CHAPTER 10 -- GRAVITATION

10.1) According to Newton, the magnitude of the gravitational force between any two bodies will always be equal to Gm_1m_2/r^2.

a.) The gravitation force *you* exert on your friend will be:

$$
\begin{aligned}
F_{on\ friend} &= G\, m_{you}\, m_{fr} / r^2 \\
&= (6.67\times10^{-11}\ nt\cdot m^2/kg^2)(80\ kg)(60\ kg)/(5\ m)^2 \\
&= 1.28\times10^{-8}\ nts.
\end{aligned}
$$

Note: If you cut a ping pong ball into a million and a half or so pieces, this force would approximately equal the weight of one piece.

b.) As for the acceleration, Newton's Second Law yields:

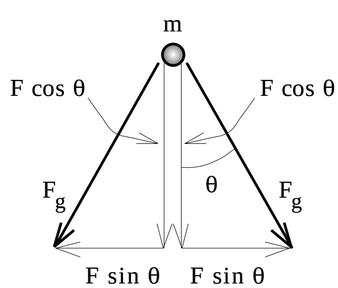
$\underline{\sum F_x:}$

$$
\begin{aligned}
F_{fr} &= m_{fr}a \\
\Rightarrow\quad a &= F_{fr} / m_{fr} \\
&= (1.28\times10^{-8}\ nt)/(60\ kg) \\
\Rightarrow\quad &= 2.13\times10^{-10}\ m/s^2.
\end{aligned}
$$

Note: You can now see why you don't feel a gravitational force when you brush past a friend on the street. The Universal Gravitational Constant G is so small that at least one of the masses has to be enormous before gravitational effects become noticeable.

10.2) Figure I below shows an f.b.d. for the forces acting on the top mass, complete with components. Figure II below shows how the radius r of the orbits can be determined.

FIGURE I



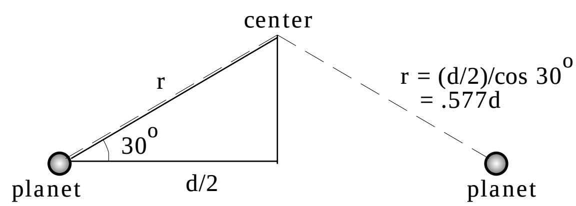


FIGURE II

Due to symmetry, the horizontal components will add to zero leaving only the vertical components with which to contend. Noting that the masses are all the same and the radius of the circular motion is *.577d,* we can use N.S.L. in the *center-seeking direction* to write:

$\underline{\sum F_{center\text{-}seeking}:}$

$$-F_g \cos\ \theta - F_g \cos\ \theta = -ma_c$$
$$\Rightarrow \quad 2[Gm^2/d^2]\cos 30^o = m(v^2/r)$$
$$= mv^2/[.577d]$$
$$\Rightarrow \quad v = (Gm/d)^{1/2}.$$

10.3) This is a straight freefall problem (i.e., although we will be using Newton's gravitational force, the force is not acting centripetally).

a.) The *earth to moon distance* (*center of mass* to *center of mass*) is $3.82\text{x}10^8$ meters plus $1.74\text{x}10^6$ meters plus $6.37\text{x}10^6$ meters, or approximately $3.9\text{x}10^8$ meters. Summing the forces in the direction of freefall, we can write:

$\underline{\sum F_r:}$

$$-Gm_m m_e/r^2 = -m_m a$$
$$\Rightarrow \quad a = Gm_e/r^2$$
$$= (6.67\text{x}10^{-11}\ nt\cdot m^2/kg^2)(5.98\text{x}10^{24}\ kg)/(3.9\text{x}10^8\ m)^2$$
$$= 2.62\text{x}10^{-3}\ m/s^2.$$

b.) The distance between the moon's center of mass and the earth's center of mass just before they strike one another will be the sum of their radii. Using that observation and repeating the calculation done in *Part a*, we get:

