Chapter 17

CAPACITORS

A.) Capacitors in General:

1.) The circuit symbol for the capacitor (see Figures 17.1a and b) evokes a feeling for what a capacitor really is. Physically, it is no more than two plates (the symbol depicts the side view) that do not touch (there is normally insulation placed between the two plates to insure no contact). In other words, a capacitor in a circuit technically effects a break in the circuit.

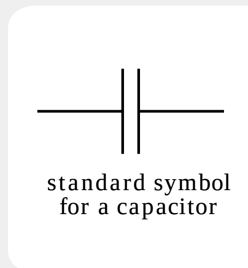


FIGURE 17.1a

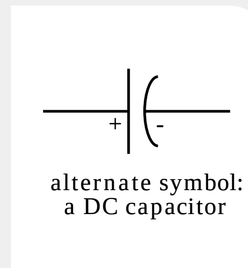


FIGURE 17.1b

Note: Although there are AC capacitors made to take high voltage at either terminal, DC capacitors have definite high and low voltage sides. When a designer of circuitry wants to specify a DC capacitor, he or she uses the symbol shown in Figure 17.1b. The straight side of that symbol is designated the high voltage side (the positive terminal) while the curved side is designated the low voltage side. We will use either symbol in DC situations.

2.) A circuit element that does not allow charge to freely flow through it probably sounds like a fairly useless device. In fact, capacitors do allow current to flow under the right conditions.

3.) Consider a circuit in which there is an initially uncharged capacitor, a power supply, a resistor, and an initially open switch (this is commonly called an RC circuit).

a.) When the switch is first closed, neither plate has charge on it. This means there is no voltage difference between the two. As the right-hand plate is connected to the ground terminal of the battery, both plates must have an initial electrical potential of zero (see Figure 17.2a).

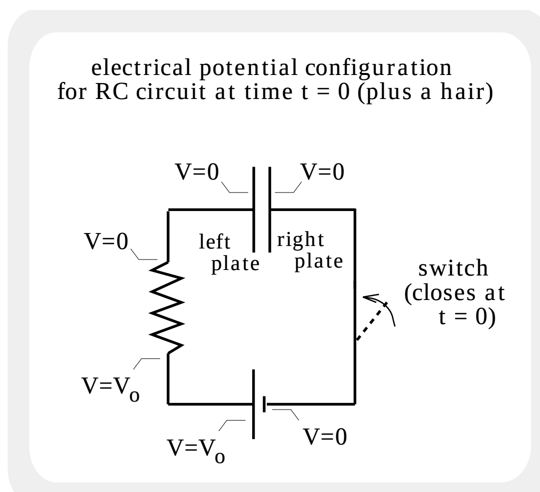


FIGURE 17.2a

b.) Just after the switch is closed, a voltage difference exists across the resistor (again, see Figure 17.2a) and, hence, current flows through the circuit.

c.) As time proceeds, positive charge accumulates on the capacitor's left plate (remember, our theory assumes that it is positive charge that moves in an electrical circuit).

d.) As it does, two things happen:

i.) Electrostatic repulsion from the positive charge accumulated on the left plate forces an equal amount of positive charge off the right plate. That leaves the right plate electrically negative.

Note: The amount of negative charge on the right plate is always equal to the amount of positive charge on the left plate. That means that current appears to be passing through a capacitor even though the capacitor's plates are not connected.

ii.) The second consequence is that the left plate's voltage begins to increase and a voltage difference begins to form across the capacitor's plates.

e.) As the voltage of the capacitor's left plate increases, the voltage on the resistor's low voltage side also begins to increase (that point and the capacitor's left plate are the same point). This decreases the voltage difference across the resistor (Figure 17.2b shows the voltage distribution around the circuit midway through the capacitor's charge-up cycle), which in turn decreases the current in the circuit (Figure 17.3 shows the Current vs. Time graph for a circuit in which a capacitor is charging).

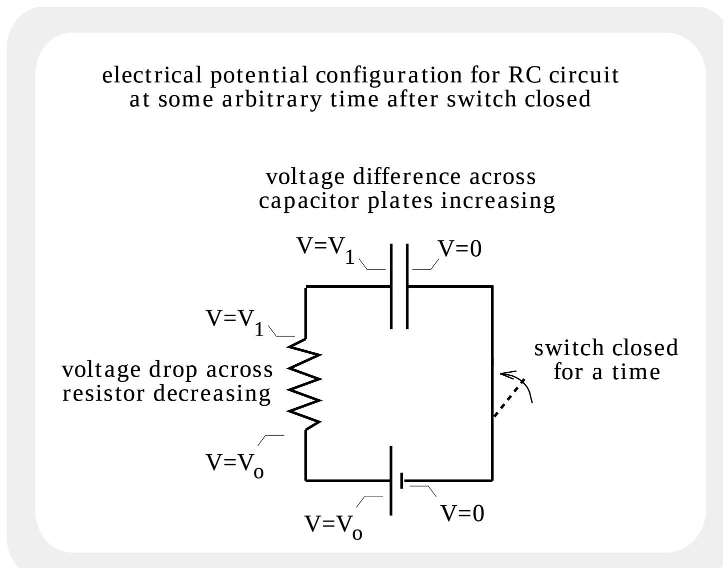


FIGURE 17.2b

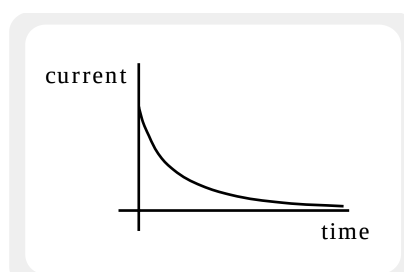


FIGURE 17.3

f.) In looking back at Figure 17.2b, it should be obvious that current will flow until V_1 builds up to and equals V_o. At that time, the voltage of the capacitor's left plate will equal the voltage of the power supply's high voltage terminal and the voltage difference across the resistor will be zero. Put another way, once the voltage across the capacitor equals the voltage across the power supply, current ceases.

Note 1: In a little different light, current will flow until the left-plate holds as much charge as it can, given the size of the power source to which it is attached.

Note 2: Does this analysis hold in theory if we switch the positions of the capacitor and resistor? Figure 17.4 shows the situation, along with the circuit's voltage distribution just after the switch has been closed. Notice that the voltage drop across the capacitor is still initially zero, and that a voltage drop across the resistor insures that current will flow. With time, the bottom plate of the capacitor accumulates positive charge, electrostatically repulsing a like charge off the top plate. That means the voltage of the top plate decreases. As the top plate of the capacitor and the left side of the resistor are one point, the voltage across the resistor diminishes with time, as does the current in the circuit.

In short, this analysis is awkward but does yield the same outward results as did the first analysis.

