CHAPTER 17 -- CAPACITORS

17.1) Initially, an uncharged capacitor will allow current to flow through it as though it had no resistance to charge flow at all (i.e., it will act like a short-circuit). As time progresses and the capacitor charges, current through the cap decreases as it becomes more and more difficult to force still more charge onto its plates. After a long enough time, current will cease completely and the totally charged capacitor will act like a break in the circuit (i.e., an open-switch circuit). We will do the entire problem for Circuit a first, then do the problem for Circuit b.

Circuit a finds two capacitors in series. Series elements have common currents. For capacitors, that means the magnitude of the charge accumulated on each capacitor plate will be the same for all caps in the series combination. It also means that for different size capacitors, the voltage across each capacitor will be different (remember, $V_1 = Q/C_1$).

a.) The initial current through the circuit will be that of a resistor in series with a battery (the uncharged caps will act like "shorts"), or:

$$\begin{aligned} i &= V_R/R \\ &= (120 \text{ volts})/(20\ \Omega) \\ &= 6 \text{ amps.} \end{aligned}$$

b.) To begin with, when the capacitors are totally charged, there will be no current through the circuit (the charged capacitors will act as open circuits). That means the entire 120 volt voltage drop will be across the capacitor combination (none across the resistor as i = 0 . . . remember, the voltage across a resistor is iR).

The charge on each individual capacitor will be the same as the charge on the circuit's equivalent capacitor. The equivalent capacitance for a series combination is such that:

$$\begin{aligned} 1/C_{eq} &= 1/C_1 + 1/C_2 + \ldots \\ C_{eq} &= [1/(6\text{x}10^{-6} \text{ f}) + 1/(12\text{x}10^{-6} \text{ f})]^{-1} \\ &= 4\text{x}10^{-6} \text{ farads.} \end{aligned}$$

From C = Q/V we get:

$$Q = CV,$$

where C is the capacitance of the capacitor in question, Q is the charge on one of the capacitor plates, and V is the voltage across the cap. With $C = C_{eq}$ and V = 120 volts, we get:

$$\begin{aligned} Q &= C_{eq}V_o \\ &= (4\ \text{x}10^{-6}\ \text{f})\ (120\ \text{volts}) \\ &= 4.8\ \text{x}10^{-4}\ \text{coulombs.} \end{aligned}$$

Each capacitor will hold $4.8\text{x}10^{-4}$ coulombs per plate when fully charged.

c.) Knowing the charge on the 6 μf cap, we can use C = Q/V to determine the voltage across the cap:

$$\begin{aligned} V_6 &= Q/C_6 \\ &= (4.8\ \text{x}10^{-4}\ \text{C})/(6\ \text{x}10^{-6}\ \text{f}) \\ &= 80\ \text{volts.} \end{aligned}$$

Note: As the total battery charge is 120 volts, that means the other capacitor has 40 volts across it when fully charged.

d.) The energy wrapped up in a charged capacitor equals:

$$\text{Energy} = (1/2)CV^2,$$

where C is the cap's capacitance and V is the voltage across the cap. Using this yields:

$$\begin{aligned} \text{Energy} &= .5(6\ \text{x}10^{-6}\ \text{f})(80\ \text{volts})^2 \\ &= .0192\ \text{joules.} \end{aligned}$$

e.) The RC time constant (τ) tells you the amount of time required for the capacitor in the circuit to charge up to 63% of its total charge. Two time constants is the time to charge up to 87%, and three time constants the time to charge to 95% of its maximum charge. (By the same token, a charged system will dump 63% of its charge in a time interval equal to one time constant, 87% in a time interval equal to two time constants, and 95% in a time interval equal to three time constants).

The relationship between the time constant, the net resistance in the circuit, and the net capacitance of the circuit is:

$$\begin{aligned}\tau &= RC_{eq} \\ &= (20\ \Omega)(4\text{x}10^{-6}\ \text{f}) \\ &= 8\text{x}10^{-5}\ \text{seconds.}\end{aligned}$$

f.) As stated above, 63% of the charge will be lost in the first time constant. That means 37% will be left. As such, we can write:

$$.37(4.8\ \text{x}10^{-4}\ \text{coul}) = 1.8\ \text{x}10^{-4}\ \text{coulombs}$$

will be left.