$\underline{\sum F_r:}$

$$-Gm_m m_e/r_{e,m}{}^2 = -m_m a$$
$$\Rightarrow \quad a = Gm_e/r_{e,m}{}^2$$

$$= (6.67\text{x}10^{-11}\ nt\cdot m^2/kg^2)(5.98\text{x}10^{24}\ kg)/(8.11\text{x}10^6\ m)^2$$
$$= 6.06\ m/s^2.$$

c.) The acceleration is not a constant (obviously), so kinematics is out. The best approach is using *conservation of energy*. Remembering that the *potential energy function* for a varying gravitational field is $U = -Gm_1m_2/r$ (a scalar), we can write:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$
$$(0) + (-Gm_m m_e/r) + (0) = (1/2)m_m v^2 + [-Gm_m m_e/(r_m + r_e)]$$
$$\Rightarrow \quad v = [2Gm_e[-1/r + 1/(r_m + r_e)]]^{1/2}$$
$$= [2(6.67x10^{-11}\ nt\cdot m^2/kg^2)(5.98x10^{24}\ kg)[-1/(3.9x10^8\ m) + 1/(1.74x10^6 + 6.37x10^6\ m)]]^{1/2}$$
$$\Rightarrow \quad v = 9814\ m/s.$$

d.) If the earth is allowed to move, we have to take into consideration its *kinetic energy* at the end of the freefall (its initial kinetic energy is zero). This means we now have a second unknown--the earth's velocity--to deal with. To get an expression relating the earth's velocity v_e to the moon's velocity v_m, note that the only force acting in the system--gravity--is internal to the system (that is, the moon exerts a gravitational force on the earth and the earth exerts an equal and opposite gravitational force on the moon). That means momentum is conserved.

With everything starting from rest, the system's initial net momentum (and the system's subsequent net momentum) is zero. Using that bit of information, we can write:

$$p_o = p_{just\ before\ impact}$$
$$\Rightarrow \quad 0 = -m_m v_m + m_e v_e$$
$$\Rightarrow \quad v_e/v_m = m_m/m_e.$$

Knowing the mass of the earth and moon, we now have the relationship we need between the earth's velocity and the moon's velocity (i.e., our second equation).

10.4)

a.) The magnitude of the force applied to m_1 a distance r from the planet's center is:

$$F = G(m_{inside\ sphere})m_1/r^2,$$

where $m_{inside\ sphere}$ is the mass inside the sphere upon which the body sits. To determine that mass, define a differential sphere of radius a and thickness da. Its differential volume will be its *surface area times its thickness*, or $dV = (4\pi a^2)da$. Defining the *volume density function* to be ρ, we can write:

$$
\begin{aligned}
m_{insidesph} &= \int \rho dv \\
&= \int_{a=0}^{r} \left[\frac{m_p}{r_p^{\ 4}} a \right] \left[4\pi a^2 da \right] \\
&= \frac{4\pi m_p}{r_p^{\ 4}} \int_{a=0}^{r} a^3 da \\
&= \frac{4\pi m_p}{r_p^{\ 4}} \left[\frac{a^4}{4} \right]_{a=0}^{r} \\
&= \frac{\pi m_p}{r_p^{\ 4}} r^4.
\end{aligned}
$$

Using this, we can write:

$$
\begin{aligned}
F &= G(m_{inside\ sphere})m_1/r^2 \\
&= G(\pi m_p r^4/r_p^{\ 4})m_1/r^2 \\
&= Gm_p m_1(\pi/r_p^{\ 4})r^2.
\end{aligned}
$$

Note that at $r = 0$, we don't find a zero in the denominator as before.

