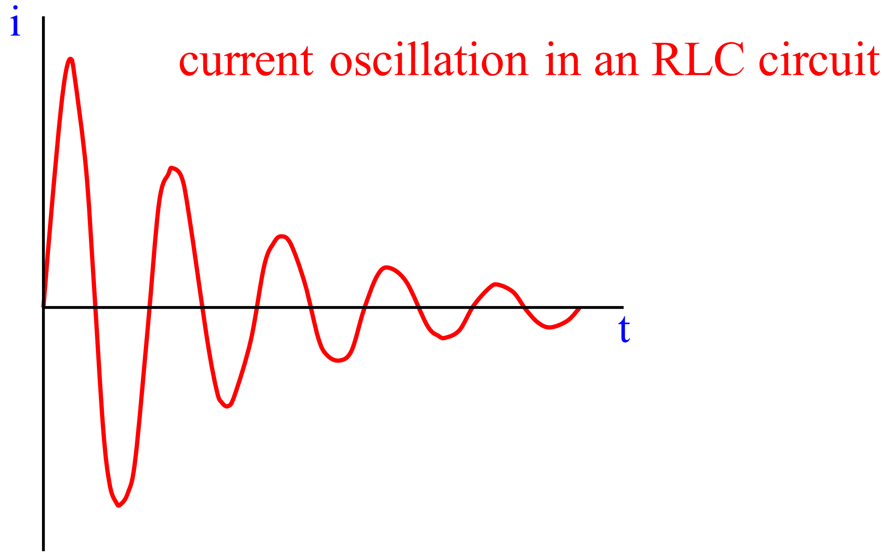
When the current stops, the cap is completely charged “going the other way.”

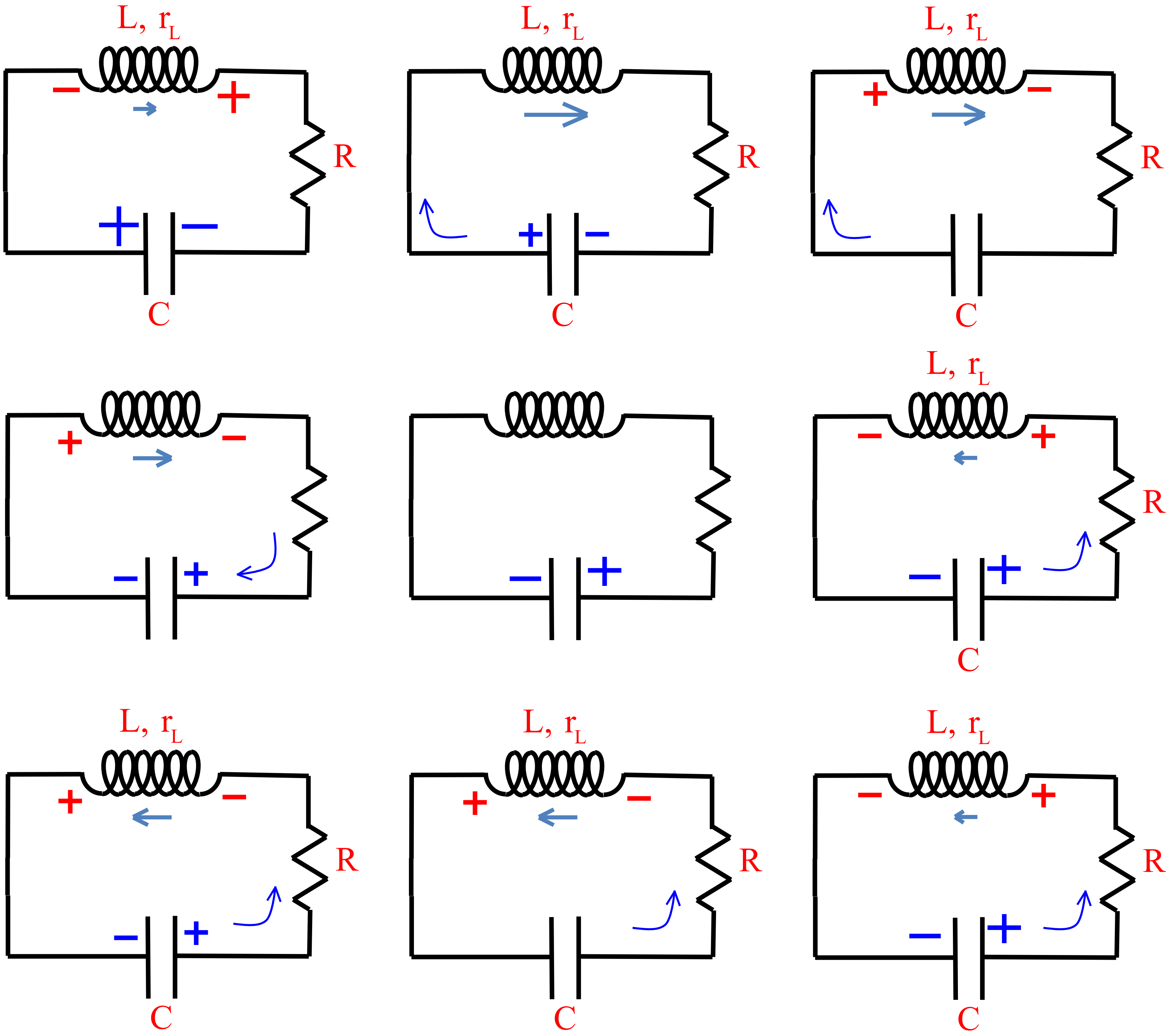


And with that, the cap will begin to discharge going the other way and the process will all over.