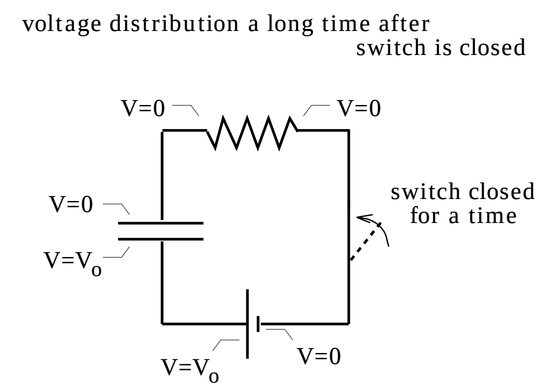


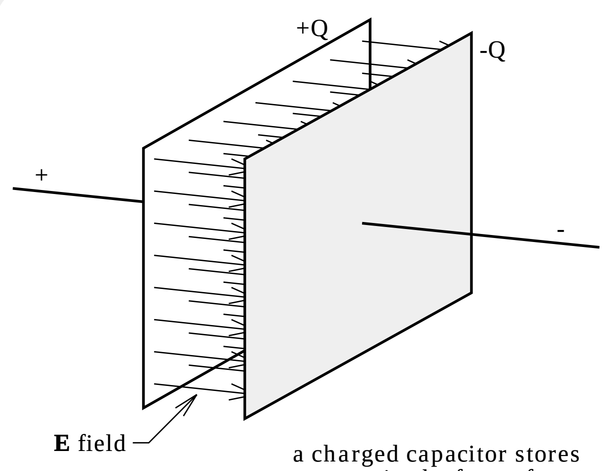
FIGURE 17.4

4.) Bottom Line:

a.) A capacitor stores charge and, in doing so, stores energy in the form of an electric field between its plates (see Figure 17.5).

b.) If a capacitor has Q's worth of positive charge on one plate, it must by its very nature have Q's worth of negative charge on its other plate.

c.) If the magnitude of the charge on ONE PLATE is Q when the



a charged capacitor stores *energy* in the form of an *electric field* between its plates

FIGURE 17.5

magnitude of the voltage drop across the capacitor's plates is V_c (this is actually $|V_- - V_+|$--as happens so often in physics, calling this V_c is sloppy but conventional notation), the capacitance of the capacitor is defined as:

$$C = Q/V_c.$$

Put another way, the magnitude of the voltage V_c across the plates of a capacitor is proportional to the charge Q on one plate. The proportionality constant is called the capacitance C, and the relationship between the variables is:

$$Q = CV_c.$$

d.) By the definition of capacitance (i.e., $C = Q/V$), the MKS units are coulombs per volt. The name given to this set of units is the farad.

One farad is an enormous capacitance. It is common to use capacitors that are much smaller. The following are the ranges most often encountered (you should know not only their prefixes and definitions but also their symbols):

i.) A millifarad is symbolized as mf and is equal to 10^{-3} farads;

ii.) A microfarad is symbolized as μf and is equal to 10^{-6} farads;

iii.) A nanofarad is symbolized as nf and is equal to 10^{-9} farads;

iv.) A picofarad is symbolized as pf and is equal to 10^{-12} farads.

5.) Example of a Capacitor In Action: Consider the camera-flash circuit shown in Figure 17.6.

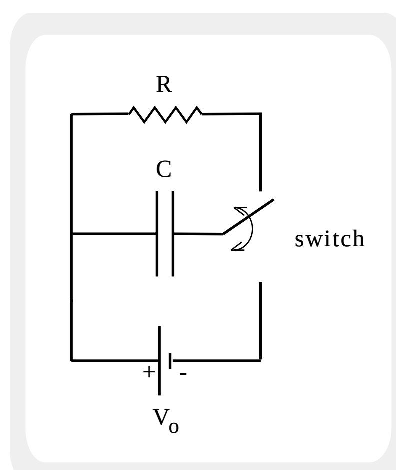


FIGURE 17.6

a.) The switch is initially connected in the down position so that the capacitor is hooked across the power supply. This allows the capacitor's plates to charge up.

b.) When the flash is activated, the switch flips to the up position. The capacitor discharges across the resistor (i.e., charge flows from one plate to the other, passing through the resistor/light bulb in the process) with the large, momentary charge-flow lighting the flashbulb.

c.) Once fired, the switch automatically flips down, allowing the capacitor to once again charge itself off the power supply.

B.) Equivalent Capacitance of Parallel and Series Combinations:

1.) The Equivalent Capacitance for Capacitors in Series:

a.) Just as current is common for all resistors connected in series, charge accumulation on capacitor plates is the common quantity for capacitors in series.

i.) Examining Figure 17.7, the positive charge electrically forced off the right plate of the first capacitor must go somewhere. Where? It accumulates on the left plate of the second capacitor.

ii.) Conclusion: The amount of charge associated with each series capacitor must be the same.

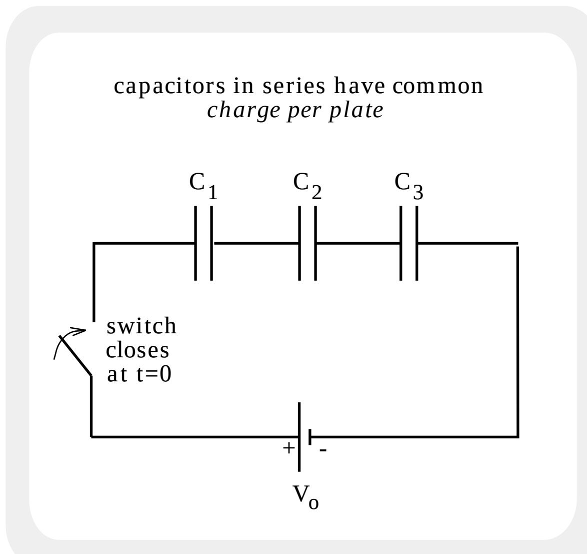


FIGURE 17.7

b.) At a given instant, the sum of the voltage drops across the three capacitors must equal the voltage drop across the power supply, or:

$$V_o = V_1 + V_2 + V_3 + \ldots$$

c.) As the voltage across a capacitor is related to the charge on and capacitance of a capacitor (V = Q/C), we can write:

$$\begin{aligned} V_o &= V_1 + V_2 + V_3 + \ldots \\ Q/C_{eq} &= Q/C_1 + Q/C_2 + Q/C_3 + \ldots \end{aligned}$$

d.) With the Q's canceling nicely, we end up with:

$$1/C_{eq} = 1/C_1 + 1/C_2 + 1/C_3.$$

e.) In other words, the equivalent capacitance for a series combination of capacitors has the same mathematical form as that of a parallel combination for resistors.

2.) The Equivalent Capacitance for Capacitors in Parallel:

a.) Just as voltage is common for all resistors connected in parallel, voltage across capacitor plates is the common quantity for capacitors in parallel (see Figure 17.8).

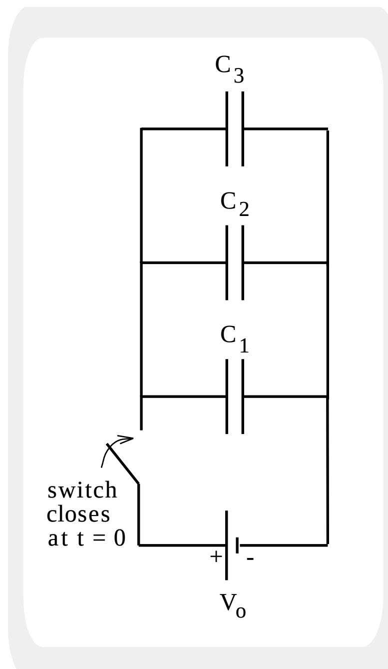


FIGURE 17.8

b.) Over time, the charge that accumulates on the various capacitors has to equal the total charge drawn from the power supply, or:

$$Q_o = Q_1 + Q_2 + Q_3 + \ldots$$

As each capacitor's charge is related to the voltage across its plates by Q = CV, we can write:

$$Q_o = Q_1 + Q_2 + Q_3 + \ldots$$
$$C_{eq}V_o = C_1V_o + C_2V_o + C_3V_o + \ldots$$

With the V_o's canceling nicely, we end up with:

$$C_{eq} = C_1 + C_2 + C_3.$$

c.) In other words, the equivalent capacitance for a parallel combination of capacitors has the same mathematical form as that of the series combination for resistors.