b.) The *potential-energy-equals-zero-point* for a variable force function is ALWAYS placed where the force is zero. In this case, that is at the planet's center. If we assume the body is sitting at some position $y = r$ up the vertical axis (i.e., we are assuming the tunnel is in the vertical), then the force on the mass m_1 when at an arbitrary point y will be $-Gm_p m_1(\pi/r_p^{\ 4})y^2$ (from *Part a*) Setting $U(y = 0) = 0$, we can write:

$$
\begin{aligned}
U(r) - U(y = 0) &= -\int \mathbf{F} \bullet d\mathbf{r} \\
\Rightarrow U(r) &= -\int_{y=0}^{r} \left[-\frac{Gm_p m_1 \pi}{r_p^{\ 4}} y^2 \mathbf{j} \right] \bullet (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\
&= \frac{Gm_p m_1 \pi}{r_p^{\ 4}} \int_{y=0}^{r} y^2 dy \\
&= \frac{Gm_p m_1 \pi}{r_p^{\ 4}} \left[\frac{y^3}{3} \right]_{y=0}^{r} \\
&= \frac{Gm_p m_1 \pi}{3r_p^{\ 4}} r^3.
\end{aligned}
$$

Again, this makes sense. At $r = 0$, the *potential energy* is zero.

10.5) We know the *orbital distance* from the earth's *center of mass* is $r = 1.3x10^6\ m + 6.3x10^6\ m = 7.6x10^6$ *meters*, and we know that this is additionally the distance between the *centers of mass* of the two bodies (the two quantities are the same as the satellite's mass is minuscule in comparison to the earth's mass).

a.) One of the things that makes orbital problems tricky is that there are a number of ways one can determine a velocity. *Conservation of energy* has velocities in it, but so does *N.S.L.* coupled with *centripetal acceleration*. Knowing which approach to use in a given situation is something that comes only with experience. In this particular case, we want to look at the *forces* acting on the satellite.

$\underline{\sum F_c}$:

$$-F_g = -ma_c$$

$$\Rightarrow \quad Gm_s m_e/r^2 = m_s(v^2/r)$$

$$\Rightarrow \quad v = (Gm_e/r)^{1/2}$$

$$= [(6.67x10^{-11}\ nt\cdot m^2/kg^2)(5.98x10^{24}\ kg)/(7.6x10^6\ m)]^{1/2}$$

$$= 7244\ m/s.$$

b.) If it takes time *T* for the satellite to travel the circumference of the circle upon which it moves, then the ratio of those two quantities (i.e., distance/time) will yield the magnitude of its velocity. As *T* is defined as the period of the motion (i.e., the time for one complete orbit), we can write:

$$v = (\text{circumference})/(\text{period})$$

$$= 2\pi r/T$$

$$\Rightarrow \quad T = 2\pi r/v$$

$$= 2\pi(7.6x10^6\ m)/(7244\ m/s)$$

$$= 6592 \text{ seconds} \quad \text{(approx. 1 hr, 50 min.).}$$

c.) The work *we* have to do in getting the satellite *up to speed* is equal to the body's *final kinetic energy*. The work *we* have to do in *lifting the satellite* into the appropriate orbit is equal to *minus* the work gravity does on the body as the body rises, or ΔU_g (remember, the work gravity *itself* does as the body rises is $-\Delta U$). If we add these two work quantities together, we have the amount of energy *we* have to provide to the system to get the satellite into orbit.

. . . Or, we could simply use the *modified conservation of energy equation*. In that expression, the $\sum W_{ext}$ term denotes the extra energy needed from *us* to put the satellite in orbit. Using that expression, we can write:

$$\sum KE_1 + \sum U_1 + \sum W_{ext} = \sum KE_2 + \sum U_2$$
$$(0) + (-Gm_s m_e/r_e) + E_{you} = (1/2)m_s v^2 + (-Gm_s m_e/r),$$

where v is the orbital velocity of the satellite and r is the orbital radius from the earth's center.

