

Chapter 15

ELECTRIC POTENTIALS and ENERGY CONSIDERATIONS

A.) Energy Considerations and the Absolute Electrical Potential:

1.) Consider the following scenario: A single, fixed, point charge Q sits in space with nothing around it. A second positive charge q is brought in a distance r units from the first charge. Once in position, q is released and, being completely free, accelerates.

a.) It should be obvious that q 's acceleration is due to its interaction with the electric field generated by the fixed charge. Although there is nothing wrong with this interpretation, there is another way we could look at the situation.

b.) Considering the idea of energy: On the assumption that static electric fields (i.e., electric fields that do not change with time) are associated with conservative forces (in fact, they are), we could claim that q 's acceleration was the consequence of its converting electrical POTENTIAL ENERGY afforded it by its presence in Q 's electric field to KINETIC ENERGY.

c.) As we have seen in previous chapters, the potential energy function for a particular force field is a mathematical contrivance designed to allow us to keep track of the amount of kinetic energy a body can potentially pick up if allowed to freefall in the force field.

i.) Put another way, if you know how much potential energy a body has when at two different points in a force field, you can determine how much work the field does as the body moves from the one point to the other. Knowing the net amount of work done, you can use the work/energy theorem to determine the body's change of kinetic energy.

d.) In the case of gravitational force fields, the amount of gravitational potential energy a body has is dependent upon the object's

mass ($U_g = mgh_1$). In the case of electric forces, the amount of electrical potential energy an object has depends upon the object's charge. The larger the charge, the more electrical potential energy the body will have.

e.) The electric field is a vector whose magnitude tells us the force per unit charge available at a given point in the field. Would it not be useful to define a similar quantity related to the concept of electrical potential energy?

f.) Just such a quantity has, indeed, been defined. It is called the absolute electrical potential of a point-of-interest due to the presence of an electric-field. It is a SCALAR quantity that tells you how much potential energy per unit charge there is available at a particular point in an electric field.

2.) Mathematically, the absolute electrical potential V_a at a Point A in space is defined as:

$$V_A = (U_A/q),$$

where U_A is the amount of potential energy a charge q will have if placed at Point A.

a.) The units for the absolute electrical potential are "energy-per-charge-unit", or joules per coulomb. This unit is given a special name: it is called a VOLT.

b.) The absolute electrical potential at an arbitrary Point A is sometimes referred to as "the absolute voltage at Point A."

c.) Example: A +4 coulomb charge is found to have 8000 joules of potential energy when placed at a particular point in an electric field. What is the absolute voltage (the absolute electrical potential) at that point?

i.) From the definition:

$$\begin{aligned} V_A &= (U/q)_A \\ &= (8000 \text{ j})/(4 \text{ C}) \\ &= 2000 \text{ joules/C} \\ &= 2000 \text{ volts.} \end{aligned}$$

Note: If you know V_A and want to know the amount of potential energy a charge q has when at A (i.e., U_A), the manipulated expression $U_A = qV_A$ will do the job. Example: A 2×10^{-12} coulomb charge is placed in an electric field at a point whose electrical potential is 2500 volts. How much potential energy will the charge have?

Solution: $U_A = qV_A = (2 \times 10^{-12} \text{ C})(2500 \text{ volts}) = 5 \times 10^{-9} \text{ joules}$.

3.) Consider a positive charge placed and released at Point A in an electric field (the field lines are shown in Figure 15.1). The charge will accelerate, picking up kinetic energy as it does, and sooner or later finds itself at Point B. The following observations can be made about Points A and B.

a.) Just as massive objects in a gravitational field naturally freefall from higher to lower potential energy, positive charges accelerate in electric fields from higher voltage points to lower voltage points. In other words, V_A must be greater than V_B .

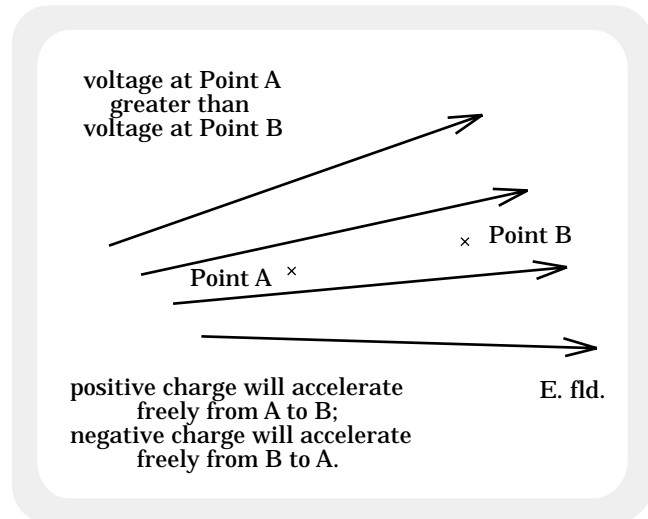


FIGURE 15.1

Note 1: With mass, one never has to worry about objects freefalling upward from a lower potential energy level to a higher potential energy level. After all, there is only one kind of mass and the earth's gravity always pulls it downward. Unfortunately, there are two kinds of charges--positive and negative ones--each of which reacts exactly the opposite to the other when put in an electric field. Our theory has been set up on the assumption that positive charges accelerate in the direction of electric field vectors, which means positive charges accelerate from higher to lower voltages. That means negative charges will do exactly the opposite. (To see this, think about the direction an electron will accelerate if put at the lower voltage, Point B).

Note 2: It does not matter whether the electric field lines are getting closer together or further apart (that is, whether the electric field intensity is getting larger or smaller)--the absolute voltage becomes less as you proceed in the direction of electric field lines.

B.) Work and Voltage Differences:

1.) The amount of work W done by a conservative force field on a body that moves from Point A to Point B in the field is intimately related to the body's change of potential energy. That is:

$$W = -\Delta U.$$

2.) In an electric field, the amount of work per unit charge done on a charged body as it moves from Point A to Point B in the field equals:

$$\begin{aligned} W/q &= -\Delta U/q \\ &= -(U_B - U_A)/q \\ &= -[(U_B/q) - (U_A/q)] \\ &= -(V_B - V_A) \\ \Rightarrow W/q &= -\Delta V. \end{aligned}$$

a.) ΔV is formally called the electrical potential difference between Points A and B. In everyday usage, it is sometimes referred to as "the voltage difference between Points A and B."

Note: Although in some texts there is very little notational delineation between the two, there is a BIG difference between a field's voltage at a point and a voltage difference between two points. In this text, an absolute electrical potential will either be subscripted to designate the point to which it is attached or made evident through the context of the situation. The Δ will occasionally be omitted when referring to a voltage difference, most notably when dealing with a power supply like a battery in an electric circuit.

When in doubt, look to the context of the problem. If the voltage value is attached to a particular point, it is an absolute electrical potential (i.e., an absolute voltage). If the voltage value is related to a voltage change between two points, you are looking at an electrical potential difference (i.e., a voltage difference).

Example: A 6 volt battery has a 6 volt electrical potential difference between the + and - terminals. As such, the number tells you how much work per unit charge the battery will do on charge-carriers as they travel through a wire from one terminal to the other.

C.) Electrical Potentials and Constant Electric Fields:

1.) For constant electric fields, we can calculate the work per unit charge as:

$$W/q = (\mathbf{F}_e \cdot \mathbf{d})/q,$$

where F_e is the force on the charge due to the electric field E , and d is a displacement vector defining the separation between the relative orientation of the Points A and B (see Figure 15.2).

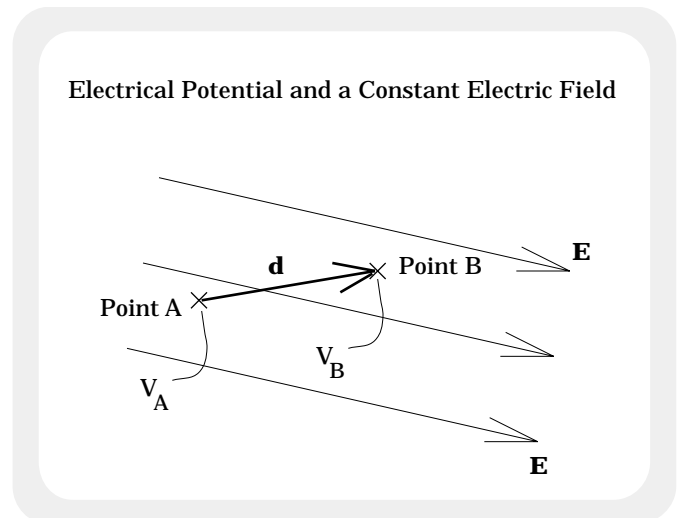


FIGURE 15.2

a.) Rearranging and manipulating yields:

$$\begin{aligned} W/q &= (\mathbf{F}_e/q) \cdot \mathbf{d} \\ &= \mathbf{E} \cdot \mathbf{d}. \end{aligned}$$

b.) Combining this result with $W/q = -\Delta V$ we get:

$$\mathbf{E} \cdot \mathbf{d} = -\Delta V.$$

c.) As can be seen, this relationship links the electrical potential difference between two points in a constant electric field to:

- i.) The electric field vector, and;
- ii.) A displacement vector between the two points-in-question.
- iii.) This is a very powerful, useful expression that will come in handy. UNDERSTAND IT WELL!