C.) The Current Characteristics of a Charging-Capacitor Circuit:

1.) Because there is no charge on the plates of an uncharged capacitor, a capacitor will initially provide no resistance to charge flow in an RC circuit. But as the capacitor charges up, it will become increasingly more difficult for additional charge to be forced onto the capacitor's plates, and the current in the circuit will decrease. We would like to derive an expression for the current in a DC-RC circuit as a function of time.

2.) Figure 17.9 shows the circuit. Remembering that the voltage drop across a capacitor will be q/C, we can use Kirchoff's Laws to write:

$$V_o - q/C - iR = 0.$$

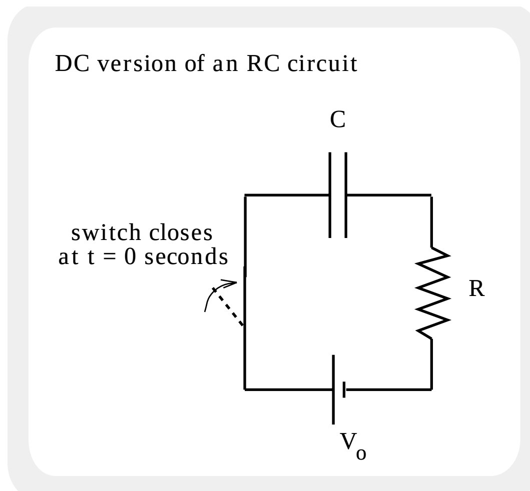


FIGURE 17.9

a.) The first thing to notice is that just after the switch is closed (i.e., at $t = 0^+$), there is effectively no charge on the capacitor. That means Kirchoff's loop equation reduces to:

$$V_o - (0) - i_oR = 0$$

and the initial current in the circuit is found to be:

$$i_o = V_o/R.$$

b.) Additionally, after a very long period, the capacitor will have charged to its maximum Q_{max} and the current in the circuit will be zero. That means Kirchoff's loop equation reduces to:

$$V_o - Q_{max}/C - (0) = 0$$

and the maximum charge on the capacitor is found to be:

$$Q_{max} = CV_o.$$

c.) The question arises, "What is the current function that defines the charge flow in the circuit as time progresses?"

To answer that question, we must deal with Kirchoff's loop equation in its most general state.

3.) Kirchoff's loop equation for this situation, written in its most convenient form and expressed showing the time dependence of q and i, states that:

$$i(t)R + q(t)/C = V_o.$$

a.) Two observations:

i.) The current in the circuit at a given instant and the rate at which charge accumulates on the capacitor's plates are related by the definition of current, or:

$$i = dq/dt.$$

ii.) Also, because we know that current, etc., is varying with time, we can for the sake of simplicity ignore the time-dependence notation. That is, from here on out we will assume that $i = i(t)$.

b.) With these observations and dividing everything by R, we can rewrite Kirchoff's loop equation as:

$$dq/dt + [1/(RC)]q = V_o/R.$$

c.) This is a differential equation. It essentially states that we are looking for a function q such that when we take its derivative dq/dt and add to it a constant times itself (i.e., (1/RC)q), we will always get the same number (in this case, V_o/R).

d.) From experience, the solution to a differential equation of this form is:

$$q = A + Be^{kt},$$

where A, B, and k are all constants to be determined.

e.) Solving for the constants is essentially a boundary value problem. That is, we must use what we know about the system at its boundaries--at $t = 0$ and at $t = \infty$. Doing so yields:

i.) At $t = 0$, the charge q on the capacitor must equal ZERO. Using that, we can write:

$$q = A + Be^{kt}$$

$$\Rightarrow \quad 0 = A + Be^{k(0)}$$
$$\Rightarrow \quad B = -A.$$

ii.) At $t = \infty$, the charge q on the capacitor must equal Q_{max}. Using that and the information gleaned above, we can write:

$$q = A - Ae^{kt}$$
$$\Rightarrow \quad Q_{max} = A - Ae^{k(\infty)}.$$

The only way this can not be infinitely large is if the constant k is negative. From physical constraints (i.e., the fact that there is a limit on the amount of charge the capacitor can hold), we can un-embed the negative sign inside k and rewrite the above equation as:

$$Q_{max} = A - Ae^{-k(\infty)}$$
$$= A - A\left(\frac{1}{e^{k(\infty)}}\right)$$
$$\Rightarrow \quad Q_{max} = A \qquad \text{(as } 1/e^{k\infty} = 0\text{)}.$$

iii.) Using the information gleaned from above, we can write the current as:

$$i = dq/dt$$
$$= \frac{d\left[Q_{max} - Q_{max}e^{-kt}\right]}{dt}$$
$$= kQ_{max}e^{-kt}.$$

iv.) At $t = 0$, we know that $i_o = V_o/R$. As such, we can write:

$$i_o = \left(\frac{V_o}{R}\right)$$

and

$$i_o = kQ_{max}e^{-k(0)}$$
$$= kQ_{max}.$$

v.) Equating the two i_o expressions, we get $k = (V_o/Q_{max})(1/R)$. Noting that $(V_o/Q_{max}) = 1/C$, we can rewrite our k expression as:

$$k = \left(\frac{1}{RC}\right).$$

vi.) With this, q(t) is:

$$q = Q_{max} - Q_{max}e^{-t/RC}.$$

As i = dq/dt, the current is:

$$i = -[-Q_{max}/(RC)]e^{-t/RC}.$$

Substituting in $Q_{max}/C = V_o$:

$$i = (V_o/R)e^{-t/RC}$$
$$= i_o e^{-t/RC}.$$

f.) See Figure 17.10.

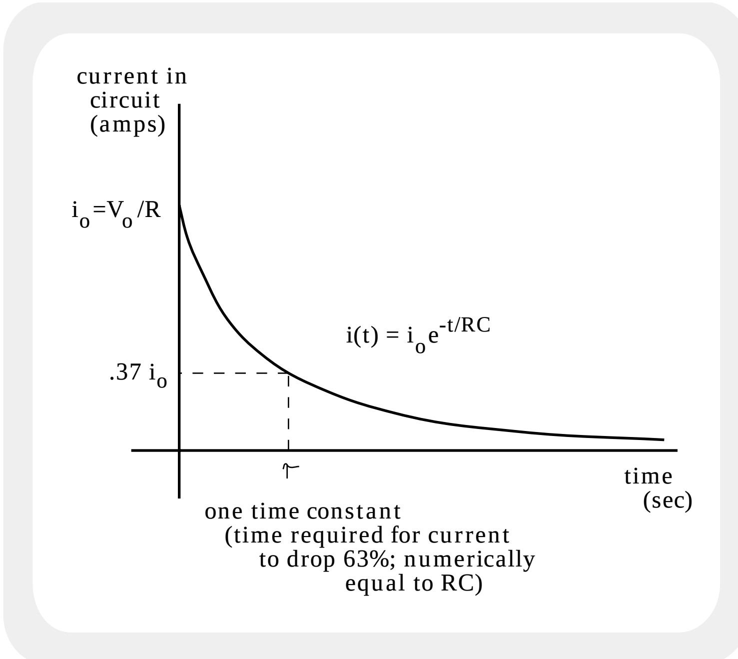


FIGURE 17.10

g.) The graph also points out a particular point in time that has been deemed important--the amount of time defined as one time constant. Consider:

i.) We would like to have some idea as to how fast the capacitor will charge, which is to say how fast the current in the system will drop (the two questions are essentially the same).

ii.) With that in mind, how much charge is on the capacitor, and how much current is in the system, at time t = RC?