Solving yields:

$$E_{you} = (1/2)m_s v^2 + Gm_s m_e[1/r_e - 1/r]$$
$$= .5(400 \text{ kg})(7244 \text{ m/s})^2 + (6.67\text{x}10^{-11} \text{ nt}\cdot\text{m}^2/\text{kg}^2)(5.98\text{x}10^{24} \text{ kg})(400 \text{ kg})[1/(6.3\text{x}10^6 \text{ m}) - 1/(7.6\text{x}10^6 \text{ m})]$$
$$\Rightarrow \quad E_{you} = 1.48\text{x}10^{10} \text{ joules.}$$

Note: If we had used the *orbital energy* expression $E = -Gm_s m_e/2r$ from the chapter, we would have ended up with $-1.05\text{x}10^{10}$ joules (this, if you will remember, is $KE_r + U_r$). So why doesn't the *orbital energy* expression equal the work required to put the satellite into orbit?

The answer is simple. The body started out with potential energy. Sitting on the earth's surface with $-Gm_e m_s/r_e = -2.53\text{x}10^{10}$ *joules* of potential energy (put in the numbers--that's what you get). Add the energy needed to get it into orbit (i.e., the $1.48\text{x}10^{10}$ joules) and you end up with the satellite's net final energy equaling $-1.05\text{x}10^{10}$ joules.

d.) The earth's angular velocity is 2π radians per 24 hours (give or take a bit). That calculates out to $\omega = 7.27\text{x}10^{-5}$ *radians/second*. At the equator, the magnitude of the earth's translational velocity is $v = r_{equ}\,\omega = (6.3\text{x}10^6 \text{ m})(7.272\text{x}10^{-5} \text{ rad/sec}) = 458 \text{ m/s}$.

A satellite launched in the direction of the earth's rotation at the equator will begin its trip with initial kinetic energy equal to $(1/2)mv_e^2$, where v_e is the velocity of the rotating earth.

In the case of our satellite, it would start out with kinetic energy equal to $(1/2)(400 \text{ kg})(458 \text{ m/s})^2 = 4.2\text{x}10^7$ *joules* worth of energy *we* wouldn't have to supply to the system to get it into its orbit.

It should now be obvious why launching pads in the U.S. (Cape Canaveral, for instance) are as close to the equator as possible.

e.) In 1800 revolutions, the energy lost to the satellite will be:

$$E_{lost} = (1800 \text{ rev})(2\text{x}10^5 \text{ joules/rev})$$
$$= 3.6\text{x}10^8 \text{ joules.}$$

The amount of energy it starts out with as it traveled in its orbit is:

$$E_1 = -Gm_e m_s/2r_1$$
$$= -(6.6\text{x}10^{-11} \text{ nt}\cdot\text{m}^2/\text{kg}^2)(5.98\text{x}10^{24} \text{ kg})(400 \text{ kg})/[2(7.6\text{x}10^6 \text{ m})]$$
$$= -1.049\text{x}10^{10} \text{ joules.}$$

After losing $3.6\text{x}10^8$ joules of energy, the satellite has:

$$E_2 = \quad E_1 \quad - \quad E_{lost}$$
$$= (-1.049\text{x}10^{10} \text{ joules}) - (3.6\text{x}10^8 \text{ joules})$$
$$= -1.085\text{x}10^{10} \text{ joules.}$$

An orbit with that amount of energy must be such that:

$$E_2 = -Gm_e m_s/2r_2$$
$$\Rightarrow \quad r_2 = -Gm_e m_s/2E_2$$
$$= -(6.67\text{x}10^{-11} \text{ nt}\cdot\text{m}^2/\text{kg}^2)(5.98\text{x}10^{24} \text{ kg})(400 \text{ kg})/[2(-1.085\text{x}10^{10} \text{ j})]$$
$$= 7.35\text{x}10^6 \text{ meters.}$$

This is the distance from the earth's center. From the earth's surface, the height will be *$7.35\text{x}10^6$ meters minus $6.3\text{x}10^6$ meters equals $1.05\text{x}10^6$ meters*.

