Chapter 10

GRAVITATION

A.) Introduction–A Little History:

1.) The study of gravitation is a particular favorite of elementary physics texts because it embraces all of classical Newtonian physics in one giant, special case. As such, prepare to meet your old favorites: Newton's Second Law, centripetal acceleration, conservation of energy, exotic potential energy functions, conservation of angular momentum . . . all the goodies.

2.) One special topic within this chapter of special topics is the work of Johannes Kepler. In 1601, Kepler essentially absconded with the celestial observations of just-deceased Tycho Brahe (Brahe was the last great astronomer to make observations without the use of a telescope). It took years, but Kepler's analysis of Brahe's data laid the mathematical foundation for what are now known as Kepler's three laws of planetary motion. We will spend a little time examining these laws later.

3.) The main event in this chapter will be centered on the work of Sir Isaac Newton. In 1665, Newton began to muse about the moon and its orbit around the earth. A brief summary of his preliminary thoughts on the famous apple is presented below (you won't be tested on this summary):

a.) The acceleration of a freefalling apple near the earth's surface is 9.8 m/s^2, where the apple is approximately 4000 miles from the earth's center. The centripetal acceleration of the moon as it orbits the earth is .002722 m/s^2 (this he determined knowing the moon's *period of revolution*, the *distance from the earth to the moon*, and the fact that the moon's acceleration is centripetal), where the moon is approximately 240,000 miles from the earth's center.

b.) Force is proportional to acceleration (this was to become his second law), which means that:

$$F_{apple} \propto a_{apple}$$

and

$$F_{moon} \propto a_{moon}.$$

c.) Since gravitational force decreases as one gets farther away from the earth, the relationship between gravitational force and the distance r from the earth must be inversely proportional. Assuming that r might be raised to an exponent other than one, it must be true that:

$$F_{apple} \ \alpha \ 1/(r_{apple})^n$$

and

$$F_{moon} \ \alpha \ 1/(r_{moon})^n.$$

d.) Putting the observations in *Parts b* and *c* together, we conclude that:

$$a_{apple} \ \alpha \ 1/(r_{apple})^n$$

and

$$a_{moon} \ \alpha \ 1/(r_{moon})^n.$$

e.) Taking the ratio of the two expressions, then putting in numbers yields:

$$\frac{a_{apple}}{a_{moon}} \propto \frac{\dfrac{1}{(r_{apple})^n}}{\dfrac{1}{(r_{moon})^n}}$$

$$\Rightarrow \quad \frac{a_{apple}}{a_{moon}} \quad \propto \quad \left(\frac{r_{moon}}{r_{apple}}\right)^n$$

$$\Rightarrow \quad \frac{\left(9.8 \ \dfrac{m}{s^2}\right)}{\left(.002722 \ \dfrac{m}{s^2}\right)} \propto \left(\frac{(240,000 \text{ mi})}{(4000 \text{ mi})}\right)^n$$

$$\Rightarrow \ 3600\alpha(60)^n.$$

f.) The exponent n that allows this to be an equality is $n = 2$.

g.) With that observation, Newton concluded that the gravitational force exerted between any two objects (the earth and moon or the earth and an apple or whatever) must be inversely proportional to the square of the distance between the two objects (i.e., $F \alpha 1/r^2$).

Note: Newton originally abandoned this analysis for a number of years because the then-accepted distance between the earth and moon was incorrect and, hence, his calculation yielded an n value that was not a whole number. It was not until a more accurate measurement was made that he went back to his theory and, lo and behold, things worked out.

B.) Newton's Gravitational Law (in general):

1.) From the observations and analysis summarized in the previous section, Newton concluded that any two masses will be attracted to one another due to a force he called *gravity*.

Note: He didn't like this idea very much. Why? Because he couldn't see any good reason why two bodies, simply by virtue of the fact that they happen to be massive, should be attracted to one another. He put forth the idea only because attraction seemed to exist and because he couldn't come up with a better explanation for the phenomenon. In fact, it wasn't until Einstein that an alternative explanation for gravitational effects (eg., objects falling toward the earth when released, the moon orbiting the earth, etc.) was presented to the world.