Note: You may wonder why this particular time was picked. It was originally picked solely because it was the amount of time required to make the exponent of the exponential equal to -1.

iii.) This amount of time is called one time constant. Its symbol is a baby tau (i.e., τ) and it equals RC. The amount of charge on the capacitor after a time interval equal to τ will be:

$$q = Q_{max} - Q_{max}e^{-RC/RC}$$
$$= Q_{max} - Q_{max}(e^{-1})$$
$$= Q_{max} - .37Q_{max}$$
$$= .63\ Q_{max}.$$

and the amount of current in the circuit will be:

$$\begin{aligned} i &= i_o e^{-RC/RC} \\ &= i_o(e^{-1}) \\ &= .37 i_o. \end{aligned}$$

h.) What does this tell us? It tells us that if we multiply the value of the capacitance and resistance together, the number we end up with will:

i.) Have the units of seconds (this has to be the case if the exponent is to be unitless);

ii.) Be the amount of time required for the capacitor to charge to 63% of its maximum; and

iii.) Be the amount of time required for the current to drop to 37% of its maximum.

Note 1: In doing the math, the time interval 2τ will give us approximately 87% charge-up for the capacitor and a current that will have dropped to approximately 13% of its initial value.

Note 2: The time it takes to charge a capacitor to $.63Q_{max}$ is the same amount of time required to discharge $.63Q_{max}$ from the capacitor.

i.) Why is τ important? It would be idiotic to build a camera flash using a resistor and capacitor whose time constant was, say, ten seconds. Waiting twenty seconds for 87% of your charge to dump through the resistor would never do. Knowing a system's time constant is helpful.

D.) Capacitance in Terms of <u>Physical Parameters</u> for a Parallel Plate Capacitor:

1.) What is the capacitance of a parallel plate capacitor in terms of its geometric parameters (i.e., its plate area, the distance between its plates, etc.)?

2.) To delve into this question, consider a typical, parallel plate capacitor (Figure 17.11a to the right). Assume the distance between the plates is d meters and the area of one plate is A_o square meters.

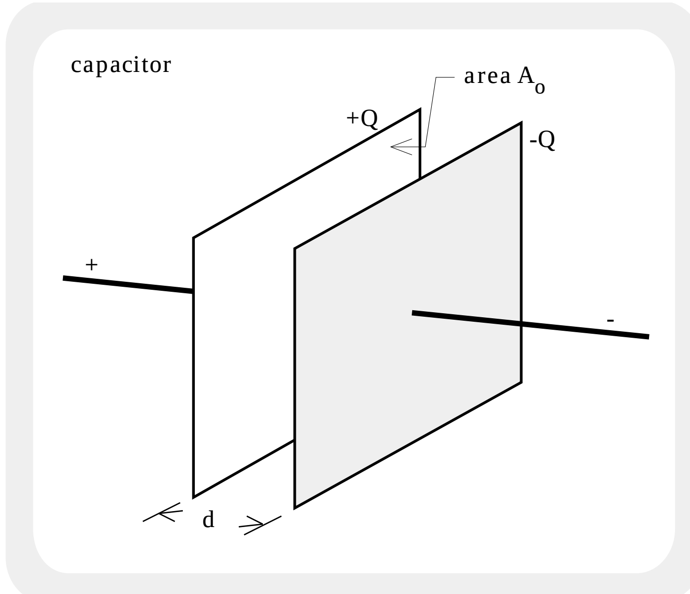


FIGURE 17.11a

3.) Assuming there is a charge Q on the high voltage plate and that the magnitude of the voltage difference across the plates is a positive V_c, we can begin with the definition of capacitance, or:

$$C = Q/V_c.$$

Note: Moving from the high voltage plate to the low voltage plate, we see a voltage difference that is $\Delta V = (V_- - V_+) = -V_c$, where V_c is defined as the magnitude of the voltage drop across the plates of the capacitor. This observation will be useful to us shortly.

4.) If we can determine the electric field function for the region between the plates, we can derive an expression for ΔV across the plates by using:

$$\Delta V = -\int \mathbf{E}\cdot d\mathbf{r}.$$

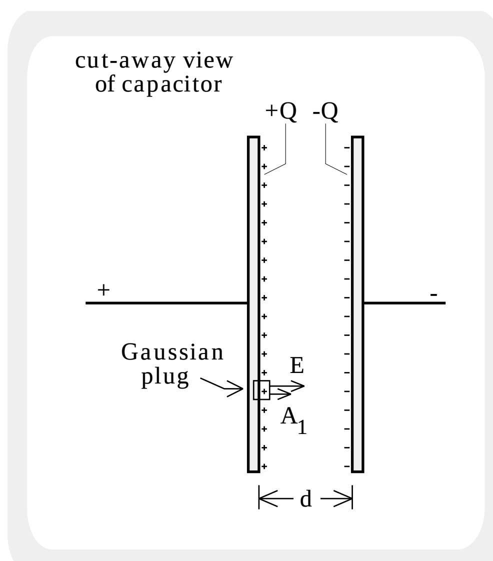


FIGURE 17.11b

5.) To get the electric field function, we must use Gauss's Law. Figure 17.11b shows a cut-away view of the capacitor, Gaussian surface and all. Note that the Gaussian plug has one of its end-faces inside the conductor (i.e., in a region in which the electric field is zero). Assuming the end-face of the Gaussian surface has an area of A_1, and noting that the surface charge density on the positive plate's surface is $\sigma = Q/A_o$, we can write:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{enclosed}}{\varepsilon_o}$$

$$\Rightarrow \quad EA_1 = \frac{\sigma A_1}{\varepsilon_o}$$

$$\Rightarrow \quad EA_1 = \frac{\left(\frac{Q}{A_o}\right)A_1}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{Q}{\varepsilon_o A_o}.$$

Note 1: Although the net electric field between the plates is the vector sum of the electric fields generated by the positive charge on the left plate and the negative charge on the right plate, the net electric field will be the same everywhere within the region (this is due to the symmetry of the situation).

The electric field very close to the left plate is produced primarily by the charges on the left plate. By using a Gaussian surface very near the left plate, we can determine the electric flux due to that charge, then deduce its resulting electric field. Knowing the field at one point, we know the field at all points between the plates.

Note 2: The constant ε_o equals 8.85×10^{-12} farads per meter. This works out to $1.26 \times 10^{-6}\ C^2 \cdot s^2/kg \cdot m^3$.