Although the problem did not ask for the velocity of the satellite when in this orbit, we will need it later. Using N.S.L., we can write:

$$\sum F_r:$$
$$F_g = ma_c$$
$$\Rightarrow \quad Gm_s m_e/r_2^{\,2} = m_s(v_2^{\,2}/r_2)$$
$$\Rightarrow \quad v_2 = (Gm_e/r_2)^{1/2}$$
$$= [(6.67\text{x}10^{-11} \text{ nt}\cdot\text{m}^2/\text{kg}^2)(5.98\text{x}10^{24} \text{ kg})/(7.35\text{x}10^6 \text{ m})]^{1/2}$$
$$= 7366 \text{ m/s.}$$

f.) The retarding force does $2\text{x}10^5$ joules *per revolution* of work on the satellite as the satellite moves along in its path. On average, the work done by the retarding force in *one orbit* will equal the average force *times* the average circumference of motion *times* cos 180^o (the retarding force and the direction of motion will be opposite one another, hence the 180^o angle). Approximating the *average radius* to be $(r_1 + r_2)/2$, we can determine the work calculation for one average orbit. Doing so yields:

$$
\begin{aligned}
2x10^5 &= F_{avg}(2\pi r_{avg}) \cos 180^o \\
&= -F_{avg}[2\pi(r_2 + r_1)/2] \\
&= -F_{avg}[2\pi(7.35x10^6 + 7.6x10^6)/2] \\
&= -(4.7x10^7)F_{avg} \\
\Rightarrow \quad F_{avg} &= -.0043 \text{ newtons}
\end{aligned}
$$

g.) The *modified conservation of angular momentum* expression is:

$$\sum L_1 + \sum \Gamma_{ext} \Delta t = \sum L_2,$$

where the L terms are angular momentum quantities and the $\Gamma \Delta t$ is a torque driven, *impulse-related* quantity (remember, the *modified conservation of linear momentum* equation was $\sum p_1 + \sum F_{ext} \Delta t = \sum p_2$, where the $F_{ext} \Delta t$ term was the <u>impulse</u> being delivered by external forces over the time interval Δt).

The average external torque about the planet's orbital axis is:

$$\Gamma_{ext} = \mathbf{r}_{s,avg} x \mathbf{F}_{avg}.$$

Noting that the frictional force $\boldsymbol{F}_{avg}$ is perpendicular to the radius vector $\boldsymbol{r}_{s,avg}$, the magnitude of the torque will simply be the product of the radius and force vectors ($\sin 90^o = 1$). Assuming the direction of the satellite's angular velocity is *positive* and noting that the torque slows the motion, our friction-induced torque will be negative. As such, we can write:

$$\Gamma_{ext} = -r_{s,avg} F_{avg}.$$

The angular momentum of the satellite will equal:

$$L = \mathbf{r}_{s,avg} x \mathbf{p},$$

where again the angle between the vectors is 90^o and the sine of the angle is *1*. As the magnitude of the momentum is $m_s v_s$, we can write in general:

$$L = r_{s,avg} m_s v_s.$$

Using all this information in our *modified conservation of angular momentum* equation, we can solve for Δt:

$$\sum L_1 \quad + \quad \sum \Gamma_{ext} \Delta t = \sum L_2$$

$$r_{s,1} m_s v_{s,1} - r_{s,avg} F_{avg} \Delta t = r_{s,2} m_s v_{s,2}$$

$$(7.6\text{x}10^6 \text{ m})(400 \text{ kg})(7244 \text{ m/s}) - (7.475\text{x}10^6 \text{ m})(.004 \text{ nt})\Delta t = (7.35\text{x}10^6 \text{ m})(400 \text{ kg})(7366 \text{ m/s})$$