Circuit b finds two capacitors in parallel. Parallel elements have voltages in common. For different size capacitors, that means the amount of charge on each cap will be different (remember, $Q_1 = C_1V$).

a.) As before, the caps will act like shorts when uncharged. Current must go through the resistor to return to the battery, so the initial current again will be governed by the size of the resistor in the circuit with the entire voltage drop occurring across that element:

$$\begin{aligned}i &= V_R/R \\ &= (120\ \text{volts})/(20\ \Omega) \\ &= 6\ \text{amps.}\end{aligned}$$

b.) When the capacitors are totally charged, there will be no current through the circuit (the charged capacitors will act as open circuits). That means the ENTIRE 120 volt voltage drop will be across EACH parallel capacitor.

For the 6 μf cap:

$$\begin{aligned}Q_6 &= C_6V_o \\ &= (6\ \text{x}10^{-6}\ \text{f})(120\ \text{volts}) \\ &= 7.2\ \text{x}10^{-4}\ \text{coulombs.}\end{aligned}$$

For the 12 μf cap:

$$\begin{aligned}Q_{12} &= C_{12}V_o \\ &= (12\ \text{x}10^{-6}\ \text{f})(120\ \text{volts}) \\ &= 1.44\ \text{x}10^{-3}\ \text{coulombs.}\end{aligned}$$

c.) When fully charged, the maximum voltage across the 6 μf cap will be V_{max} = 120 volts (as stated above).

d.) The energy wrapped up in the 6 μf cap when fully charged equals:

$$\begin{aligned} \text{Energy} &= .5(6 \times 10^{-6} \text{ f})(120 \text{ volts})^2 \\ &= .0432 \text{ joules.} \end{aligned}$$

e.) The previous Part e explained what the time constant means. Determining it for this circuit requires that we determine the equivalent capacitance for the circuit. For parallel combinations of capacitors, we just add the capacitances. That means:

$$\begin{aligned} C_{eq} &= (6 \times 10^{-6} \text{ f}) + (12 \times 10^{-6} \text{ f}) \\ &= 18 \times 10^{-6} \text{ farads.} \end{aligned}$$

Knowing that:

$$\begin{aligned} \tau &= RC_{eq} \\ &= (20\ \Omega)(18 \times 10^{-6} \text{ f}) \\ &= 3.6 \times 10^{-4} \text{ seconds.} \end{aligned}$$

f.) As stated above, 63% of the charge will be lost in the first time constant. That means 37% will be left. As such, we can write:

$$.37(7.2 \times 10^{-4} \text{ C}) = 2.7 \times 10^{-4} \text{ coulombs}$$

will be left.

17.2)

a.) By inspection, the equivalent capacitance for each combination is: (a.) (2/3)C; (b.) 3C; (c.) (2/3)C (this configuration is exactly the same as Part a--only the sketch has been rendered differently); (d.) C/3; (e.) (3/2)C. That means the order should be: d, a and c, e, and b.

b.) The energy content of a capacitor combination is such that:

$$(1/2)C_{eq}V^2,$$

where C_{eq} is the equivalent capacitance of the capacitor combination in question. For a given voltage, that means the most energy-storing capacity will go to the combination with the largest equivalent capacitance. That will be the parallel capacitor combination in Part b.

17.3) The capacitance of a single parallel plate capacitor is:

$$C = \varepsilon_o A/d,$$

where A is the area of one plate, d is the distance between the plates, and ε_o is the permeability of free space and is equal to $4\pi x10^{-7}$ farads per meter.

a.) Neither ε_o nor A is changing in this situation, whereas d gets larger by a factor of 4. That means, according to $C = \varepsilon_o A/d$, the capacitance should diminish by a factor of 4, giving us $(1/4)C_{original}$.

b.) The charge on the capacitor is related to the voltage across the capacitor and the size of the capacitor by the relationship:

$$C = Q/V \quad \text{or} \quad Q = CV.$$

V hasn't changed but C is now a quarter of its original value. That means Q_{new} must equal $(1/4)Q_{old}$.

c.) The energy equals:

$$\text{Energy} = (1/2)CV^2.$$

This implies that the energy in the cap will decrease by a factor of 4 also.