We use Newton's theory of gravity because it is a model that works. Nevertheless, it is generally conceded that gravitational forces do not, in reality, exist (gravitational *effects* exist but the force postulated by Newton . . . no!). We'll talk about Einstein's view later.

2.) The magnitude of Newton's gravitational force is proportional to the mass of the two attracting bodies and inversely proportional to the square of the distance between the *center of mass* of each.

a.) Newton's relationship in its most general form is:

$$\mathbf{F}_{grav} = G\frac{m_1 m_2}{r^2}(-\mathbf{r}),$$

where m_1 and m_2 are the masses involved, r is the distance between the center of masses of the two bodies, and $\boldsymbol{r}$ is a unit vector directed along the line between the two bodies (this is called a *radial* unit vector).

Note: A *radial* unit vector is drawn *from the field producing mass* toward the mass experiencing the field (see Figure 10.1a). The NEGATIVE SIGN in front of the radial unit vector in Newton's expression means that the gravitational force is *opposite* the direction of $\boldsymbol{r}$. That is, it is an attractive force.

People get confused with the negative sign when they use the force function in conjunction with a Cartesian coordinate system because a negative sign doesn't mean the

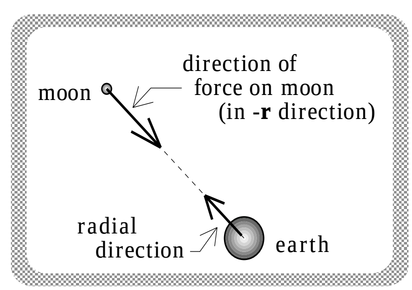

FIGURE 10.1a

same thing in that system as it does in a polar spherical system (i.e., the system in which radial vectors exist).

Bottom Line: When doing gravitational problems, use Newton's expression to determine the force *magnitude*, then decide whether the force direction is positive or negative *relative to the coordinate system being used.*

Example: The earth is placed at the origin of a Cartesian coordinate system and the moon is placed on the *x axis* to the left of the earth (see Figure 10.1b). In that case, the magnitude of the attractive gravitational force on the moon is $Gm_e m_m/x^2$, and the direction of the attractive force is *positive*.

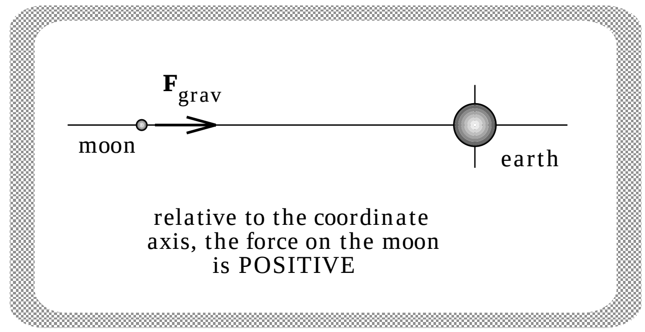

FIGURE 10.1

b.) The proportionality constant G, called the *Universal Gravitional Constant*, is equal to $6.67\text{x}10^{-11}$ nt·m^2/kg^2.

c.) Gravitational forces are always attractive.

Note: Newton correctly reasoned that if a gravitational-type force did exist, the net force on an apple would be the *vector sum* of all the individual gravitational forces exerted by all the individual pieces of matter that make up the earth (see Figure 10.2).

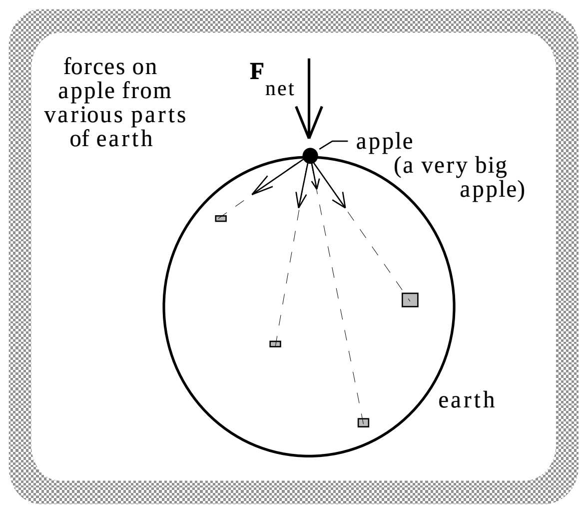

FIGURE 10.2

Newton circumvented the problem of vectorially adding billions of tiny force quantities by assuming that the net gravitational effect from all of the pieces was the same as if all the earth's mass was centered at the earth's *center of mass*. In that case, all he had to worry about was the point-mass apple and the point-mass earth.