6.) We are now ready to use $\Delta V = -\int E \cdot dr$. Noting that we will move in the direction of the electric field (i.e., from the left plate with its higher electrical potential to the right plate with its lower electrical potential), and denoting the voltage on the positive plate as V_+ and the voltage of the negative plate as V_-, we can write:

$$\begin{aligned} V_c &= -\left[V_- - V_+\right] \qquad \text{(i.e., } -\Delta V) \\ &= -\left[-\int \mathbf{E} \bullet d\mathbf{r}\right] \\ &= \int_{x=0}^{d} \left[\frac{Q}{\varepsilon_o A_o}\,\mathbf{i}\right] \bullet \left[dx\mathbf{i}\right] \\ &= \frac{Q}{\varepsilon_o A_o} \int_{x=0}^{d} dx \\ &= \frac{Q}{\varepsilon_o A_o}\, d. \end{aligned}$$

7.) With a general expression for the voltage difference V_c across the capacitor's plates, we can return to the general expression for capacitance and write:

$$C = \frac{Q}{V_c} = \frac{Q}{\left[\frac{Qd}{\varepsilon_o A_o}\right]} = \varepsilon_o \frac{A_o}{d}.$$

8.) As involved as this may seem, the approach allows us to derive expressions for the capacitance of a capacitor in terms of the capacitor's geometric characteristics. Reviewing, the approach is straightforward:

a.) Assuming a charge Q on the plates, begin with:

$$C = Q/V_c.$$

b.) The magnitude of the voltage drop across the plates (i.e., V_c) is:

$$V_c = -(V_- - V_+) = +\int \mathbf{E}\cdot d\mathbf{r},$$

where E is the electric field expression for the region between the plates and dr is a differential path directed <u>from the higher to the lower voltage plates</u>.

c.) If the electric field function is not known, use Gauss's Law to determine it.

d.) Once you have E, determine V_c and put it back into the relationship $C = Q/V_c$.

E.) Dielectrics:

1.) Consider the situation in which a piece of insulating material, called a dielectric, is placed between the plates of the capacitor (see Figure 17.12a).

The capacitor is charged, then isolated (that is, once charged it is disconnected from the power supply). What must be true?

a.) Let E_o be the electric field without the dielectric between the capacitor's plates.

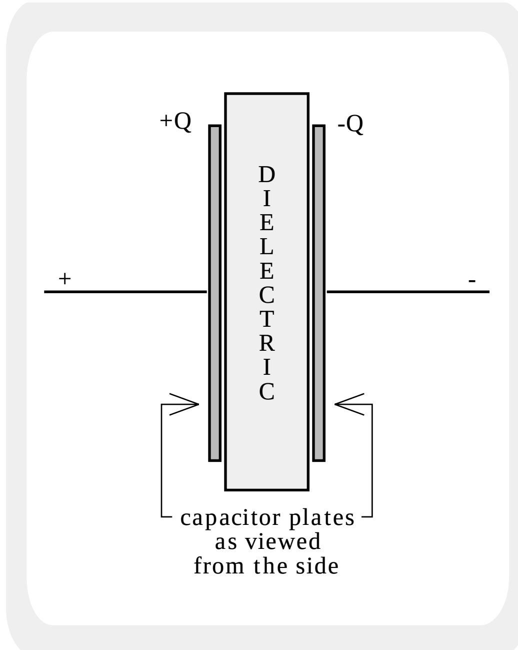


FIGURE 17.12a

b.) When the insulator is placed between the plates, the surface of the insulator facing the positive plate of the capacitor will experience a Van der Waal-type charge separation that makes that face appear negative. A similar effect will be found on the other face making it appear positive (see Figure 17.12b).

c.) The charge separation in the dielectric creates a second electric field E_d between the plates (again, see Figure 17.12b) in the opposite direction of E_o. Although E_d is considerably smaller than E_o, the effect is to decrease the net electric field between the plates.

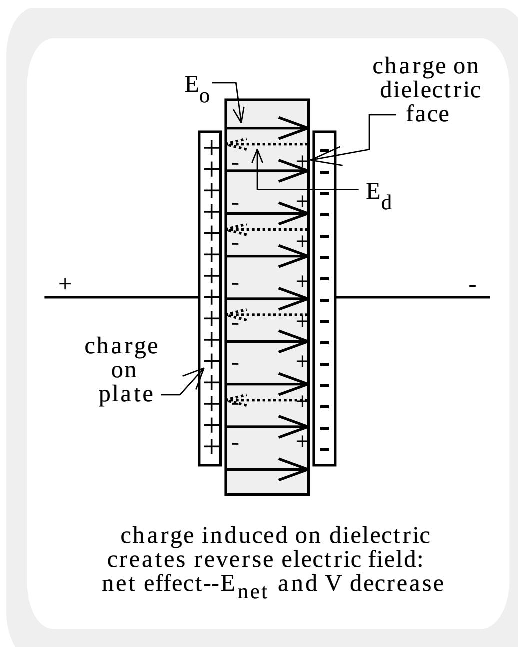


FIGURE 17.12b

d.) As the electric field between the plates is proportional to the voltage difference across the plates, decreasing the electric field by inserting the dielectric effectively decreases the voltage across the plates.

e.) We know that $C = Q/V_c$. If the charge on the plates stays the same while the voltage across the plates goes down the capacitance C increases.

f.) <u>Bottom Line:</u> Inserting a dielectric between the plates of a capacitor INCREASES THE CAPACITANCE.

F.) Derivation of Capacitance in Terms of <u>Physical Parameters</u> for a Dielectric-Filled, Parallel Plate Capacitor:

1.) There is a small problem that arises when we try to use the geometric parameters of a dielectric-filled capacitor to derive its capacitance.

Note: Sections a through e below are provided solely to allow students to understand the origin and theoretical underpinnings of what is called the dielectric constant. You will not be required to duplicate the information presented; you will be expected to <u>understand</u> the bottom line presented in Section f.

a.) We have already established that when a dielectric surface is placed near or against a positively charged capacitor plate, a negative surface charge is induced on the dielectric's surface.

b.) What that means is that if we use Gauss's Law to derive an expression for the electric field between the plates, the charge enclosed within the Gaussian surface will not be made up solely of the charge on the capacitor plate. We will also have to take into account the charge induced on the dielectric.

c.) Mathematically speaking, this means that the charge enclosed inside the Gaussian structure will be:

$$q_{encl} = q_{plate} - q_{diel}$$

leaving Gauss's equation looking like:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{plate} - q_{diel}}{\varepsilon_o}.$$

d.) We know nothing about q_{diel}. To get around this problem, we can define a dielectric constant κ_d (some books use the symbol ε_d) such that:

$$\frac{1}{\kappa_d} = \frac{q_{plate} - q_{diel}}{q_{plate}}.$$

With this, we can write:

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{plate} - q_{diel}}{\varepsilon_o}$$

$$= \left(\frac{q_{plate} - q_{diel}}{q_{plate}} \right) \frac{q_{plate}}{\varepsilon_o}$$

$$= \left(\frac{1}{\kappa_d} \right) \frac{q_{plate}}{\varepsilon_o}.$$

e.) As the dielectric constants for commercially produced dielectrics are known values, we can safely use the dielectric constant in Gauss's Law, ignore the surface charge on the dielectric, and deal solely with the charge on the capacitor's plates.

f.) A more conventional way of writing Gauss's Law with the dielectric constant included is:

$$\kappa_d \int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{encl}}{\varepsilon_o},$$

where $q_{enclosed}$ is assumed to be the free charge on the capacitor's plates inside the Gaussian surface.