2.) Consequences of $\mathbf{E} \cdot \mathbf{d} = -\Delta V$: Consider the electric field lines shown in Figure 15.3 on the next page (ignore gravity). Given that $V_A = 12$ volts, $V_B = 3$ volts, and the distance between Points A and B, Points A and C, and Points A and D is 2 meters:

a.) Determine the magnitude of the electric field:

i.) Moving from A to B along the E field lines, we can write:

$$E \cdot d = -\Delta V$$

$$|E| |d_{AB}| \cos \xi = -(V_B - V_A),$$

where ξ is the angle between the line of E and the line of d_{AB} .

ii.) We know d_{AB} , θ , and the voltages at Points A and B. All we do not know is the magnitude of the electric field. Solving for that quantity yields:

$$E = -(V_B - V_A)/(d_{AB} \cos \theta)$$

$$= -(3 \text{ v} - 12 \text{ v})/[(2 \text{ m})(\cos 0^\circ)]$$

$$= 4.5 \text{ volts/meter.}$$

Note: A volt per meter is equal to a newton per coulomb.

b.) Determine voltage V_C :

i.) Once again, we know:

$$E \cdot d = -\Delta V$$

$$|E| |d_{AC}| \cos \theta = -(V_C - V_A).$$

ii.) Looking at Figure 15.4, $\theta = 120^\circ$ (θ is supposed to be the angle between the line of E and the line of d_{AC} , where d_{AC} is a vector directed

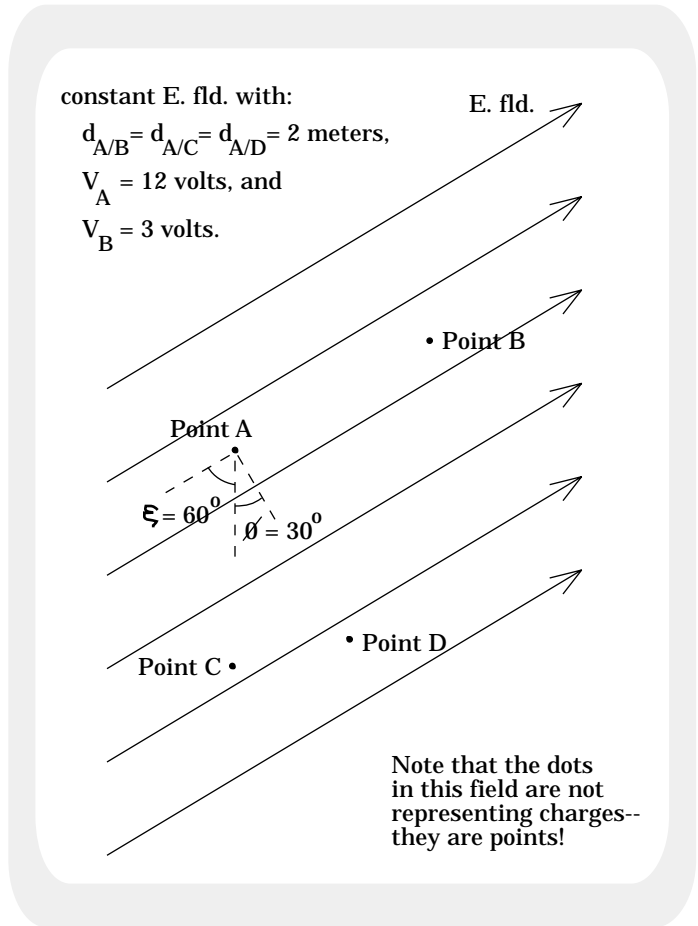


FIGURE 15.3

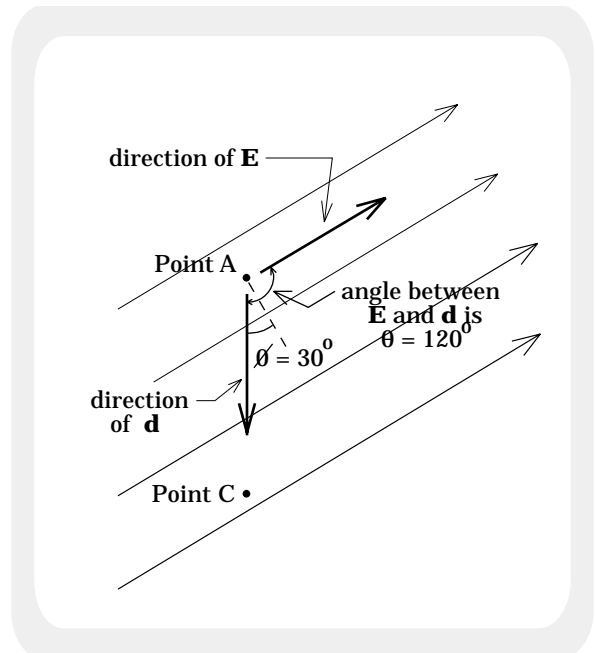


FIGURE 15.4

from the starting Point A to the finishing Point C). Plugging in the known values and solving, we get:

$$\begin{aligned} V_C &= V_A - E d_{AC} \cos \theta \\ &= (12 \text{ v}) - (4.5 \text{ nt/C})(2 \text{ m}) \cos 120^\circ \\ &= 16.5 \text{ volts.} \end{aligned}$$

Note: The fact that the voltage at Point C is larger than the voltage at Point A makes sense: the voltage should be larger "upstream," so to speak, in the electric field.

c.) Determine V_D :

i.) Moving from A to D, we have:

$$\begin{aligned} E \cdot d &= -\Delta V \\ |E| |d_{AD}| \cos \theta &= -(V_D - V_A). \end{aligned}$$

ii.) Noting that the angle between the line of E and the line of d_{AD} is $\theta = 90^\circ$ (although this is not explicitly stated in Figure 15.3, this should nevertheless be obvious from looking at the sketch), we solve for V_D :

$$\begin{aligned} V_D &= V_A - E d_{AD} \cos \theta \\ &= (12 \text{ volts}) - (4.5 \text{ volts/m})(2 \text{ m}) \cos 90^\circ \\ &= 12 \text{ volts.} \end{aligned}$$

Important Note 1: The line between Points A and D is perpendicular to the line of the electric field. That means the dot product $E \cdot d$ is zero (the dot product of vectors perpendicular to one another will always be zero), which in turn means ΔV is zero. If the change of the electrical potential between two points is zero, the voltage at each point must be the same. That is exactly what we have here: $V_A = V_D = 12$ volts.

Important Note 2: There is a group of points each of which has a voltage of 12 volts. Connecting those points defines an equipotential line. Equipotential lines are always PERPENDICULAR to electric field lines (in three dimensional situations, you can have whole equipotential surfaces).

Previously, we determined the electric field lines generated by two equal, positive charges. Equipotential lines for that situation are shown in Figure 15.5 to the right.

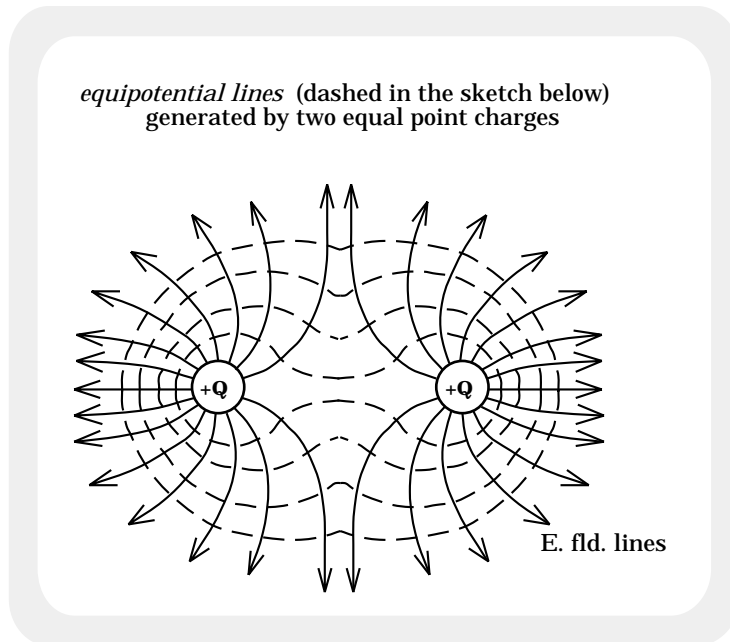


FIGURE 15.5

D.) The Electrical Potential of a Point Charge:

1.) Just as it was possible to derive a general algebraic expression for the size of the electric field created by a SINGLE POINT CHARGE Q , it is also possible to derive a general algebraic expression for the absolute electrical potential (i.e., the amount of potential energy per unit charge) generated by a field-producing point-charge Q as a function of the distance r from the charge.

2.) The general relationship between a constant electric field and its electrical potential function was previously derived as:

$$\Delta V = -E \cdot d \quad (\text{Equation 1}),$$

where $\Delta V = V_B - V_A$ is the electrical potential difference between two points A and B in the electric field, E is the electric field vector, and d is a vector beginning at Point A and ending at Point B. Figure 15.6 re-caps the set-up.

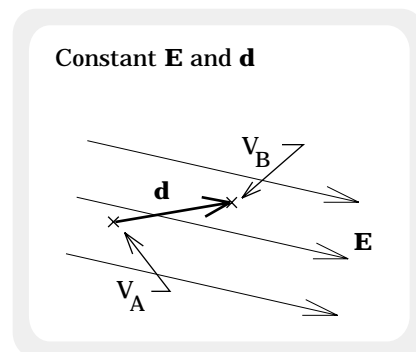


FIGURE 15.6

3.) How does this expression change when either the field or the path varies?

a.) The W/q done as a test charge moves from Point A to Point B will always equal the change of electrical potential between the two points. As such, the left-hand side of Equation 1 will not change at all.

b.) The $E \cdot d$ term is really $(F/q) \cdot d$ (i.e., the work-- $F \cdot d$ --per unit charge q). If F varies, d varies, or the angle between F and d varies, the work done along one section of the path will not be the same as the work done along another section.

c.) To deal with this problem:

i.) Define a differential path length dr along an arbitrarily selected section of the path (see Figure 15.7);

ii.) Determine the amount of work per unit charge (i.e., $E \cdot dr$) done on a charge moving over dr ;

iii.) Then sum over the entire path using integration (i.e., $\int E \cdot dr$).

d.) Summarizing:

$$\Delta V = -E \cdot d$$

becomes:

$$\Delta V = -\int \mathbf{E} \cdot d\mathbf{r},$$

where $d\mathbf{r}$ is an arbitrary, differential vector along the path.

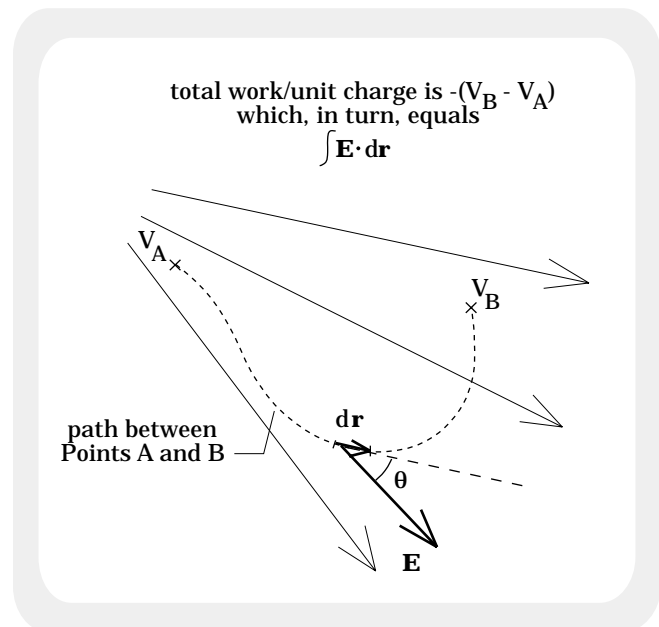


FIGURE 15.7

4.) With all this in mind, we want to derive a general algebraic expression for the electrical potential a distance r units from a point charge Q .

a.) To begin with, we must decide where the electrical potential should be zero. A potential energy function is traditionally defined as zero where its associated force function is zero. As an electrical potential is nothing more than a modified potential energy function, we will define its zero where its associated electrical field is zero. In the case of a point charge, that is at infinity.

b.) Having made that decision, we must determine the amount of work per unit charge available due to the electric field-produced by a field-producing point charge as we move from infinity to a point r units from the point charge (see Figure 15.8).