Unfortunately, there existed no mathematics at the time from which he could justify that assumption. His solution? He took a few years off from his physics pursuits and created Calculus. That's right, folks, that mathematical discipline you have been banging your head into for the last year or two--it didn't exist before Newton. He made it up so he could finish a gravitation problem on which he was working (actually, Leibniz independently created his own version at about the same time).

3.) To see how well this force function matches experimental observation, consider the following: What is the gravitational force between the earth and a 100 kg man standing on the earth's surface?

a.) The mass of the earth is $5.97\text{x}10^{24}$ kg while its radius is $6.37\text{x}10^{6}$ meters. According to Newton's force function, the magnitude of the force felt by the man is:

$$\begin{aligned} F_g &= G\,[m_{man}\;\; m_e \;\; / \;\; r^2\,] \\ &= (6.67\text{x}10^{-11}\ \text{nt}\cdot\text{m}^2/\text{kg}^2)[(100\ \text{kg})(5.97\text{x}10^{24}\ \text{kg})/(6.37\text{x}10^{6}\ \text{meters})^2] \\ &= 9.81\text{x}10^{2}\ \text{newtons.} \end{aligned}$$

b.) As the radius of the earth is enormous, this force will be approximately the same whether the man is located on the earth's surface or some small distance above the earth's surface (i.e., a few hundred meters). That is why we say the gravitational force on a body near the earth's surface is, for all intents and purposes, a constant.

c.) Because the gravitational force on a body will be nearly constant near the earth's surface, we can define a function that allows us to determine that force more easily. Specifically, as the gravitational force is proportional to the mass m_{man}, we can write:

$$F_{grav} = m_{man}g,$$

where g is the proportionality constant required to make the proportion an equality.

i.) Let's determine g by using the force value calculated with Newton's general gravitational function. According to our calculation, a 100 kg man will feel a force due to the earth's gravitational attraction equal to $9.81\text{x}10^{2}$ newtons (in my country, we call this *the man's weight*).

ii.) If that be the case, we can write:

$$\begin{aligned} F_g &= m_{man}\ g \\ \Rightarrow\quad 9.81\text{x}10^{2}\ \text{nt} &= (100\ \text{kg})g \\ \Rightarrow\quad g &= 9.8\ \text{m/s}^2. \end{aligned}$$

iii.) This constant is equal to the experimentally observed value for the *gravitational acceleration* of an object located near the surface of the earth.

4.) An additional example: How far from the earth's surface must an astronaut in space be if she is to feel a gravitational acceleration that is half what she would feel on the earth's surface?

a.) This is a tricky question in one respect only. The gravitational expression defined by Newton had a distance term in it, but the distance was defined as that between the *center of masses* of the two bodies. Our question asks for the distance r between the astronaut and the earth's surface, not between the astronaut and the earth's center of mass (i.e., $r + r_e$, where r_e is the radius of the earth). Using the appropriate distance, we can write:

$$\underline{\sum F_c}:$$

$$GmM/(r+r_e)^2 = ma_1$$

$$r = (GM/a_1)^{1/2} - r_e.$$

b.) We know that a_1 is one-half the gravitational acceleration here on Earth, or 4.9 m/s^2. We know that the universal gravitational constant $G = 6.67x10^{-11}\ nt{\cdot}m^2/kg^2$, the mass of the earth $M = 5.97x10^{24}\ kg$, and the radius of the earth $r_e = 6.37x10^6$ *meters*. Using all this information, we can write:

$$r = [(6.67\text{x}10^{-11}\ \text{nt}{\cdot}\text{m}^2/\text{kg}^2)(5.97\text{x}10^{24}\ \text{kg})/(4.9\ \text{m/s}^2)]^{1/2} - (6.37\text{x}10^6\ \text{m})$$

$$= 2.64\text{x}10^6 \text{ meters.}$$

This is approximately four-tenths of the radius of the earth.