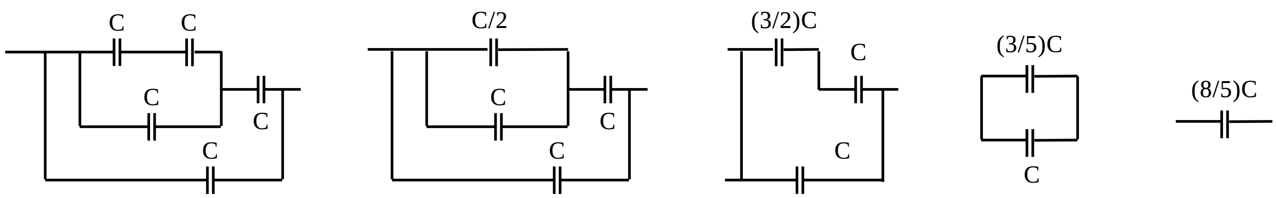
d.) Knowing the dielectric constant allows us to determine the new capacitance knowing the old capacitance. That is:

$$C_{new} = \kappa_d C_{old}.$$

That means $C_{new} = 1.6\ C_{old}$.

17.4)

a.) The circuit evaluation to determine C_{eq} is shown below (remember that the equivalent capacitance rules are the mirror image of equivalent resistance rules).



b.) The energy stored in a capacitor of capacitance (8/5)C is:

$$E = (1/2)CV^2$$
$$= .5[(8/5)(25\text{x}10^{-3}\ f)](120\ v)^2$$
$$= 288 \text{ joules.}$$

17.5.)

a.) This question is just down right tricky.

--When the switch is closed, the 30 Ω resistor is in parallel with the two series-connected capacitors. Being in parallel, the net voltage drop across the capacitors and across the resistor must be the same.

--Initially, the capacitors have no charge on them. That means the capacitors initially have no voltage drop across them.

--No initial voltage drop across the capacitors means no initial voltage drop across the 30 Ω resistor.

--With no initial voltage drop across the 30 Ω resistor, there will be no initial current through that resistor. As such, the initial current i_3 equals zero.

--Meanwhile, the capacitors initially act like open circuits (with no charge on them, there is nothing to motivate them to do otherwise), which means the initial current will flow freely through them and $i_1 = i_2$.

--That means the entire 120 volts from the battery is initially dropped across the 20Ω resistor, and we can write initial currents as:

$i_1 = (120\ v)/(20\ \Omega) = 6$ amps;
$i_2 = i_1 = 6$ amps; and
$i_3 = 0$.

b.) After a long period of time:

--The capacitors will be fully charged so that $i_2 = 0$;

--The currents i_1 and i_3 will be equal to one another.

--The full 120 volt drop will be across the 30 Ω and 20 Ω resistors in series. As such, the steady-state current will be:

$$i_1 = i_3 = (120 \text{ v})/(50\ \Omega) = 2.4 \text{ amps, and } i_2 = 0.$$

c.) Note first that this is also a bit tricky. Why? Because the fact that there are three unknown currents might lead you to believe that you need only three equations. The problem is that there are also capacitors and unknown charge quantities with which to deal. There are three loop equations possible, but only two are independent of one another. In short, we need two loop equations, one node equation, and one other equation. Starting with Kirchoff's Laws (presented in general algebraic terms first) and noting that the equivalent capacitance of the series combination of capacitors is 4 μC, we can write:

Node equation: $i_1 = i_2 + i_3$.

Left inner loop: $V_o - q/C_{eq} - i_1R_{20} = 0$

$\Rightarrow \quad 120 - q/(4\text{x}10^{-6}) - 20i_1 = 0.$

Right inner loop: $-i_3R_{30} + q/C_{eq} = 0$

$\Rightarrow \quad -30i_3 + q/(4\text{x}10^{-6}) = 0.$

Where does the last equation come from? The rate at which charge q is deposited on the capacitor's plates and the current dq/dt in that part of the circuit are the same, so we can write:

$$i_2 = dq/dt.$$

These are the four equations we need to determine i_1, i_2, and i_3.