Bottom line: Current oscillates back and forth, dampening out due to the resistor-like resistance in the circuit, at a resonant frequency governed by the value of the cap and inductor.

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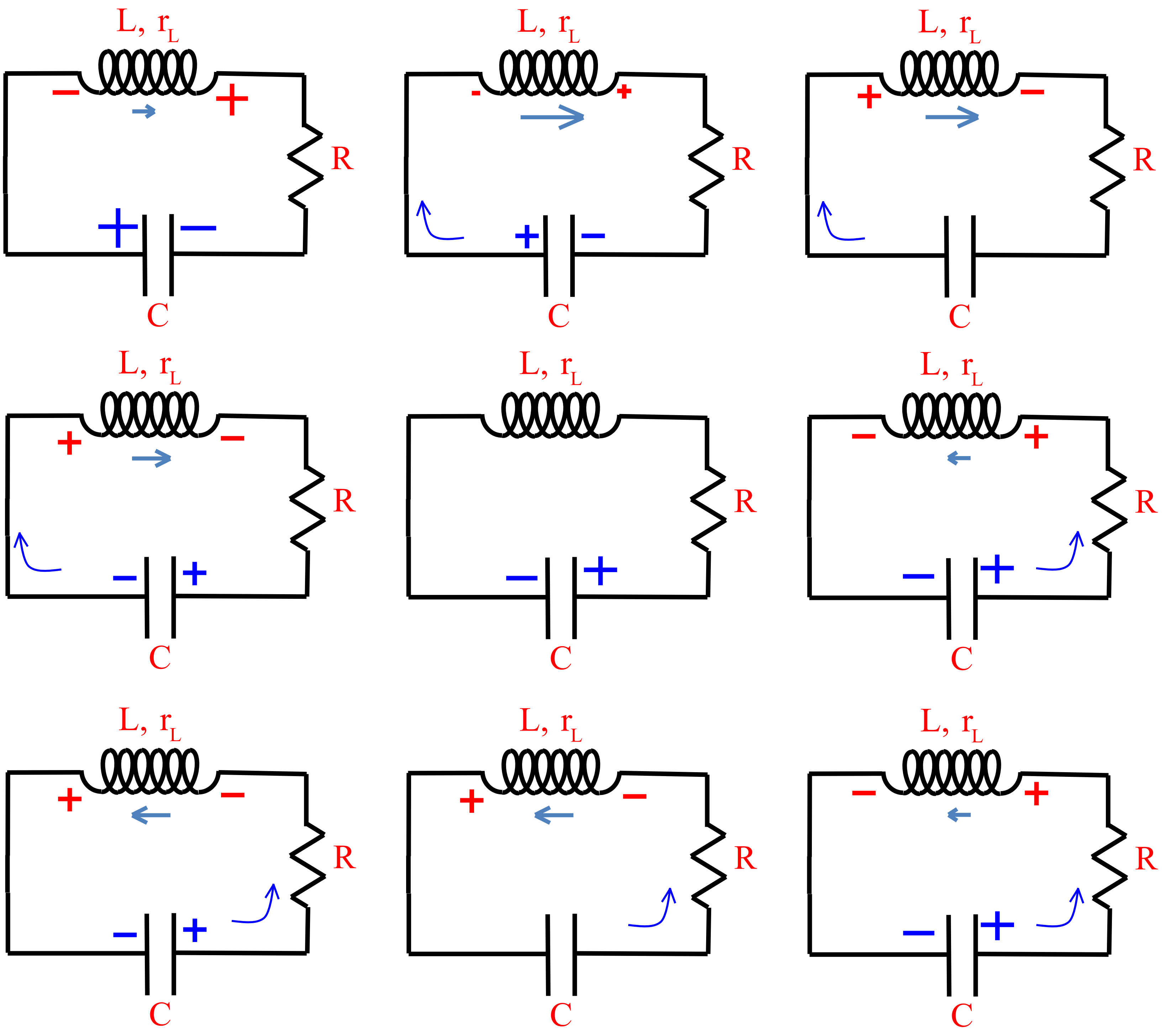
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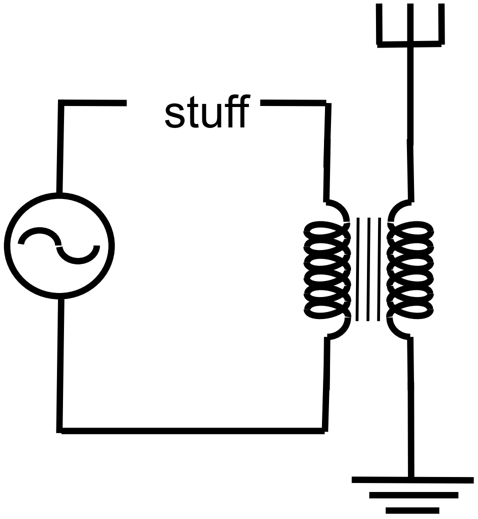
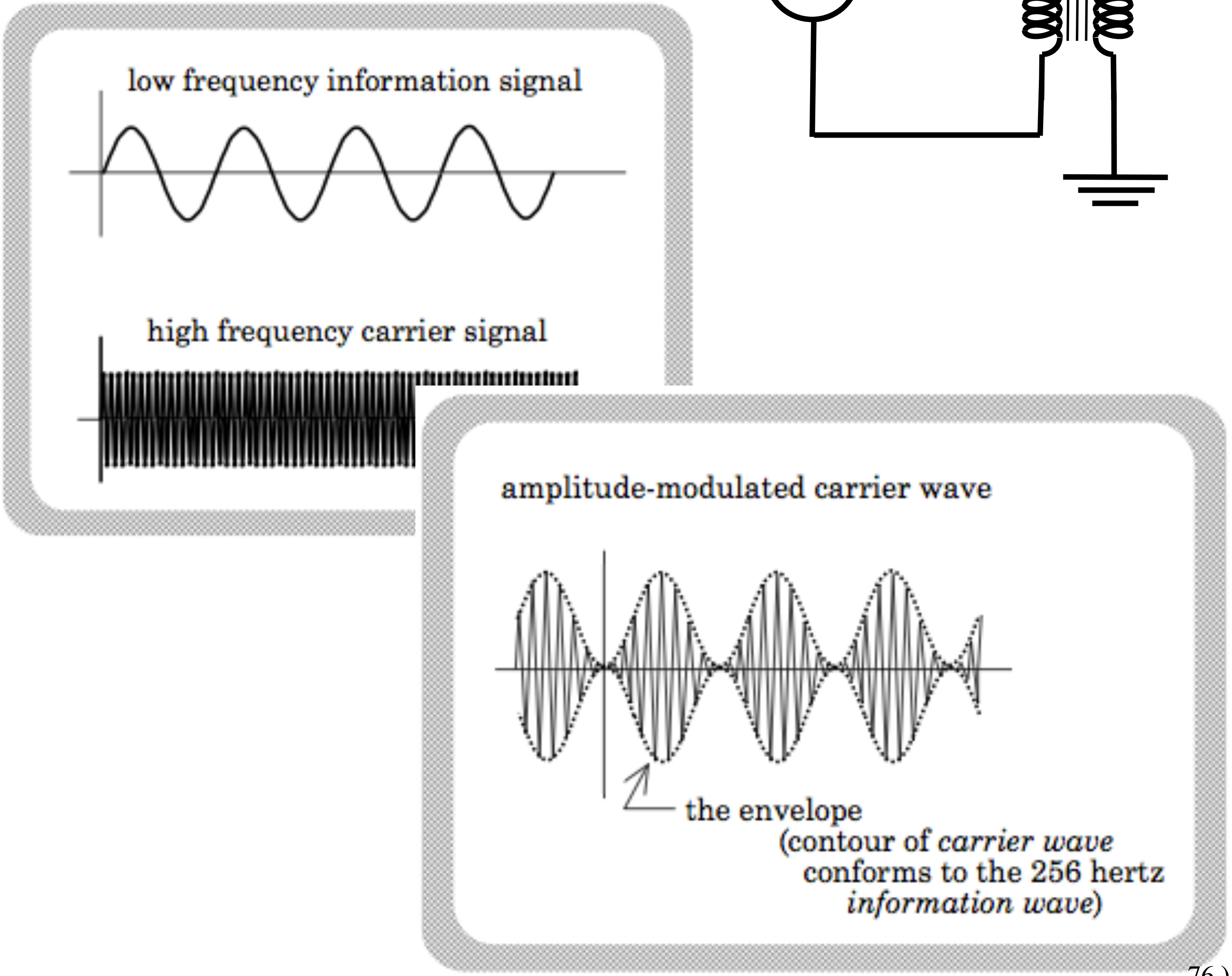
AM Radio Circuit:



1.) A radio station takes it’s high frequency carrier wave provided to it by the FCC (say 980,000 Hz) and amplitude modulates that wave so its envelope matches a low frequency information wave (say, middle C at 256 Hz).



radio circuit

2.) The antenna of a receiving set absorbs the signal.

3.) Transferred to the RC tuning circuit via a transformer. Being an RC circuit, there will be only one frequency at which charge will want to oscillate. If the cap and inductor of that circuit produce a resonance frequency that matches the frequency of the station you want to listen to, that station’s signal will drive current in the tuning circuit and that signal will proliferate in that circuit. All others will dampen out.