2.) How does this affect the derivation we did to determine the capacitance of a parallel plate capacitor in terms of its geometric parameters? With a dielectric filling up ALL the space between the plates, that calculation looks like:

$$\kappa_d \int \mathbf{E} \bullet d\mathbf{S} = \frac{q_{encl}}{\varepsilon_o}$$

$$\Rightarrow \quad EA_1 = \frac{\sigma A_1}{\kappa_d \varepsilon_o}$$

$$\Rightarrow \quad EA_1 = \frac{\left(\frac{Q}{A_o} \right) A_1}{\kappa_d \varepsilon_o}$$

$$\Rightarrow \quad E = \frac{Q}{\kappa_d \varepsilon_o A_o}.$$

With this E, we can write:

$$
\begin{aligned}
V_{c,d} &= -\left[V_{-} - V_{+}\right] \\
&= -\left[-\int \mathbf{E} \bullet d\mathbf{r}\right] \\
&= \int_{x=0}^{d} \left[\frac{Q}{\kappa_d \varepsilon_o A_o}\mathbf{i}\right] \bullet \left[dx\mathbf{i}\right] \\
&= \frac{Q}{\kappa_d \varepsilon_o A_o} \int_{x=0}^{d} dx \\
&= \frac{Q}{\kappa_d \varepsilon_o A_o} d.
\end{aligned}
$$

That yields a final expression for the capacitance of a parallel plate capacitor with a dielectric in place between its plates (call this C_d). It is:

$$
\begin{aligned}
C_d &= \frac{Q}{V_{c,d}} \\
&= \frac{Q}{\left[\dfrac{Qd}{\kappa_d \varepsilon_o A_o}\right]} \\
&= \kappa_d \varepsilon_o \frac{A_o}{d}.
\end{aligned}
$$

3.) There are two observations that can be made when dealing with this kind of situation:

a.) As $C_d = \kappa_d \varepsilon_o (A_o/d)$ and $C_{w/o} = \varepsilon_o (A_o/d)$, it would appear that the dielectric constant κ_d links the capacitance of an air-filled capacitor to the capacitance of a dielectric-filled capacitor by the relationship:

$$C_d = \kappa_d C_{w/o}.$$

b.) Put in a little different way, the dielectric constant for a material is simply a number that tells you how much the capacitance of an air-filled capacitor will be boosted when the dielectric is placed between its plates.

4.) Dielectrics used in conjunction with capacitors are useful for three reasons:

a.) As explained above, the presence of a dielectric between a capacitor's plates inherently increases the capacitance of the capacitor.

b.) A piece of insulating material (a dielectric) placed between the plates acts like a gap-jumping barrier for electricity. That means much larger voltages, hence much larger electric fields, can be dealt with using a capacitor that would not otherwise have been able to handle the situation. Put another way, more charge can be stored on the plates without fear of breakdown that would otherwise have been the case (breakdown occurs when the electric field between the plates is so large that charge leaps the gap--once breakdown is achieved in a dielectric-filled capacitor, the capacitor is ruined).

c.) Due to its insulating properties, dielectrics allow plates to be brought very close to one another. As the capacitance is inversely proportional to the distance d between the plates, this allows for both the miniaturization of capacitors as well as the increasing of a capacitor's capacitance per unit of plate area.

G.) Capacitance in Terms of <u>Physical Parameters</u> for a Partial, Dielectric-Filled Parallel Plate Capacitor:

1.) Figure 17.13 shows a parallel plate capacitor (plate area A_o, distance between plates d) with a dielectric material (dielectric constant κ_d) filling the space between the plates to a distance 2d/3 units from the left-hand plate. In terms of geometric parameters of the capacitor, what is the capacitor's capacitance?

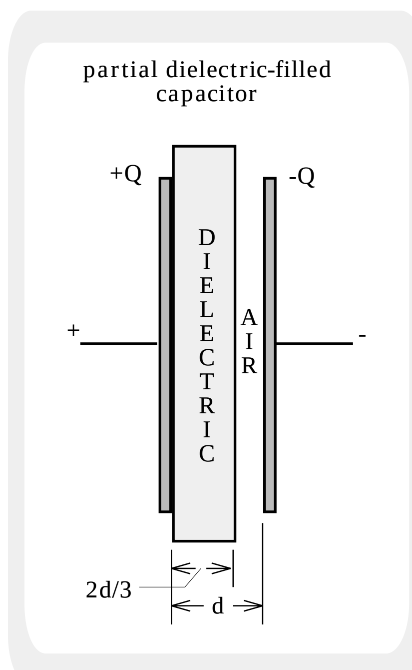


FIGURE 17.13

2.) Until now, we have proceeded in capacitance in terms of physical parameters-type problems as follows. We have:

a.) Used Gauss's Law to determine an expression for the electric field in the region between the plates; then

b.) Used $V_c = -(V_- - V_+) = -[-\int E \cdot dr]$ to determine the voltage across the capacitor's plates; then c.) Used the relationship $C = Q/V_c$ to determine the capacitance of the capacitor.

3.) The twist here is in the fact that the dielectric does not completely fill the space between the plates, which means there are two distinct electric fields in that region--one in the dielectric-filled region and one in the air-filled region. To accommodate this, we must:

a.) Use Gauss's Law to determine an expression for the electric field in both regions between the plates (the only difference between the two expressions will be the limits and the presence of the dielectric constant in the electric field expression for the dielectric-filled region); then . . .

b.) Note that the voltage difference across the dielectric added to the voltage difference across the air-filled region will yield the net voltage difference across the plates. Combining this with our electric field expressions yields:

$$\begin{aligned} V_c &= -\Delta V \\ &= -\left[\ \left(V_{\text{end of diel}} - V_+\right) \ + \ \left(V_- - V_{\text{end of diel}}\right) \ \right] \\ &= -\left[\left(-\int_{\text{pos plate}}^{\text{end of diel}} \mathbf{E}_{\text{diel}} \bullet d\mathbf{r}\right) + \left(-\int_{\text{end of diel}}^{\text{neg plate}} \mathbf{E}_{\text{air}} \bullet d\mathbf{r}\right)\right]. \end{aligned}$$

c.) With V_c we can use the relationship $C = Q/V_c$ to determine the capacitance of the capacitor.