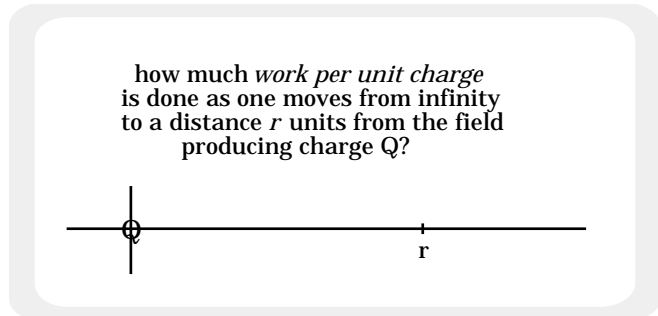


FIGURE 15.8

c.) Noting that the electrical potential is ZERO at infinity, (i.e., that $V(\infty) = 0$)

and the electric field function for a point charge is $\left[\frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \mathbf{r} \right]$ (where \mathbf{r} is a unit vector in the radial direction), the math yields:

$$\begin{aligned} \Delta V &= -\int \mathbf{E} \cdot d\mathbf{r} \\ \Rightarrow V(r) - V(\infty) &= -\int_{r=\infty}^r \left[\frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \mathbf{r} \right] \cdot [(dr)\mathbf{r}] \\ \Rightarrow V(r) &= -\frac{Q}{4\pi\epsilon_0} \int_{r=\infty}^r \left[\frac{1}{r^2} \right] dr \\ &= -\frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{r} \right]_{r=\infty}^r \\ &= \frac{1}{4\pi\epsilon_0} \frac{Q}{r}. \end{aligned}$$

5.) There are a couple of things to notice about this expression:

a.) The equation looks something like the magnitude of the electric field equation for a point charge (i.e., it looks like kQ/r^2). The resem-

blance is superficial. Electrical potentials are energy-related scalar quantities. That means they can be negative, but being so has nothing to do with direction.

Note: Gravitational potential energy is meaningful only in the context of determining the work done by a conservative gravitational force as a body moves from one point to another in the field. It is perfectly acceptable for a potential energy value to be negative at some point--all that HAS TO BE TRUE is that $W_{\text{fld.}} = -\Delta U$. The same is true of electrical potentials. It makes no difference what their values are, positive or negative, just as long as the amount of work per unit charge (W/q) available to a charge as it moves through a given electric field equals minus the change in electrical potential (i.e., minus the voltage change, or $-\Delta V$) between the beginning and end points of the motion.

b.) $V_{\text{at A due to pt.chg. } Q} = kQ/r$ works equally well for positive or negative charges. But unlike the electric field equation in which charge quantities are always inserted as magnitudes, the sign of the charge must be included when solving for the electrical potential of a point charge.

In other words, a positive point charge will produce a positive electrical potential and a negative point charge will produce a negative electrical potential.

Note: The variable r tells you the distance between the point of interest and the point charge. When used in the electrical potential equation for a point charge, this value must always be positive. Why? A charge at the origin of a coordinate axis is going to provide as much potential energy per unit charge at $x = 2$ meters as it will at $x = -2$ meters.

E.) Electrical Potential of a System of Point Charges:

1.) A group of point charges will produce some net electric potential field in the space around them. That field will be the SCALAR SUM of the electrical potentials generated by the individual point charges involved in the configuration.

2.) Example: A 3 microcoulomb (μC) charge is placed at $x = -3$ meters and a $-3 \mu\text{C}$ charge is placed at $x = 3$ meters (see Figure 15.9 to the right).

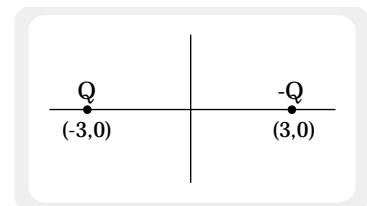


FIGURE 15.9

a.) What is the net electrical potential at $x = 0$?

$$\begin{aligned}V_o &= V_{\text{due to } Q_1} + V_{\text{due to } Q_2} \\&= k Q_1/r_1 + k Q_2/r_2 \\&= (9 \times 10^9)(3 \times 10^{-6} \text{ C})/(3 \text{ m}) + (9 \times 10^9)(-3 \times 10^{-6} \text{ C})/(3 \text{ m}) \\&= 0.\end{aligned}$$

Note: At first glance, this probably seems bizarre. After all, if you released a positive charge at $x = 0$, it will certainly accelerate toward the negative charge. That is to say, it will certainly have potential energy.

The response to this is, "Not necessarily." If you put a mass 1 meter above the ground, you could define 1 meter above the ground to be the zero potential energy level. Let the mass loose and it accelerates even though its potential energy has been defined as zero.

Potential energy is meaningful only in the context of potential energy changes. Only then are they related to the work done by the field as the body moves through it. The same is true of electrical potentials. You can have zero electrical potential at some spot as long as the electrical potential downstream (i.e., further down along the electric field lines) is even lower (in this case it would have to be negative).

b.) What is the net electric potential at $x = +1$ meter?

$$\begin{aligned}V_1 &= V_{\text{due to } Q_1} + V_{\text{due to } Q_2} \\&= k Q_1/r_1 + k Q_2/r_2 \\&= (9 \times 10^9)(3 \times 10^{-6} \text{ C})/(4 \text{ m}) + (9 \times 10^9)(-3 \times 10^{-6} \text{ C})/(2 \text{ m}) \\&= -6.75 \times 10^3 \text{ volts.}\end{aligned}$$

Note: The electric field along the x -axis between the charges is to the right. Notice that makes $x = 1$ "downstream" relative to the origin. Notice also that the electrical potential at $x = 1$ is less than at the origin, just as was predicted in the Note above.

F.) Deriving the Electrical Potential Function for an Extended, Charged Object Using a Differential Charge Approach:

1.) Determine a general expression for the electrical potential along the central axis of a hoop of radius R and upon which resides a net charge Q .

a.) Define a differential amount of charge dq on the hoop. Also, define the distance between the differential charge and an arbitrary point $(x, 0)$ on the central axis (all of this is shown in Figure 15.10).

b.) The differential electrical potential dV at $(x, 0)$ due to the differential charge dq is:

$$\begin{aligned} dV &= \frac{1}{4\pi\epsilon_0} \frac{dq}{r} \\ &= \frac{1}{4\pi\epsilon_0} \frac{dq}{(R^2 + x^2)^{1/2}}. \end{aligned}$$

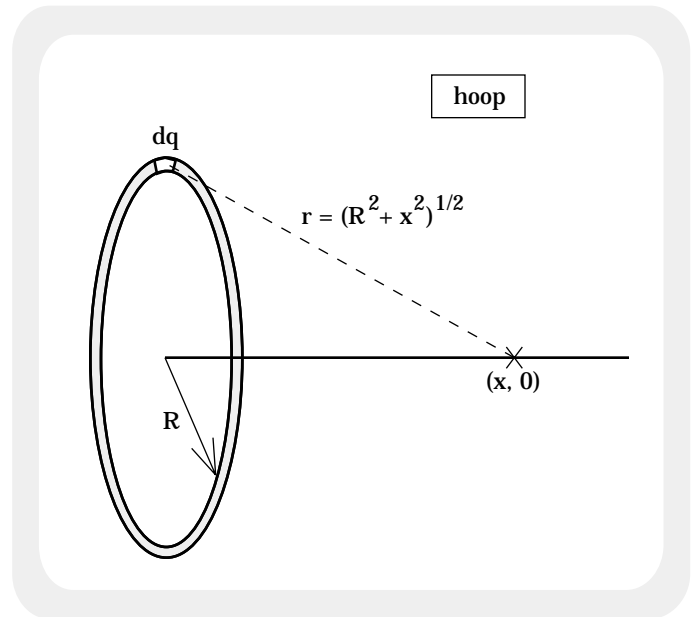


FIGURE 15.10

c.) The only variable in this expression is dq (all else is fixed in the problem--even x has been defined as a specific coordinate). As such, we can pull out the constants and integrate, yielding:

$$\begin{aligned} V &= \int dV \\ &= \int \frac{dq}{4\pi\epsilon_0 (R^2 + x^2)^{1/2}} \\ &= \frac{1}{4\pi\epsilon_0 (R^2 + x^2)^{1/2}} \int dq \\ &= \frac{Q}{4\pi\epsilon_0 (R^2 + x^2)^{1/2}}. \end{aligned}$$

2.) Consider now a flat disk of radius R with a net charge Q on its surface. What is the electrical potential at an arbitrary point $(x, 0)$ along its central axis (see Figure 15.11 on the next page)?

a.) If we assume a differential charge dq resides within a hoop of radius a and thickness da , we can use the expression derived above for the electrical potential of a hoop by substituting dq for Q and a for R . Doing so yields a differential electrical potential dV of:

$$dV = \frac{1}{4\pi\epsilon_0} \frac{dq}{(a^2 + x^2)^{1/2}}.$$

b.) The problem?
The variable dq varies as a varies. To accommodate, we have to express dq in terms of a .

c.) A disk with charge Q uniformly distributed over its surface will have a surface charge density σ equal to:

$$\sigma = Q/(\pi R^2).$$

d.) Knowing the radius a and differential thickness da of the hoop, and having defined σ , we can write:

$$dq = \sigma dA,$$

where dA is the differential area of the hoop. This differential area will be the product of the hoop's circumference and thickness, or:

$$dA = (2\pi a)da.$$

e.) Putting it all together, we get:

$$\begin{aligned} dq &= \sigma dA \\ &= \left[\frac{Q}{\pi R^2} \right] [(2\pi a)da] \\ &= \left[\frac{2Q}{R^2} \right] (a)da \\ \Rightarrow dV &= \frac{1}{4\pi\epsilon_0} \frac{\left[\frac{2Q}{R^2} \right] (a)da}{(a^2 + x^2)^{1/2}} \\ &= \frac{Q}{2\pi\epsilon_0 R^2} \frac{(a)}{(a^2 + x^2)^{1/2}} da. \end{aligned}$$