C.) Kepler's Laws:

Note: Kepler determined all of his laws from observational data. The mathematical justifications presented below were generated much later.

1.) Kepler's First Law: Planets move in *elliptical orbits* having the *sun at one focal point.*

a.) This is called the Law of Orbits.

b.) Justification: Using the *conservation of angular momentum* and the *conservation of energy* with a *potential energy function* appropriate to an inverse-square force field, it is possible to derive an expression for $r(\theta)$ (i.e., the distance between the sun and a planet as a function of angular position within the orbit). Although it is not a pleasant integral to evaluate, doing so yields a position function $r(\theta)$ that turns out to be the equation of an ellipse.

2.) Kepler's Second Law: A line joining any planet to the sun sweeps out *equal areas in equal times* (i.e., $dA/dt = constant$).

a.) This is called the Law of Areas.

b.) Proof (although you won't be expected to reproduce this, it is instructional to see how it is achieved):

i.) The area of a triangle is one-half its *base* times its *height*. The differential area of the triangle shown to the right in Figure 10.3 is:

$$\begin{aligned} dA &= (1/2)(\text{height})(\text{base}) \\ &= (1/2)\quad r \quad [r(d\theta)] \\ &= \ .5\, r^2(d\theta). \end{aligned}$$

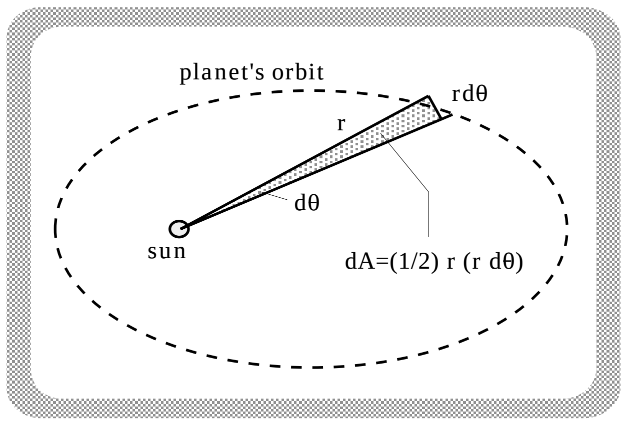

FIGURE 10.3

ii.) IF the area swept out by a planet were to *increase at a constant rate* (i.e., dA/dt is constant) no matter where the planet was in its orbit, then in a specific time interval (say, 10 seconds) the area swept out would always be the same.

Put another way, if we can show that dA/dt is a constant, we will have proved Kepler's Second Law.

To start:

$$\begin{aligned} dA/dt &= d[.5\, r^2 d\theta]/dt \\ &= .5(2r)(dr/dt)d\theta + .5r^2(d\theta/dt). \end{aligned}$$

iii.) A derivative implies that Δt goes to zero. When that happens, $d\theta$ goes to zero. Noting that $d\theta/dt = \omega$, our expression becomes:

$$\begin{aligned} dA/dt &= .5(2r)(dr/dt)d\theta + .5r^2(d\theta/dt). \\ &= .5(2r)(dr/dt)(0) + .5r^2(\omega) \\ &= .5r^2(\omega). \end{aligned}$$

iv.) We know that both r and ω are varying in time, but does their product vary in time? To see, consider the planet's *angular momentum*.

$$\begin{aligned} |\mathbf{L}| &= |\mathbf{r}\text{x}\mathbf{p}| \\ &= r\,(mv)\sin\,\theta. \end{aligned}$$