4.) Doing the problem:

a.) In Section D-5, we used Gauss's Law to derive the electric field expression for an air-filled region between capacitor plates. That expression was:

$$E = Q/(\varepsilon_o A_o).$$

b.) In Section F-2, we used Gauss's Law to derive the electric field expression for a dielectric-filled region between capacitor plates. That expression was:

$$E = Q/(\kappa_d \varepsilon_o A_o).$$

c.) Using this with our electrical potential difference relationship, we get:

$$
\begin{aligned}
V_c &= -\left[\ \left(V_{2d/3} - V_+\right) \ + \ \left(V_- - V_{2d/3}\right)\ \right] \\
&= -\left[\left(-\int_{x=0}^{2d/3} \mathbf{E}_{diel} \bullet d\mathbf{r}\right) + \left(-\int_{2d/3}^{d} \mathbf{E}_{air} \bullet d\mathbf{r}\right)\right] \\
&= \left[\left(\int_{x=0}^{2d/3} \left(\frac{q_{plate}}{\kappa_d \varepsilon_o A_o}\mathbf{i}\right) \bullet dx\mathbf{i}\right) + \left(\int_{x=2d/3}^{d} \left(\frac{q_{plate}}{\varepsilon_o A_o}\mathbf{i}\right) \bullet dx\mathbf{i}\right)\right] \\
&= \left[\frac{q_{plate}}{\kappa_d \varepsilon_o A_o}\left(\int_{x=0}^{2d/3} dx\right) + \frac{q_{plate}}{\varepsilon_o A_o}\left(\int_{x=2d/3}^{d} dx\right)\right] \\
&= \left[\frac{q_{plate}}{\kappa_d \varepsilon_o A_o}\left(\frac{2d}{3} - 0\right) + \frac{q_{plate}}{\varepsilon_o A_o}\left(d - \frac{2d}{3}\right)\right] \\
&= \frac{q_{plate} d}{3\varepsilon_o A_o}\left[\frac{2}{\kappa_d} + 1\right] \\
&= \frac{q_{plate} d\left[2 + \kappa_d\right]}{3\kappa_d \varepsilon_o A_o}.
\end{aligned}
$$

d.) Using our definition for capacitance, we get:

$$
\begin{aligned}
C &= \frac{q_{plate}}{\left[\dfrac{q_{plate} d\left[2 + \kappa_d\right]}{3\kappa_d \varepsilon_o A_o}\right]} \\
&= \frac{3\kappa_d \varepsilon_o A_o}{d\left[2 + \kappa_d\right]}.
\end{aligned}
$$

H.) Capacitance in Terms of <u>Physical Parameters</u> for Geometries Other Than That of the Parallel Plate:

1.) Figure 17.14 (next page) shows an air-filled coaxial cable with inside radius R_1 and outside radius R_2. In terms of geometric parameters of the capacitor, what is the capacitor's capacitance per unit length?

2.) Notice that the problem does not define charge for the plates. That is something we must assume on our own. As such, assume there is a linear charge density $+\lambda$ on the inner cylinder and an equal and opposite linear charge density $-\lambda$ on the outer cylindrical shell.

3.) We need an expression for the electric field between the plates. Using a Gaussian cylinder of length L and radius r, where $R_2 > r > R_1$, and assuming there is Q/L's worth of charge inside the Gaussian surface (i.e., λ = Q/L), we can write:

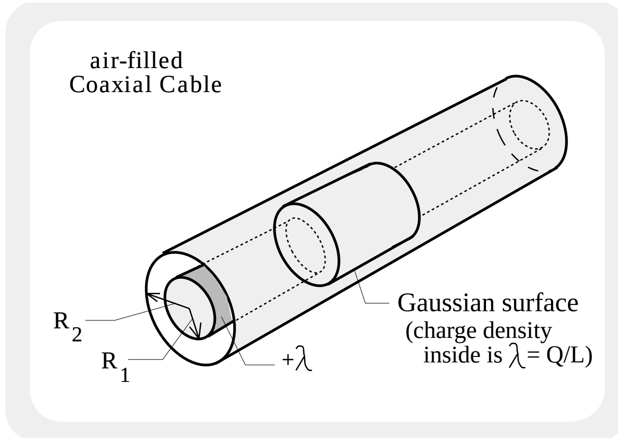


FIGURE 17.14

$$\int_S \mathbf{E} \bullet d\mathbf{S} = \frac{q_{encl}}{\varepsilon_o}$$

$$\Rightarrow \quad E(2\pi rL) = \frac{\lambda L}{\varepsilon_o}$$

$$\Rightarrow \quad E = \frac{\left(\frac{Q}{L}\right)}{2\pi\varepsilon_o r}.$$

4.) Using our electrical potential difference relationship, we can write:

$$\begin{aligned} V_c &= -[V_- - V_+] \\ &= -\left[-\int_{r=R_1}^{R_2} \mathbf{E} \bullet d\mathbf{r}\right] \\ &= \int_{r=R_1}^{R_2} \left[\frac{Q}{2\pi\varepsilon_o rL}\,\mathbf{r}\right] \bullet dr\mathbf{r} \\ &= \frac{Q}{2\pi\varepsilon_o L}\left[\int_{r=R_1}^{R_2} \left(\frac{1}{r}\right) dr\right] \\ &= \frac{Q}{2\pi\varepsilon_o L}[\ln r]_{r=R_1}^{R_2} \\ &= \frac{Q}{2\pi\varepsilon_o L}[\ln R_2 - \ln R_1]. \end{aligned}$$

5.) Using our derived expression for V_c and the definition of capacitance, we can write:

$$
\begin{aligned}
C &= \frac{Q}{V_c} \\
&= \frac{Q}{\left[\dfrac{Q(\ln R_2 - \ln R_1)}{2\pi\varepsilon_o L}\right]} \\
\Rightarrow \quad C/L &= \frac{2\pi\varepsilon_o}{\ln\left(\dfrac{R_2}{R_1}\right)}.
\end{aligned}
$$

6.) This is the capacitance per unit length for an air-filled coaxial cable. Coaxial cables used in industry (i.e., for TV and VCR hook-ups) are dielectric-filled. You will have the thrill of determining their capacitance per unit length in the Questions section.

Big Note: We haven't said anything about spherical capacitors. Think about how you could put the above theory to use in determining the capacitance of a partial dielectric-filled spherical capacitor (you won't find the answer anywhere in this book--this is something for you to chew on on your own). Sounds like a great little test question.

I.) Energy Stored in a Capacitor:

1.) The work done to charge a capacitor is stored as electrical potential energy in the electric field created between the capacitor's plates. As such, we can determine the amount of energy stored in a capacitor by determining the amount of work required to charge the capacitor.

a.) To be as general as possible, assume a capacitor of capacitance C initially has charge q on its high voltage plate and -q on its low voltage plate. If the voltage across the plates is initially V_c, how much work must be done to add an additional dq's worth of charge to the positive plate?

b.) The amount of work we are looking for will equal the amount of work required to move the charge dq from one plate to the other (that is effectively what is happening as electrostatic repulsion pushes dq's worth of positive charge off the capacitor's low voltage plate).

c.) The relationship between the work dW done on charge dq as it moves through a potential difference $(V_- - V_+) = -V_c$ is:

$$\begin{aligned} dW/dq &= -\Delta V \\ &= +V_c. \end{aligned}$$

d.) Remembering that $V_c = q/C$, where q is the charge already on the plates, we can rewrite this as:

$$\begin{aligned} dW &= (V_c)dq \\ &= (q/C)dq. \end{aligned}$$

e.) The total amount of energy required to place a net charge Q on the capacitor's plates will be the sum (i.e., integral) of all the differential work dW quantities evaluated from q = 0 to q = Q. Doing that operation yields:

$$\begin{aligned} W &= \int dW \\ &= \int_{q=0}^{Q} \left[\frac{q}{C}\right] dq \\ &= \left(\frac{1}{C}\right)\left[\frac{q^2}{2}\right]_{q=0}^{Q} \\ &= \left(\frac{1}{C}\right)\left[\frac{Q^2}{2}\right]. \end{aligned}$$

f.) As $Q = CV_c$, the work expression can be re-written as:

$$W = \frac{1}{2} C V_c^{\,2}.$$

This is the amount of ENERGY wrapped up in a capacitor whose capacitance is C and across whose plates a voltage V_c is impressed.