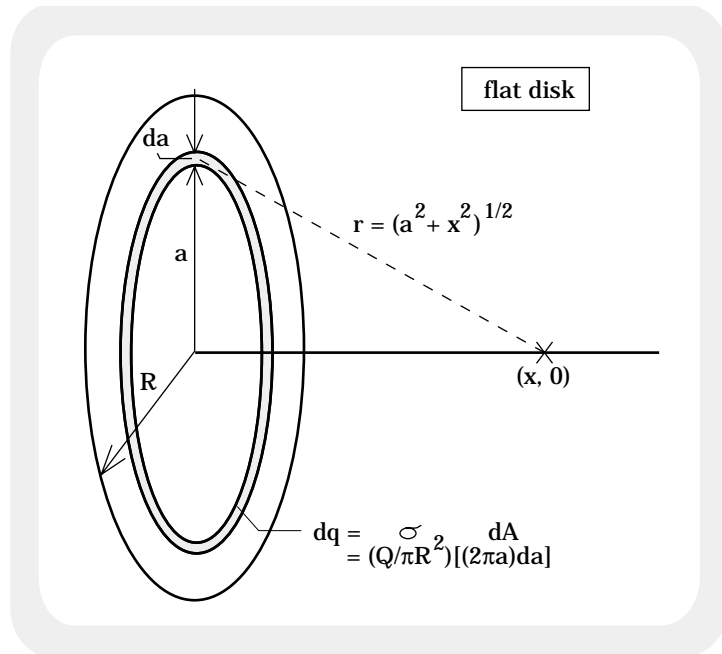


FIGURE 15.11

f.) With this, the net electrical potential for the entire disk along its central axis becomes:

$$\begin{aligned}
 V &= \int dV \\
 &= \frac{Q}{2\pi\epsilon_0 R^2} \int_{a=0}^R \frac{(a)}{(a^2 + x^2)^{1/2}} da \\
 &= \frac{Q}{2\pi\epsilon_0 R^2} \left[(a^2 + x^2)^{1/2} \right]_{a=0}^R \\
 &= \frac{Q}{2\pi\epsilon_0 R^2} \left[(R^2 + x^2)^{1/2} - x \right].
 \end{aligned}$$

G.) More Fun With Extended, Charged Objects:

1.) So far, we have dealt with extended objects by determining the differential electrical potential due to a differential point charge on the object, then by integrating to determine the total electrical potential due to all charges on the object.

Continuing on in this fashion, consider a rod of length $2L$ with a total charge Q distributed uniformly upon its surface (see Figure 15.12). Determine the electrical potential at point $(b, 0)$ on the axis.

2.) We must begin by defining a differential charge dq located some distance a units up the vertical axis. Assuming the section length upon which dq resides has a differential length da , and defining λ as the charge per unit length on the rod, we can write the differential charge as $dq = (\lambda)da$.

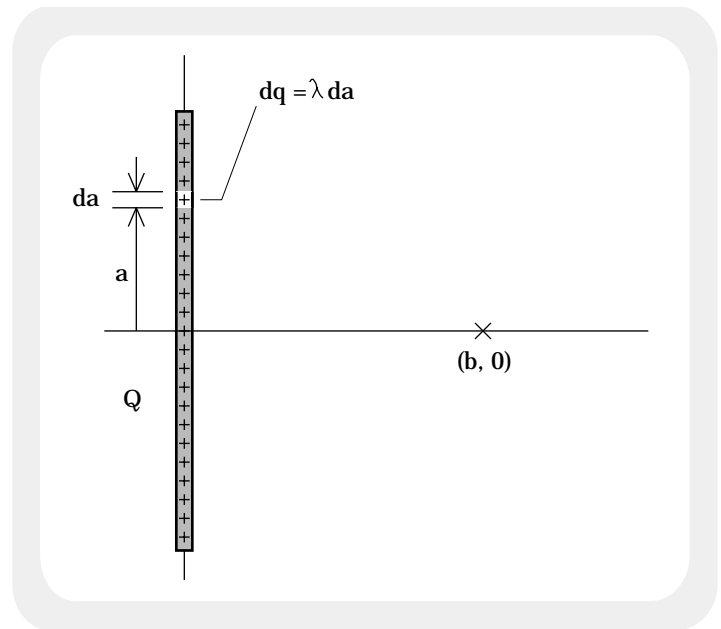


FIGURE 15.12

Note: The charge per unit length in this case will equal $Q/2L$.

3.) To determine the electrical potential at (b, 0):

a.) The electrical potential due to a point charge is:

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r},$$

where the charge q is written with the sign of its charge intact and r is the distance between the field-producing charge q and the point of interest.

b.) In our case, the point charge is the differential charge dq . Additionally, $r = (a^2 + b^2)^{1/2}$.

c.) Putting it all together, we can write the differential electrical potential at (b, 0) as:

$$dV = \frac{1}{4\pi\epsilon_0} \frac{dq}{(a^2 + b^2)^{1/2}}.$$

d.) Due to the symmetry of the problem, we can integrate from $a = 0$ to $a = L$, then double that result. Substituting in for dq and λ and doing this integration, the net electrical potential at (b, 0) is:

$$\begin{aligned} V &= \int dV \\ &= \int \left[\frac{1}{4\pi\epsilon_0} \frac{dq}{(a^2 + b^2)^{1/2}} \right] \\ &= (2) \frac{1}{4\pi\epsilon_0} \int_{a=0}^L \frac{(\lambda)da}{(a^2 + b^2)^{1/2}} \\ &= \frac{1}{2\pi\epsilon_0} \int_{a=0}^L \frac{(Q/2L)}{(a^2 + b^2)^{1/2}} da \\ &= \frac{Q}{4\pi\epsilon_0 L} \int_{a=0}^L \left[\frac{1}{(a^2 + b^2)^{1/2}} \right] da. \end{aligned}$$

Note: You may need a book of integrals to solve this, depending upon how much experience you have had with Calculus. Continuing:

$$\begin{aligned}
&= \frac{Q}{4\pi\epsilon_0 L} \left[\ln \left[a + (a^2 + b^2)^{1/2} \right] \right]_{a=0}^L \\
&= \frac{Q}{4\pi\epsilon_0 L} \left[\ln \left[L + (L^2 + b^2)^{1/2} \right] - \ln \left[0 + (0^2 + b^2)^{1/2} \right] \right] \\
&= \frac{Q}{4\pi\epsilon_0 L} \ln \left[\frac{L + (L^2 + b^2)^{1/2}}{b} \right].
\end{aligned}$$

H.) Absolute Electrical Potentials for a Spherically Symmetric Charge Configuration:

1.) There are times when it is not convenient to define a differential bit of charge, determine dV for that charge at a point of interest, then integrate to determine the net electrical potential at the point. In such cases, we must revert to a more fundamental approach.

2.) How did we derive the general expression for the absolute electrical potential of a point charge back in Section D-3-c? The procedure is presented below:

a.) We calculated the amount of work per unit charge (i.e., $\int \mathbf{E} \cdot d\mathbf{r}$) available as we moved from infinity to some arbitrary point r in the point charge's field.

b.) We noticed that $W/q = -\Delta U/q = -\Delta V = -[V(r) - V(\infty)]$, where $V(\infty)$ was defined as zero.

c.) We combined Part a and Part b to generate the equation:

$$V(r) = - \int_{r=\infty}^r \mathbf{E}_{\text{pt. chg.}} \cdot d\mathbf{r},$$

where $E_{\text{pt. chg.}} = Q/(4\pi\epsilon_0 r^2)$.

Note: If this isn't clear, look back at the section in which this derivation was actually done.

3.) The procedure outlined above was used to determine the electrical potential at some point r units from a point charge, but the procedure itself can be used on any charge configuration.

4.) To see this, consider the following example: A thick spherical shell of inside radius R_1 and outside radius R_2 (see Figure 15.13) is shot full of charge such that its volume charge density is $\rho = ka$, where a is an arbitrary distance from the center of the sphere to a point inside the shell, and k has a magnitude of one with the appropriate units. Determine the electrical potential function $V(r)$ for $r > R_2$, $R_2 > r > R_1$, and for $R_1 > r$.

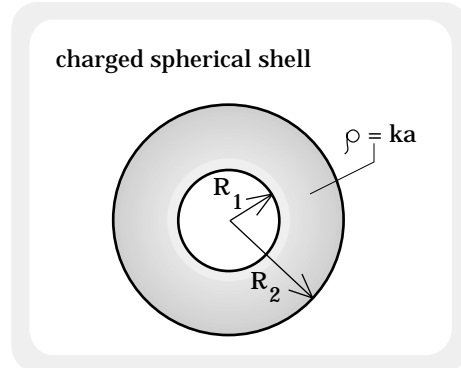


FIGURE 15.13

5.) For $r > R_2$:

a.) What is the first thing that has to be done? Decide where the electrical potential is to be zero. In this case, the electric field generated by the charge configuration is zero at infinity, so that is where we will define our zero electrical potential point.

b.) We want to use the relationship:

$$\Delta V = -\int \mathbf{E} \cdot d\mathbf{r}.$$

c.) By setting that expression into the context of our problem--that is, by evaluating it between the zero electrical potential point and an arbitrary point r , we can write:

$$V(r) - V(\infty) = -\int_{r=\infty}^r \mathbf{E} \cdot d\mathbf{r}.$$

Note: Why is this desirable? Because by our own definition, $V(\infty) = 0$. That means that the evaluation of the integral will leave us with a general expression for $V(r)$, which is exactly what we want.