Note: You could have used the outside loop instead of either of the two loops used. Doing so would have yielded the equation:

$$V_o - i_3R_{30} - i_1R_{20} = 0.$$

d.) The charge on the 6 μf capacitor will be the same as the charge on the 12 μf capacitor (they are in series). By definition, this will also be the same as the charge on the equivalent capacitor.

When the capacitors are fully charged, the current through the circuit will be $i_1 = i_3 = 2.4$ amps, as calculated above. That means the voltage drop across the 20 Ω resistor will be:

$$\begin{aligned} V_{20} &= i_1 R_{20} \\ &= (2.4 \text{ amps})(20\ \Omega) \\ &= 48 \text{ volts.} \end{aligned}$$

Adding voltage drops as we go, the voltage drop across the battery will equal the sum of the voltage drops across the two capacitors (or their single equivalent capacitor) and the 20 Ω resistor. At maximum, that is:

$$\begin{aligned} &V_o = V_{c,max} + V_{20} \\ \Rightarrow\quad &120 = V_{c,max} + 48 \\ \Rightarrow\quad &V_{c,max} = 72 \text{ volts.} \end{aligned}$$

Knowing the voltage across the equivalent capacitor, we can use the definition of capacitance to determine the charge on that capacitor:

$$\begin{aligned} q_{max} &= C_{equ} V_{c,max} \\ &= (4\text{x}10^{-6} \text{ farads})(72 \text{ volts}) \\ &= 2.88\text{x}10^{-4} \text{ coulombs.} \end{aligned}$$

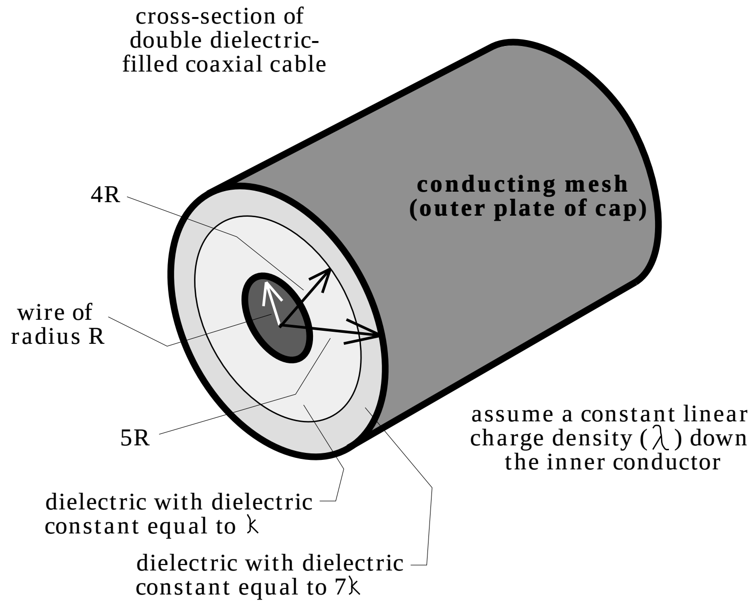
e.) The total energy on the capacitors when fully charged will be:

$$\begin{aligned} \text{Energy} &= (1/2) C_{equ} V_{c,max}{}^2 \\ &= .5(4\text{x}10^{-6} \text{ farads})(72 \text{ volts})^2 \\ &= 1.04\text{x}10^{-2} \text{ joules.} \end{aligned}$$

f.) It will take the capacitor two time constants to dump 87% of its charge. As the 20 Ω resistor is out of the circuit when the switch is open:

$$\begin{aligned} t &= 2\tau \\ &= 2[RC_{equ}] \\ &= 2(30\ \Omega)(4\text{x}10^{-6} \text{ farads}) \\ &= 2.4\text{x}10^{-4} \text{ seconds.} \end{aligned}$$

17.6) A sketch of the setup is shown on the next page. To determine the capacitance expression, we need to determine q/V_c, where q is an arbitrary



amount of charge on the capacitor and V_c is the voltage across the capacitor plates when q is present. V_c is the electrical potential difference between the inner surface and the interface between the two dielectrics added to the electrical potential difference between the dielectric interface and the outer surface.