QUESTIONS

17.1) Assuming there is no charge initially on any capacitor, answer all the following questions for the capacitor circuit in sketch a. When done, repeat the process for the circuit shown in sketch b:

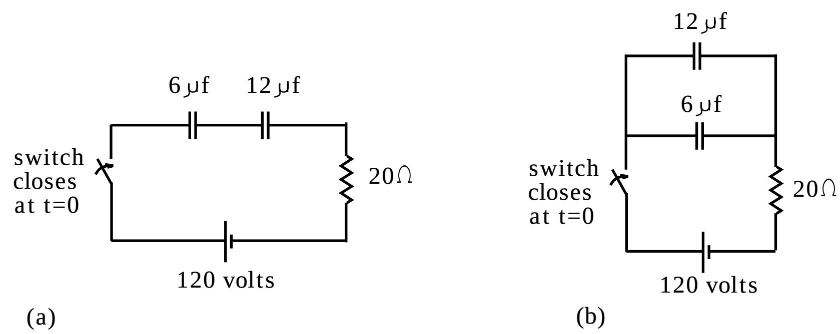


FIGURE I

a.) Determine the initial current in the circuit when the switch is first thrown.

b.) A long time after the switch is thrown (i.e., by the time the caps are charged up fully), how much charge is there on each plate?

c.) What is the voltage across the 6 μf capacitor when fully charged?

d.) How much energy does the 6 μf capacitor hold when completely charged?

e.) Determine the RC circuit's time constant. What does this information tell you?

f.) How much charge is there on the 6 μf capacitor after a time interval equal to one time constant passes?

17.2) Three identical capacitors are connected in several ways as shown in Figure II.

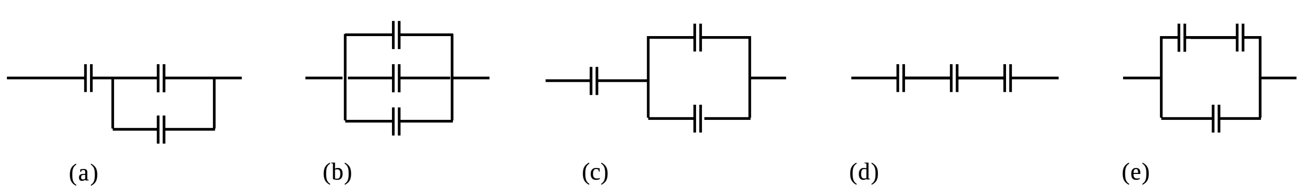


FIGURE II

a.) Order the combinations from the smallest equivalent capacitance to the largest; and

b.) Which combination has the potential of storing the most energy?

17.3) A parallel plate capacitor is connected to a 20 volt power supply. Once charged to its maximum possible Q, the capacitor's plates are separated by a factor of four (that is, the distance between the plates is quadrupled) while the capacitor is kept hooked to the power supply. As a consequence of this change in geometry:

a.) How will the capacitor's capacitance change?

b.) How will the charge on the capacitor change?

c.) How will the energy stored in the capacitor change?

d.) If a dielectric (κ_d = 1.6) had been placed between the plates of the original setup, what would the new capacitance have been?

17.4) Determine:

a.) The equivalent capacitance of the circuit shown in Figure III.

b.) Assuming each capacitor's capacitance is 25 mf, how much energy can this system store if it is hooked across a 120 volt battery?

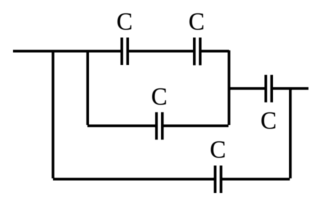


FIGURE III

17.5) The capacitors in the circuit shown in Figure IV are initially uncharged. At t = 0, the switch is closed. Knowing the resistor and capacitor values:

a.) Determine all three initial currents in the circuit (i.e., the currents just after the switch is closed).

b.) Determine all three currents in the circuit after a long period of time (i.e., at the theoretical point t = ∞).

c.) Without solving them, write out the equations you would need to solve if you wanted to determine the currents in the circuit at any arbitrary point in time. Be sure you are complete.

d.) Determine the total charge the 6 μf capacitor will accumulate (i.e., the amount of charge on its plates at t = ∞).

e.) Once totally charged, how much energy do the capacitors hold?

f.) After a very long time (i.e., long after the capacitors have fully charged), the switch is opened. How long will it take for the two capacitors to dump 87% of their charge across the 30 Ω resistor?

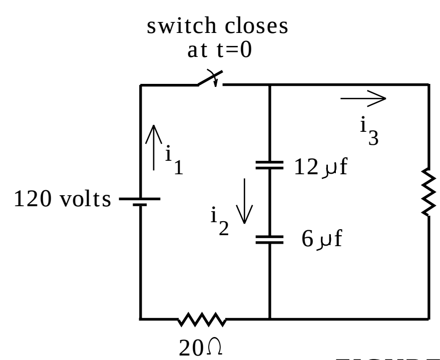
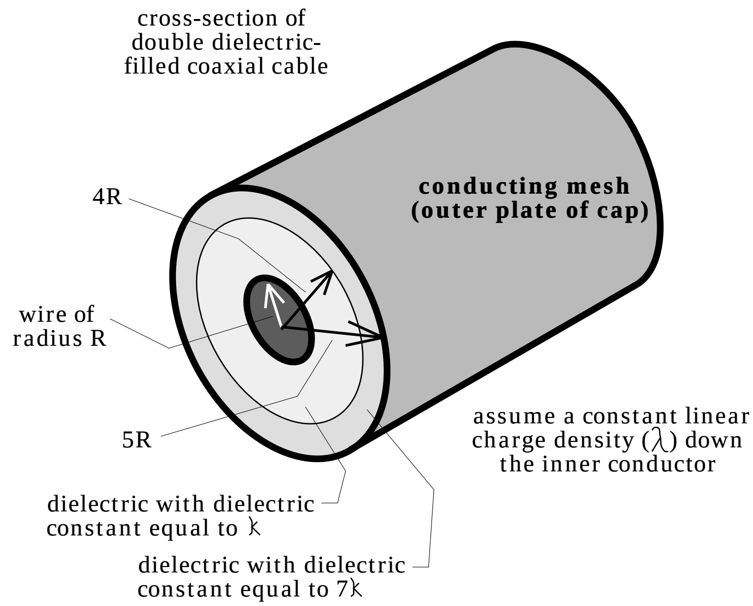


FIGURE IV

17.6) A very long, cylindrical, coaxial cable has a certain amount of capacitance per unit length associated with it (the sketch to the right shows a cross-section of the cable). The cable is made up of a wire of radius R (this acts like the inner plate of the capacitor) and an outer, cylindrical, conducting mesh of radius 5R (this mesh looks like a metal pipe and acts as the outer plate of the capacitor). Additionally, between R and 4R there is a dielectric whose dielectric constant is κ, and between 4R and 5R there is a second dielectric whose dielectric constant is 7κ. Derive from scratch an expression for the capacitance per unit length for this cable.



17.7) THE PROBLEM THAT WASN'T (Hint: You haven't done anything with spherical capacitors . . . that should make you wonder!).