d.) Minor problem: To do this integral, we need to have a general expression for the electric field \mathbf{E} IN THE REGION OVER WHICH THE INTEGRAL IS TO BE TAKEN. To get that, we will use GAUSS'S LAW. Defining a spherical Gaussian surface whose radius is $r > R_2$, we get:

$$\begin{aligned}
\int_S \mathbf{E}_{\text{outside}} \cdot d\mathbf{S} &= \frac{\int \rho dV}{\epsilon_0} \\
\Rightarrow \mathbf{E}_{\text{out}} (4\pi r^2) &= \frac{\int_{a=R_1}^{R_2} (ka) [(4\pi a^2) da]}{\epsilon_0} \\
\Rightarrow \mathbf{E}_{\text{out}} &= \frac{4\pi k}{4\pi \epsilon_0 r^2} \int_{a=R_1}^{R_2} (a^3) da \\
&= \frac{k(R_2^4 - R_1^4)}{4\epsilon_0} \left(\frac{1}{r^2} \right).
\end{aligned}$$

e.) Knowing the electric field in the region outside the sphere, we can now proceed to determine the electrical potential function for this charge configuration:

$$\begin{aligned}
V(\mathbf{r}) - V(\infty) &= -\int_{r=\infty}^r \left[\frac{k(R_2^4 - R_1^4)}{4\epsilon_0} \left(\frac{1}{r^2} \right) \mathbf{r} \right] \cdot [(d\mathbf{r})\mathbf{r}] \\
\Rightarrow V(\mathbf{r}) &= -\frac{k(R_2^4 - R_1^4)}{4\epsilon_0} \int_{r=\infty}^r \left(\frac{1}{r^2} \right) dr \\
&= -\frac{k(R_2^4 - R_1^4)}{4\epsilon_0} \left[-\frac{1}{r} \right]_{r=\infty}^r \\
&= -\frac{k(R_2^4 - R_1^4)}{4\epsilon_0} \left[\left(-\frac{1}{r} \right) - \left(-\frac{1}{\infty} \right) \right] \\
&= \frac{k(R_2^4 - R_1^4)}{4\epsilon_0 r}.
\end{aligned}$$

6.) For $R_2 > r > R_1$:

a.) This is considerably more interesting because it introduces a problem with which we have not yet had to cope. We want to determine ΔV between infinity and a point r units from the sphere's center, where r is inside the sphere. The difficulty is that the electric field function inside the sphere is different from the electric field function outside the sphere. So what E do we use in $-\int \mathbf{E} \cdot d\mathbf{r}$?

b.) We must use E for the region in which it is applicable. That means:

i.) To determine the voltage difference between $r = \infty$ and $r = R_2$ (this will be $V(R_2) - V(\infty)$), we must use the electric field function derived for the field outside the shell (i.e., E_{outside});

ii.) To determine the voltage difference between $r = R_2$ and an arbitrary point r in between R_1 and R_2 (this will be $V(r) - V(R_2)$), we must derive an electric field function for the field inside the shell (call this E_{inside}).

iii.) The net electrical potential difference between $r = \infty$ (i.e., where the electrical potential is zero) and an arbitrary position r units from the center and inside the sphere will be:

$$[V(r) - V(R_2)] + [V(R_2) - V(\infty)] = V(r).$$

c.) To determine E_{in} , we must use Gauss's Law. Shown in truncated form below, this yields:

$$\begin{aligned} \int_S \mathbf{E}_{\text{inside}} \cdot d\mathbf{S} &= \frac{\int \rho dV}{\epsilon_0} \\ \Rightarrow E_{\text{in}} (4\pi r^2) &= \frac{\int_{a=R_1}^r (ka) [(4\pi a^2) da]}{\epsilon_0} \\ \Rightarrow E_{\text{in}} &= \frac{4\pi k}{4\pi \epsilon_0 r^2} \int_{a=R_1}^r (a^3) da \\ &= \frac{k(r^4 - R_1^4)}{4\epsilon_0 r^2} \\ &= \frac{k r^2}{4\epsilon_0} - \frac{k R_1^4}{4\epsilon_0 r^2}. \end{aligned}$$

d.) We can now use our approach to determine $V(r)$:

$$\begin{aligned}
V(r) &= [V(R_2) - V(\infty)] + [V(r) - V(R_2)] \\
&= \left[-\int_{r=\infty}^{R_2} \mathbf{E}_{\text{out}} \cdot d\mathbf{r} \right] + \left[-\int_{r=R_2}^r \mathbf{E}_{\text{in}} \cdot d\mathbf{r} \right] \\
&= \left[-\int_{r=\infty}^{R_2} \left[\frac{k(R_2^4 - R_1^4)}{4\epsilon_0} \left(\frac{1}{r^2} \right) \right] \mathbf{r} \cdot [(dr)\mathbf{r}] \right] + \left[-\int_{r=R_2}^r \left[\frac{k(r^2)}{4\epsilon_0} - \frac{k(R_2^4)}{4\epsilon_0 r^2} \right] \mathbf{r} \cdot [(dr)\mathbf{r}] \right] \\
&= \left[-\frac{k(R_2^4 - R_1^4)}{4\epsilon_0} \int_{r=\infty}^{R_2} \left(\frac{1}{r^2} \right) dr \right] + \frac{k}{4\epsilon_0} \left[-\int_{r=R_2}^r r^2 dr + R_2^4 \int_{r=R_2}^r \left(\frac{1}{r^2} \right) dr \right] \\
&= \left[-\frac{k(R_2^4 - R_1^4)}{4\epsilon_0} \left[-\frac{1}{r} \right]_{r=\infty}^{R_2} \right] + \frac{k}{4\epsilon_0} \left[-\left[\frac{r^3}{3} \right]_{r=R_2}^r + R_2^4 \left[-\frac{1}{r} \right]_{r=R_2}^r \right] \\
&= \left[-\frac{k(R_2^4 - R_1^4)}{4\epsilon_0} \left[\left(-\frac{1}{R_2} \right) - \left(-\frac{1}{\infty} \right) \right] \right] + \frac{k}{4\epsilon_0} \left[-\left[\frac{r^3}{3} - \frac{R_2^3}{3} \right] + R_2^4 \left[\left(-\frac{1}{r} \right) - \left(-\frac{1}{R_2} \right) \right] \right].
\end{aligned}$$

7.) For $R_1 > r$:

a.) Using the approach outlined above, we can write:

$$V(r) = [V(r) - V(R_1)] + [V(R_1) - V(R_2)] + [V(R_2) - V(\infty)].$$

That is, the voltage jump from infinity to R_2 plus the voltage jump between R_2 and R_1 plus the voltage jump from R_1 to r will give you the net voltage jump between infinity (where $V = 0$) and r inside the hollow.

b.) We already know the electric field function for the outer and inside-shell portions. What is the electric field function inside the hollow?

A Gaussian surface that resides inside the hollow will have no charge enclosed within its bounds. That means the electric field in that region must be zero.

c.) Zero electric field in a region means no electrical potential difference from point to point in the field. This, in turn, implies a constant electrical potential throughout the region.

d.) Electrical potential functions, being energy related, must be continuous functions (electric field functions, on the other hand, do not have to be continuous). As such, the electrical potential at the edge of the hollow will be the same as the electrical potential anywhere inside the hollow. Knowing that, we can write:

$$\begin{aligned}
V(\mathbf{r}) &= [V(R_2) - V(\infty)] + [V(R_1) - V(R_2)] + [V(r) - V(R_1)] \\
&= \left[-\int_{r=\infty}^{R_2} \mathbf{E}_{\text{out}} \cdot d\mathbf{r} \right] + \left[-\int_{r=R_2}^{R_1} \mathbf{E}_{\text{in}} \cdot d\mathbf{r} \right] + [0] \\
&= \left[-\int_{r=\infty}^{R_2} \left[\frac{k(R_2^4 - R_1^4)}{4\epsilon_0} \left(\frac{1}{r^2} \right) \right] \mathbf{r} \cdot [(d\mathbf{r})\mathbf{r}] \right] + \left[-\int_{r=R_2}^{R_1} \left[\frac{k(r^2)}{4\epsilon_0} - \frac{k(R_2^4)}{4\epsilon_0 r^2} \right] \mathbf{r} \cdot [(d\mathbf{r})\mathbf{r}] \right] \\
&= \left[-\frac{k(R_2^4 - R_1^4)}{4\epsilon_0} \left[\left(-\frac{1}{R_2} \right) - \left(-\frac{1}{\infty} \right) \right] \right] + \frac{k}{4\epsilon_0} \left[-\left[\frac{R_1^3}{3} - \frac{R_2^3}{3} \right] + R_2^4 \left[\left(-\frac{1}{R_1} \right) - \left(-\frac{1}{R_2} \right) \right] \right].
\end{aligned}$$

Note: This is important. The electrical potential will be the same at every point in a region in which the electric field is zero. **THIS INCLUDES REGIONS INSIDE CONDUCTORS.** Although you haven't seen a problem with a conductor in it yet, you will.

I.) Absolute Electrical Potential for a Cylindrically Symmetric Charge Configuration:

1.) The approach we have used for the determination of the electrical potential function of a complex charge configuration with spherical symmetry will also work for charge configurations that have cylindrical symmetry (with a twist that will become evident shortly). The following example will illustrate this.

2.) A conducting cylindrical shell of radius R_2 has a constant surface charge density σ on it (see Figure 15.14). Down its central axis runs a wire of radius R_1 whose linear charge density $-\lambda$ is constant. Determine a function for the absolute electrical potential at r , where $r < R_1$.

a.) Assuming the electrical potential is zero at infinity, we need to execute the following operation:

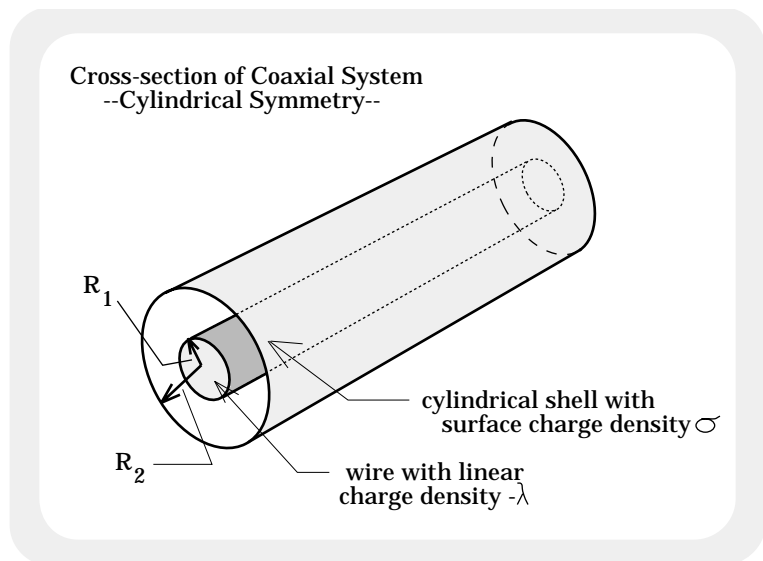


FIGURE 15.14

$$\begin{aligned}
 V(r) &= [V(r) - V(R_1)] + [V(R_1) - V(R_2)] + [V(R_2) - V(\infty)] \\
 &= \left[-\int_{R_1}^r \mathbf{E}_{\text{inner}} \cdot d\mathbf{r} \right] + \left[-\int_{R_2}^{R_1} \mathbf{E}_{\text{middle}} \cdot d\mathbf{r} \right] + \left[-\int_{\infty}^{R_2} \mathbf{E}_{\text{outer}} \cdot d\mathbf{r} \right].
 \end{aligned}$$