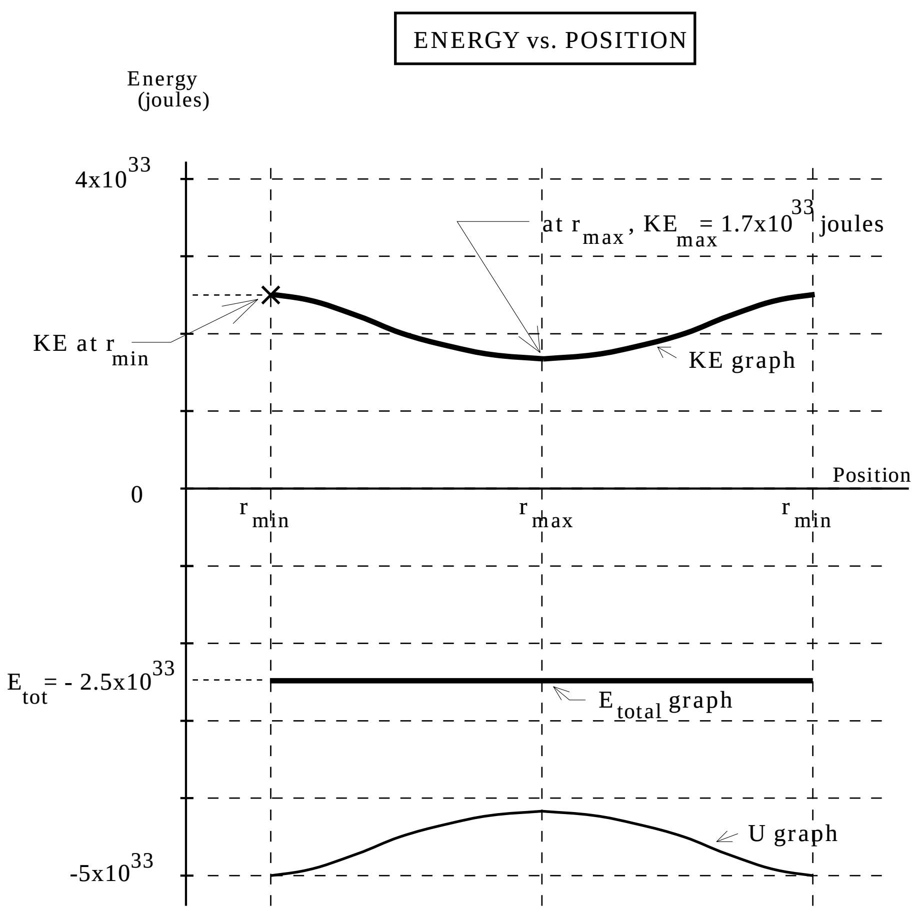
$$\Rightarrow \quad \Delta t = 1.223\text{x}10^7 \text{ seconds.}$$

This rounded-off value is approximately 141 days, 13.5 hours.

10.6)

a.) The conserved quantities for planetary motion are *angular momentum* (there are no external torques acting on the planet), and *energy* (there is practically no frictional effect in space, and there are generally no appreciable non-conservative forces acting on planets--asteroid collisions excepted).

b.) At r_{min}, the star has U_1 = *-5x10*^{33} *joules* (see graph) and kinetic energy KE_1 = *2.5x10*^{33} *joules*. That means the total energy of the system is KE_1+ U_1 = *-2.5x10*^{33} *joules*. In addition, the sum of the potential and kinetic energies must *always* equal that number (energy *is* conserved). Because we know the *total energy* in



the system, and because we have a graph of the *potential energy* as a function of position, we can determine the *kinetic energy* as a function of position for any point in the orbit. Doing that for key points (example: at r_{max}, the graph tells us that the *potential energy* is approximately U_{max} = $-4.2x10^{33}$ *joules* . . . that means KE_{max} - $4.2x10^{33}$ *joules* = *total energy* = $-2.5x10^{33}$ *joules*, or KE_{max} = $1.7x10^{33}$ *joules* at that point), and remembering that the *kinetic energy function* mirrors the *potential energy function* when the *total energy* is constant, we can draw the *kinetic energy* graph as shown below.