To determine these electrical potential differences, we need to know the electric fields in these areas.

Doing the problem in pieces, begin by determining a general expression for the electric field between R and 4R (call this region I). Begin by assuming a linear charge density of $\lambda = +q/L$ on the inside cylinder (i.e., the wire), where L is an arbitrary length and q is the amount of charge on the wire along that length. Defining a cylindrical Gaussian surface of length L, we can write Gauss's Law as:

$$(\kappa)\int_S \mathbf{E}_I \bullet d\mathbf{S} = \frac{q_{encl}}{\varepsilon_o}$$

$$\Rightarrow \quad E_I\kappa(2\pi r L) = \frac{\lambda L}{\varepsilon_o}$$

$$\Rightarrow \quad E_I = \frac{\lambda}{2\pi r\kappa\varepsilon_o}.$$

The dielectric constants in the two regions are different, but aside from that, the electric field expressions should have the same form. As such, we can write:

$$(7\kappa)\int_S \mathbf{E}_{II} \bullet d\mathbf{S} = \frac{q_{encl}}{\varepsilon_o}$$

$$\Rightarrow \quad E_{II}(7\kappa)(2\pi r L) = \frac{\lambda L}{\varepsilon_o}$$

$$\Rightarrow \quad E_{II} = \frac{\lambda}{14\pi r\kappa\varepsilon_o}.$$

With these electric field expressions, we can now determine the net electrical potential difference across the plates. If we start at the higher voltage plate (the inside plate) and move in the direction of the electric field (i.e., outward) toward the outside plates, the calculated V will be negative. As V_c in the capacitance definition is defined as positive, we must write:

$$
\begin{aligned}
V_C &= -(\Delta V)_{\text{from R to 5R(i.e., from high to low voltage)}} \\
V_C &= -(\Delta V_{\text{from R to 4R}} + \Delta V_{\text{from 4R to 5R}}) \\
&= -\left[(V_{4R} - V_R) + (V_{5R} - V_{4R})\right] \\
&= -\left(-\int_{r=R}^{4R} \mathbf{E}_I \bullet d\mathbf{r} - \int_{r=4R}^{5R} \mathbf{E}_{II} \bullet d\mathbf{r}\right) \\
&= \int_{r=R}^{4R} \left(\frac{\lambda}{2\pi\kappa\varepsilon_o r}\right) \bullet (dr\mathbf{r}) + \int_{r=4R}^{5R} \left(\frac{\lambda}{14\pi\kappa\varepsilon_o r}\right) \bullet (dr\mathbf{r}) \\
&= \left(\frac{\lambda}{2\pi\kappa\varepsilon_o}\right)\left[\int_{r=R}^{4R} \frac{dr}{r} + \frac{1}{7}\int_{r=4R}^{5R} \frac{dr}{r}\right] \\
&= \left(\frac{\lambda}{2\pi\kappa\varepsilon_o}\right)\left[\ln\left(\frac{4R}{R}\right) + \frac{1}{7}\ln\left(\frac{5R}{4R}\right)\right] \\
&= 1.418\left(\frac{\lambda}{2\pi\kappa\varepsilon_o}\right).
\end{aligned}
$$

This expression, the fact that $\lambda = +q/L$, and the definition of capacitance yield:

$$
\begin{aligned}
C &= \frac{q}{V_C} \\
&= \frac{q}{\left(\frac{1.418(q/L)}{2\pi\kappa\varepsilon_o}\right)} \\
&= \frac{2\pi\kappa\varepsilon_o L}{1.418} \\
\Rightarrow \quad C/L &= \frac{2\pi\kappa\varepsilon_o}{1.418}.
\end{aligned}
$$

17.7) The problem that wasn't. THINK ABOUT SPHERICAL CAPACITORS ON YOUR OWN!!!