3.) To execute the operation, we need to use Gauss's Law to determine the various electric fields. Doing so yields:

a.) For $r < R_1$:

i.) Because we are dealing with a conducting wire, free charge on the wire distributes itself along the wire's outer edge leaving no charge inside that surface. As such, the electric field is zero.

b.) For $R_1 < r < R_2$:

i.) There is a negative charge per unit length along the central axis. If we define a Gaussian cylinder of radius $R_1 < r < R_2$ and length L , we get:

$$\begin{aligned}
 \int_S \mathbf{E}_{\text{middle}} \cdot d\mathbf{S} &= \frac{q_{\text{encl}}}{\epsilon_0} \\
 \Rightarrow E_{\text{middle}} (2\pi r L) &= \frac{-\lambda L}{\epsilon_0} \\
 \Rightarrow E_{\text{middle}} &= -\frac{\lambda}{2\pi\epsilon_0 r}.
 \end{aligned}$$

c.) For $r > R_2$:

i.) If we define a Gaussian cylinder of radius r ($r > R_2$) and length L , we get:

$$\begin{aligned}
 \int_S \mathbf{E}_{\text{out}} \cdot d\mathbf{S} &= \frac{q_{\text{encl}}}{\epsilon_0} \\
 \Rightarrow E_{\text{out}} (2\pi r L) &= \frac{-\lambda L + \sigma(2\pi R_2 L)}{\epsilon_0} \\
 \Rightarrow E_{\text{out}} &= -\frac{\lambda}{2\pi\epsilon_0 r} + \frac{\sigma R_2}{\epsilon_0 r}.
 \end{aligned}$$

4.) We are now ready to use our technique.

$$\begin{aligned}
 V(\mathbf{r}) &= [V(\mathbf{r}) - V(\mathbf{R}_1)] + [V(\mathbf{R}_1) - V(\mathbf{R}_2)] + [V(\mathbf{R}_2) - V(\infty)] \\
 &= \left[-\int_{r=R_1}^r \mathbf{E}_{\text{inner}} \cdot d\mathbf{r} \right] + \left[-\int_{r=R_2}^{R_1} \mathbf{E}_{\text{middle}} \cdot d\mathbf{r} \right] + \left[-\int_{r=\infty}^{R_2} \mathbf{E}_{\text{outer}} \cdot d\mathbf{r} \right] \\
 &= [0] + \left[-\int_{r=R_2}^{R_1} \left[-\frac{\lambda}{2\pi\epsilon_0} \frac{1}{r} \right] \mathbf{r} \cdot [(d\mathbf{r})\mathbf{r}] \right] + \left[-\int_{r=\infty}^{R_2} \left[-\frac{\lambda}{2\pi\epsilon_0} \frac{1}{r} + \frac{\sigma R_2}{\epsilon_0} \frac{1}{r} \right] \mathbf{r} \cdot [(d\mathbf{r})\mathbf{r}] \right] \\
 &= [0] + \left[\frac{\lambda}{2\pi\epsilon_0} \int_{r=R_2}^{R_1} \frac{1}{r} dr \right] + \left[\left(\frac{\lambda}{2\pi\epsilon_0} - \frac{\sigma R_2}{\epsilon_0} \right) \int_{r=\infty}^{R_2} \frac{1}{r} dr \right] \\
 &= [0] + \left[\frac{\lambda}{2\pi\epsilon_0} [\ln r]_{r=R_2}^{R_1} \right] + \left[\left(\frac{\lambda}{2\pi\epsilon_0} - \frac{\sigma R_2}{\epsilon_0} \right) [\ln r]_{r=\infty}^{R_2} \right] \\
 &= \left[\frac{\lambda}{2\pi\epsilon_0} [\ln R_1 - \ln R_2] \right] + \left[\left(\frac{\lambda}{2\pi\epsilon_0} - \frac{\sigma R_2}{\epsilon_0} \right) [\ln R_2 - \ln \infty] \right] \\
 &= \left[\frac{\lambda}{2\pi\epsilon_0} \ln \left(\frac{R_1}{R_2} \right) \right] + \left[\left(\frac{\lambda}{2\pi\epsilon_0} - \frac{\sigma R_2}{\epsilon_0} \right) \ln \left(\frac{R_2}{\infty} \right) \right].
 \end{aligned}$$

5.) We haven't done anything technically wrong, but we have come up with an expression that makes no sense ($\ln(R_2/\infty) = \ln(0) \dots$ this doesn't exist). What happened?

a.) The electric field function we derived using Gauss's Law (i.e., $E \propto 1/r$) is actually the field for an infinitely long wire. Such a wire would have an infinite amount of charge on it, which automatically puts it outside the realm of possibility. For a finite length wire, Gaussian symmetry works (i.e., \vec{E} is in a radial direction out from the wire) as long as one doesn't get too far from the wire. At infinity, the electric field expression isn't good.

If we had used the differential point charge approach to determine E , we would have ended up with an electric field expression that was good for all space around a finite wire. That function turns out to be $E \propto 1/r^3$. Using our approach with this electric field function would have yielded a general electrical potential function whose value at infinity would have been zero, but that wouldn't have exploded for finite r .

b.) Was this exercise a waste of time? Nope, not if it made you think about the approach used to derive $V(r)$ from E .

J.) Deriving an Electric Field Function from an Electrical Potential Function:

1.) So far, we have used the fact that the change of electrical potential is related to the area under an electric field versus position graph. That is, knowing the electric field for a charge configuration, we have determined V by being clever with the relationship $\Delta V = -\int \mathbf{E} \cdot d\mathbf{r}$.

We now want to go the other way, determining \mathbf{E} knowing something about its associated electrical potential function $V(\mathbf{r})$.

2.) Consider the relationship between the differential potential change dV , the electric field \mathbf{E} that produces the potential field, and a differential displacement $d\mathbf{r}$ over which the change occurs.

a.) Assuming \mathbf{E} and $d\mathbf{r}$ are in the same direction (let's assume it is in the x direction), we can write:

$$dV = -(\mathbf{E}_x)dx.$$

b.) It should be obvious from examination that if the above expression is correct, the x component of the electric field must equal:

$$\mathbf{E}_x = -dV/dx.$$

c.) Additionally, similar expressions should be true for the y and z directions.

3.) Put in words, the electric field component at a particular point in a particular direction equals the rate at which the electrical potential changes as one moves some differential distance in that direction.

Put another way, the electric field must equal the slope of the electrical potential versus position graph, evaluated at a point of interest.

4.) The electric field function is a vector. The operator that executes a rate of change with position calculation and makes the result into a vector is the del operator. In short,

$$\mathbf{E} = -\nabla V.$$

K.) Electric Fields and Point Charge Configurations:

1.) Take an easy example first. We know that the electrical potential for a positive point charge is:

$$V = \frac{q}{4\pi\epsilon_0 r},$$

where r is the distance between the field-producing charge q and the point of interest. What is the electric field function associated with this electrical potential function?

2.) We have a small problem here as the function is in radial symmetry. The del operator in spherical coordinates is (no, you needn't know this):

$$\nabla V = \left[\frac{\partial V}{\partial r} \mathbf{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \Theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \Phi \right].$$

a.) Using this with our potential function, we get:

$$\begin{aligned} \mathbf{E} &= -\nabla V \\ &= -\frac{q}{4\pi\epsilon_0} \left[\frac{\partial(1/r)}{\partial r} \mathbf{r} + \frac{1}{r} \frac{\partial(1/r)}{\partial \theta} \Theta + \frac{1}{r \sin \theta} \frac{\partial(1/r)}{\partial \phi} \Phi \right] \\ &= -\frac{q}{4\pi\epsilon_0} \left[\frac{\partial(1/r)}{\partial r} \mathbf{r} + 0\Theta + 0\Phi \right] \\ &= -\frac{q}{4\pi\epsilon_0} \left(-\frac{1}{r^2} \right) \mathbf{r} \\ &= \frac{q}{4\pi\epsilon_0 r^2} \mathbf{r}. \end{aligned}$$

b.) This is the most general form of the electric field due to a point charge, complete with radial symmetry. The problem? You are not supposed to know the del operator in spherical coordinates. To circumvent the difficulty, we could as well have assumed the point of interest was along the x -axis and used rectangular coordinates. Doing so would yield an electrical potential function of:

$$V = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{x} \right).$$

and an electric field expression of:

$$\begin{aligned}
 \mathbf{E} &= -\nabla V \\
 &= -\frac{q}{4\pi\epsilon_0} \left[\frac{\partial V}{\partial x} \mathbf{i} + \frac{\partial V}{\partial y} \mathbf{j} + \frac{\partial V}{\partial z} \mathbf{k} \right] \\
 &= -\frac{q}{4\pi\epsilon_0} \left[\frac{\partial(1/x)}{\partial x} \mathbf{i} + \frac{\partial(1/x)}{\partial y} \mathbf{j} + \frac{\partial(1/x)}{\partial z} \mathbf{k} \right] \\
 &= \frac{q}{4\pi\epsilon_0} \frac{1}{x^2} \mathbf{i}.
 \end{aligned}$$

c.) Again, this is the electric field function for a point charge (though in restricted form as \mathbf{E} for a point charge isn't solely a function of x).

3.) Consider the two equal point charges shown in Figure 15.15.

a.) The electrical potential at point $(x, 0)$ due to the top charge q is:

$$V = \frac{q}{4\pi\epsilon_0 r},$$

where $r = (a^2 + x^2)^{1/2}$.

b.) The electrical potential at point $(x, 0)$ due to the bottom charge q is:

$$V = \frac{q}{4\pi\epsilon_0 r},$$

where $r = (a^2 + x^2)^{1/2}$.

c.) The total electrical potential at $(x, 0)$ is the scalar sum of those two electrical potentials, or:

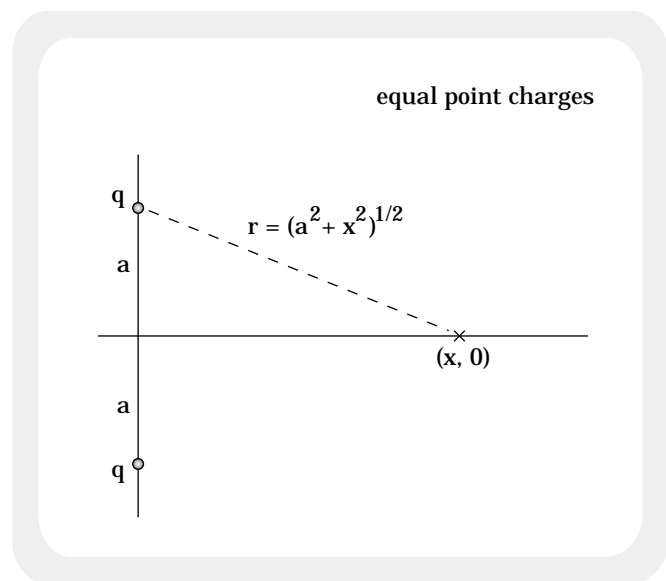


FIGURE 15.15

$$V = 2 \frac{q}{4\pi\epsilon_0 r}$$

$$= \frac{q}{2\pi\epsilon_0 (a^2 + x^2)^{1/2}}.$$

Note: What is so nice about using electrical potential functions is that they are not vectors. We can add them like scalars--we don't have to hassle with breaking them into components before using them.

d.) According to theory, the electric field should be:

$$\mathbf{E} = -\nabla V$$

$$= -\frac{q}{2\pi\epsilon_0} \left[\frac{\partial[(a^2 + x^2)^{-1/2}]}{\partial x} \mathbf{i} + \frac{\partial[(a^2 + x^2)^{-1/2}]}{\partial y} \mathbf{j} + \frac{\partial[(a^2 + x^2)^{-1/2}]}{\partial z} \mathbf{k} \right]$$

$$= -\frac{q}{2\pi\epsilon_0} \left[\frac{\partial[(a^2 + x^2)^{-1/2}]}{\partial x} \mathbf{i} \right]$$

$$= -\frac{q}{2\pi\epsilon_0} \left(-\frac{1}{2} \right) (a^2 + x^2)^{-3/2} (2x) \mathbf{i}$$

$$= \frac{qx}{2\pi\epsilon_0 (a^2 + x^2)^{3/2}} \mathbf{i}.$$

e.) Lo and behold, this is exactly the expression we derived using the definition of the electric field for a point charge and the vector approach introduced in the Electric Field chapter.

f.) UNFORTUNATELY, WE'VE DETERMINED THE RIGHT ANSWER BUT WE HAVE DONE SOMETHING DIRTY IN THE PROCESS. To see the difficulty, consider the next problem.

4.) Consider the two opposite point charges shown in Figure 15.16 on the next page. Determine the electric field at point $(x=b, 0)$, using the electrical potential function for a point charge.

a.) The electrical potential at point $(x, 0)$ due to the top charge q is:

$$V_q = \frac{q}{4\pi\epsilon_0 r},$$

where $r = (a^2 + x^2)^{1/2}$.

b.) The electrical potential at point $(x, 0)$ due to the bottom charge $-q$ is:

$$V_{-q} = \frac{-q}{4\pi\epsilon_0 r},$$

where $r = (a^2 + x^2)^{1/2}$.

c.) The total electrical potential at $(x, 0)$ is the scalar sum of those two electrical potentials, or ZERO!

d.) This isn't a problem as far as the electrical potential is concerned. As long as the potential energy characteristics are maintained, finding a zero electrical potential point is perfectly acceptable. The problem is in trying to use that function in conjunction with the electric field expression $E = -\nabla V$.

Why won't it work? The problem is rooted in the fact that $E = -\nabla V$ requires a GENERAL expression for the electrical potential (i.e., $V(x, y)$) to work properly. We didn't use such a function in the two positive charges problem done in the previous section, and we got away with it due to the problem's charge symmetry. Without that symmetry, we would have lost.

5.) Consider the two opposite point charges shown in Figure 15.17. Determine the electric field at point $(x = b, 0)$, using the electrical potential function for a point charge.

a.) We must begin by determining the electrical potential at an arbitrary point (x, y) . For the top charge q ,

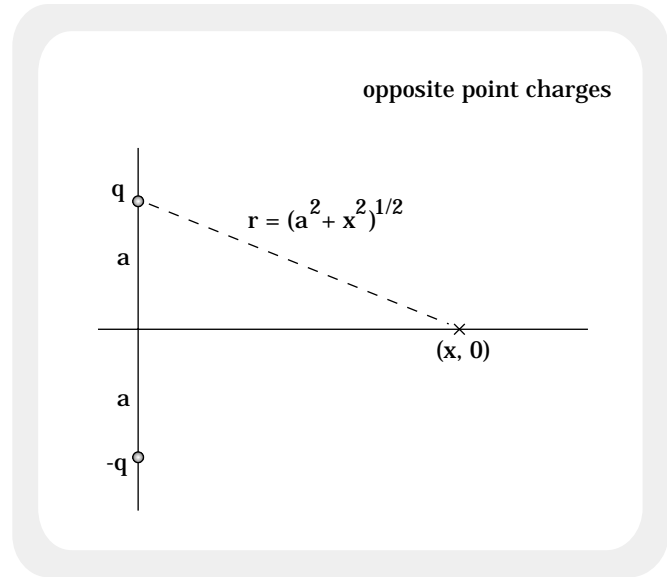


FIGURE 15.16

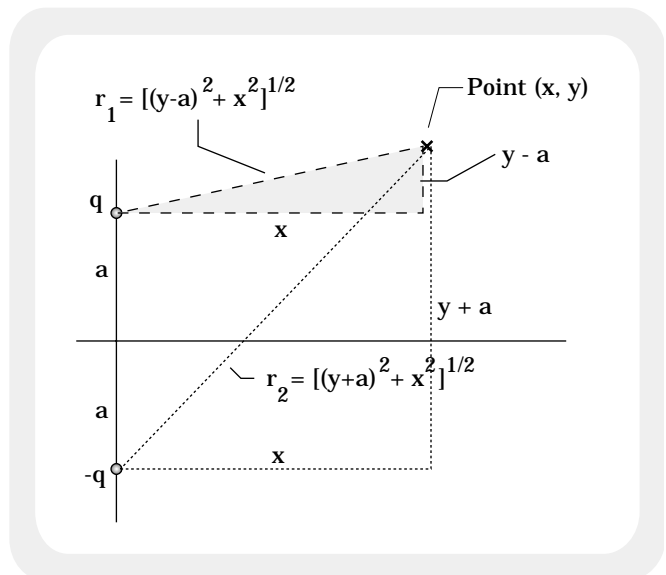


FIGURE 15.17

the electrical potential at (x, y) is:

$$\begin{aligned} V_q &= \frac{q}{4\pi\epsilon_0 r_1} \\ &= \frac{q}{4\pi\epsilon_0 [(y-a)^2 + x^2]^{1/2}}. \end{aligned}$$

b.) The electrical potential at point (x, y) due to the bottom charge $-q$ is:

$$\begin{aligned} V_{-q} &= \frac{-q}{4\pi\epsilon_0 r_2} \\ &= \frac{-q}{4\pi\epsilon_0 [(y+a)^2 + x^2]^{1/2}}. \end{aligned}$$

c.) The net electrical potential at (x, y) is the scalar sum of those two electrical potentials, or:

$$\begin{aligned} V &= V_q + V_{-q} \\ &= \frac{q}{4\pi\epsilon_0 [(y-a)^2 + x^2]^{1/2}} + \frac{-q}{4\pi\epsilon_0 [(y+a)^2 + x^2]^{1/2}} \\ &= \frac{q}{4\pi\epsilon_0} \left[[(y-a)^2 + x^2]^{-1/2} - [(y+a)^2 + x^2]^{-1/2} \right]. \end{aligned}$$

d.) The electric field at (x, y) is:

$$\begin{aligned} \mathbf{E} &= -\nabla V \\ &= -\frac{q}{4\pi\epsilon_0} \left[\frac{\partial \left[[(y-a)^2 + x^2]^{-1/2} - [(y+a)^2 + x^2]^{-1/2} \right]}{\partial x} \mathbf{i} + \frac{\partial \left[[(y-a)^2 + x^2]^{-1/2} - [(y+a)^2 + x^2]^{-1/2} \right]}{\partial y} \mathbf{j} \right] \\ &= -\frac{q}{4\pi\epsilon_0} \left[\left[\frac{-x}{[(y-a)^2 + x^2]^{3/2}} + \frac{x}{[(y+a)^2 + x^2]^{3/2}} \right] \mathbf{i} + \left[\frac{-(y-a)}{[(y-a)^2 + x^2]^{3/2}} + \frac{(y+a)}{[(y+a)^2 + x^2]^{3/2}} \right] \mathbf{j} \right]. \end{aligned}$$

e.) This expression is general for any point in the x - y plane.

f.) It would be interesting to see if the field is correct for a position along the x-axis. To find out, substitute $x = b$, $y = 0$ into our expression. Doing so yields:

$$\begin{aligned} \mathbf{E} &= -\nabla V \\ &= -\frac{q}{4\pi\epsilon_0} \left[\left[\frac{-x}{[(y-a)^2 + x^2]^{3/2}} + \frac{x}{[(y+a)^2 + x^2]^{3/2}} \right] \mathbf{i} + \left[\frac{-(y-a)}{[(y-a)^2 + x^2]^{3/2}} + \frac{(y+a)}{[(y+a)^2 + x^2]^{3/2}} \right] \mathbf{j} \right] \\ &= -\frac{q}{4\pi\epsilon_0} \left[\left[\frac{-b}{[(0-a)^2 + b^2]^{3/2}} + \frac{b}{[(0+a)^2 + b^2]^{3/2}} \right] \mathbf{i} + \left[\frac{-(0-a)}{[(0-a)^2 + b^2]^{3/2}} + \frac{(0+a)}{[(0+a)^2 + b^2]^{3/2}} \right] \mathbf{j} \right] \\ &= -\frac{q}{4\pi\epsilon_0} \left[\left[\frac{-b}{[a^2 + b^2]^{3/2}} + \frac{b}{[a^2 + b^2]^{3/2}} \right] \mathbf{i} + \left[\frac{a}{[a^2 + b^2]^{3/2}} + \frac{a}{[a^2 + b^2]^{3/2}} \right] \mathbf{j} \right] \\ &= -\frac{q}{4\pi\epsilon_0} \frac{2a}{[a^2 + b^2]^{3/2}} \mathbf{j}. \end{aligned}$$

g.) Does this make sense? You bet. Looking at Figure 15.18, the charge configuration is shown. Due to symmetry, the x components of the electric fields add to zero while the y components (after being added together) equal:

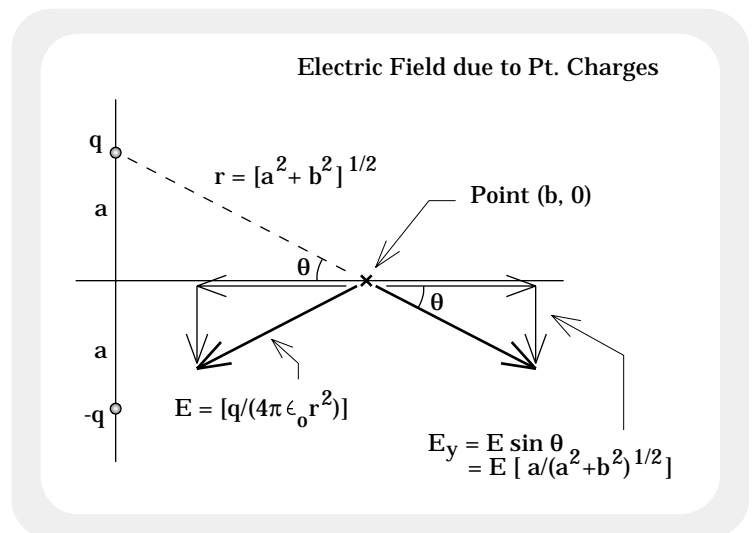


FIGURE 15.18

$$\begin{aligned} \mathbf{E} &= \left[(2) \frac{q}{4\pi\epsilon_0 [(a^2 + b^2)^{1/2}]^2} \cos \theta \right] (-\mathbf{j}) \\ &= \left[(2) \frac{q}{4\pi\epsilon_0 [(a^2 + b^2)^{1/2}]^2} \frac{a}{(a^2 + b^2)^{1/2}} \right] (-\mathbf{j}) \\ &= -\frac{2qa}{4\pi\epsilon_0 [(a^2 + b^2)^{3/2}]} \mathbf{j}. \end{aligned}$$

h.) Bottom line: Technically, using $E = -\nabla V$ to determine E requires a general expression for V . BUT, if all the charge in the configuration is the same kind (i.e., it is all positive or all negative), and if you want E at a point of symmetry (i.e., along the x-axis, for instance, in the two positive charge problem from the previous section), and assuming you are not squeamish about exploiting a mathematical anomaly that allows you to do a problem wrong but get the answer right, then you can get away with setting the problem up as was originally done in Part 3 of Section K.

QUESTIONS

15.1) The diameter of a typical hydrogen atom is approximately 10^{-10} meters across. The charge on an electron is the same as the charge on a proton (one elementary charge unit-- 1.6×10^{-19} coulomb) and the mass of an electron is 9.1×10^{-31} kilograms. If we assume that the electron follows a circular path around the proton:

- a.) What is the electric potential at the edge of the atom's boundary due to the presence of the proton at the atom's center?
- b.) How much electrical potential energy does the electron have due to the presence of the proton?
- c.) Assuming it is moving with velocity 2.25×10^6 m/s, what is the electron's total energy?

15.2) The areas A and B shown in Figure I are found to have electrical potential values of $V_A = -12$ volts and $V_B = +20$ volts:



FIGURE I

- a.) Re-drawing the sketch larger, draw in the electric field lines for the region between and around A and B; and
- b.) Draw the region's equipotential lines at eight-volt intervals.

15.3) A particle made up of two protons and two neutrons is called an α particle (the mass of either a proton or a neutron is 1.67×10^{-27} kilograms--the

charge on a proton is 1.6×10^{-19} coulombs; and there is no charge on a neutron). An α particle is accelerated through an 18 Megavolts electrical potential difference (one megavolt equals 10^6 volts). If the acceleration takes place along a 12 meter long track:

- Assuming the α particle has no potential energy by the end of its run, how much potential energy does it have at the beginning of its run?
- How much work per unit charge does the field do on the α particle during its run?
- Assuming it starts from rest, what is the particle's velocity magnitude at the end of the run (approach this using conservation of energy, not the work/energy theorem)?

15.4) The following information is known about the constant electric field shown in Figure II to the right and below: the electric field intensity is 80 nts-per-coulomb; the voltage $V_A = 340$ volts; the voltage $V_E = 320$ volts; distance $d_{AB} = .25$ meters; the distance $d_{DE} = .50$ meters and is perpendicular to the electric field; and Point C's vertical position is half-way between A and B.

- Is V_A greater or less than V_B ?
- Determine the distance d_{AD} some other way than just eyeballing it.
- Determine V_B .
- There are a number of ways to determine V_C . Pick two ways and do it.
- How much potential energy will be available to a $6 \mu\text{C}$ charge when placed at Point A?
- How much work per unit charge is done by the field as a $6 \mu\text{C}$ charge moves from Point A to Point E?
- How much work is done by the field on a $6 \mu\text{C}$ charge that moves from Point A to Point B?
- If V_A had been 340 volts and V_B had been 290 volts:
 - What would the electric field's direction have been?
 - What would the electric field's magnitude have been?

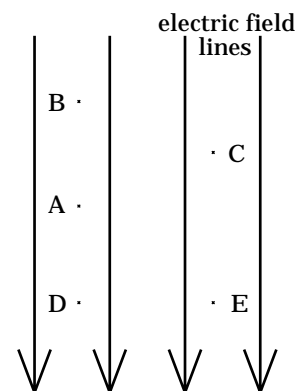


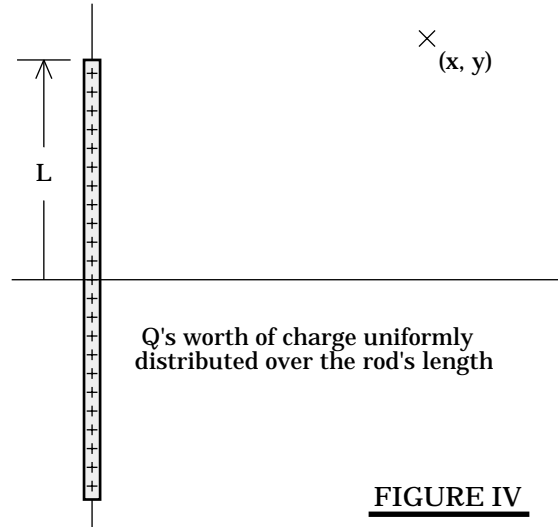
FIGURE II

15.5) Two equal point-charges ($q = 10^{-16}$ C each) are placed at opposite corners of a square whose edges are .4 meters long. Assuming we neglect gravity, if $q_1 = 10^{-18}$ C is placed at the exact center of the square and given a

slight nudge toward one of the unoccupied corners, it will accelerate. Letting q_1 's mass be $m = 7 \times 10^{-22}$ kilograms, what will q_1 's velocity be by the time it reaches the unoccupied corner toward which it accelerates? Do the problem algebraically first, then put in the numbers.

15.6) A rod of length $2L$ has a total charge Q distributed uniformly upon its surface (see Figure IV).

- a.) Determine the electrical potential at an arbitrary point (x, y) in the field.
- b.) Knowing the electrical potential function, how could you determine the electric field at (x, y) ? (You don't need to actually do it).



15.7) An electrical potential function:

$$V = k_1 e^{-kx} + k_2 / y^3$$

exists within a region. Assuming the k terms are constants, what is the electric field function for the region?

15.8) An electric field function:

$$\mathbf{E} = k_1 e^{-kx} \mathbf{i} + (k_2 / y^3) \mathbf{j}$$

exists within a region that excludes $y = 0$. If the k terms are constants:

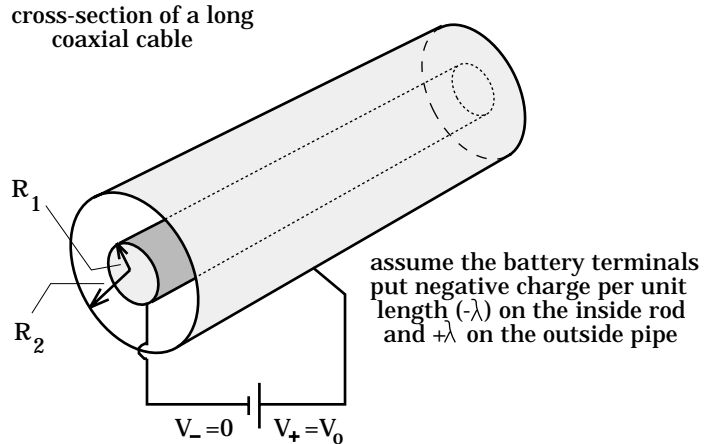
- a.) At what x and y coordinate must the electrical potential be zero?
- b.) What is the electrical potential function for the region?

15.9) A very large (essentially infinite) flat conductor has on its surface a surface charge density equal to 10^{-10} coulombs per square meter. Equipotential surfaces that differ by 12 volts are plotted in the vicinity of the conductor's face.

- a.) Relative to the conductor's face, how are the equipotential surfaces oriented?
- b.) How far apart are the surfaces?

15.10) Electric field functions are not continuous. Electrical potential functions are continuous. Explain how you know this must be true.

15.11) A conducting rod of radius R_1 is surrounded by a coaxially positioned pipe of radius R_2 . A battery whose voltage is V_0 is connected such that the rod is attached to the ground of a battery (assume the electrical potential of the ground is zero) while the pipe is attached to the high voltage side of the power source. Derive a general expression for the electrical potential between the rod and the pipe. In other words, what is $V(r)$ for $R_2 > r > R_1$?



15.12) A charge Q exists (it is hung from an inconsequential thread) at the center of a spherical conducting shell of inside radius R_1 and outside radius R_2 . Knowing that the shell has $2Q$'s worth of free charge placed on it, plot the electric field versus position graph and the electrical potential versus position graph for this configuration.

After having used the theory to determine the field expressions for the various regions, substitute the following values into your expressions: $R_1 = 2$ meters, $R_2 = 3$ meters, and $[Q/(4\pi\epsilon_0)] = 1$ (that is, assume Q is the right size for this to be true).

Note: The above problem could as easily have been a spherical configuration in which there were surface or volume charge densities, or it could have been a cylindrical configuration in which there were linear, surface, or volume charge densities. Be prepared to deal with any contingency.

TRANSLATION: Review GAUSS'S LAW for both spherical and cylindrical symmetry before your next test!