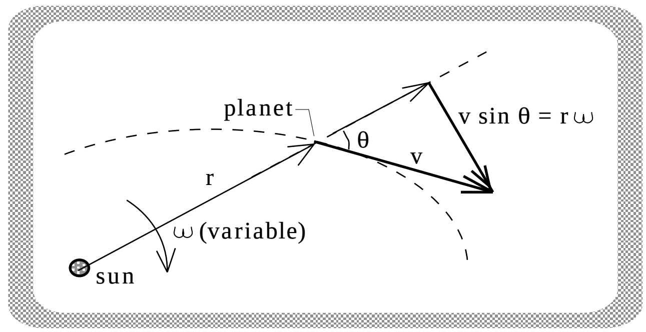

FIGURE 10.4

v.) The component of velocity *perpendicular to r* is $v \sin\,\theta$. It is also equal to $r\omega$ (see Figure 10.4). As such, we can write:

$$\begin{aligned} |\mathbf{L}| &= r\,[m(v\sin\,\theta)] \\ &= r\,[m\quad(r\omega)\quad] \\ &= mr^2\omega. \end{aligned}$$

vi.) *Angular momentum* for planetary motion *is constant*. How do we know this? The only force exerted on the planet is gravitational. That means the force is directed radially along the line between it and the sun (we are assuming there are no other large celestial objects--planets included--exerting a gravitational force on the planet). If F is parallel to r, the angle between the line of the two vectors is zero and the cross product $\boldsymbol{rxF}$ is zero. In other words, there are no external torques acting on the planet and, hence, its *angular momentum* must be constant.

vii.) The planet's *rate of change of area* is:

$$dA/dt = (.5)r^2\omega,$$

whereas its CONSTANT *angular momentum* is:

$$L = (m)r^2\omega.$$

That is, both are proportional to the same quantity $r^2\omega$.

viii.) Conclusion: The *angular momentum* of a planet is proportional to $r^2\omega$ and is a CONSTANT. As dA/dt is also proportional to $r^2\omega$, it must also be a CONSTANT. Thus, Kepler's Second Law is secure.

Note: As was stated at the beginning, this proof is *not* something on which you will be tested.

3.) Kepler's Third Law: The *square of the period* of any planet about the sun is proportional to the *cube of the planet's mean distance* from the sun.

a.) This is called the Law of Periods.

b.) Theoretical justification (the ins-and-outs of the following ARE IMPORTANT--you will see problems based on it on your next test):

i.) Consider two celestial objects (a planet and its moon or two stars in a binary star system . . . whatever) of arbitrary mass M and m moving in circular orbits. Assume also that $M > m$ (see Figure 10.5).

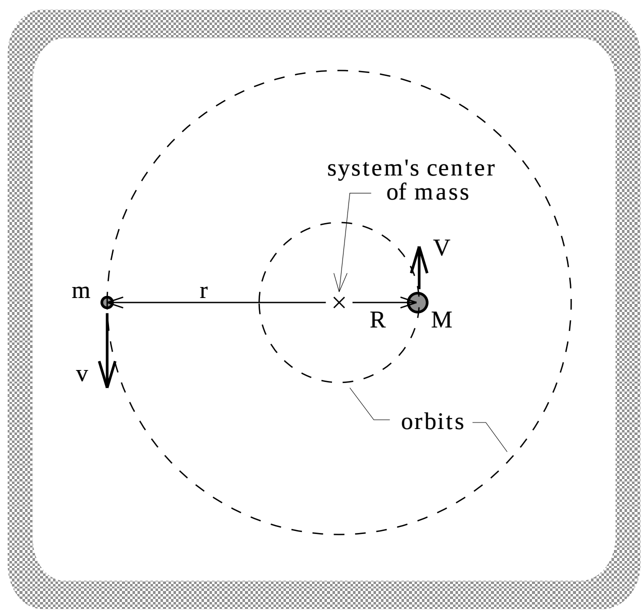

FIGURE 10.5

ii.) In such cases, the two bodies will *not* orbit one another but rather will orbit around the *center of mass* of the two-body system (our moon appears to be orbiting a stationary Earth only because the earth is so massive that the *center of mass* of the earth-moon system is *inside* the earth about two-thirds of the way from the earth's center).

The analysis we are about to do is general to all elliptical orbits even though we have chosen a circular path for simplicity.

iii.) The only force accelerating mass *m* is the gravitational pull provided by mass *M*. Noting that that force is centripetal, we can use N.S.L. to write:

$$\underline{\sum F_c:}$$

$$GmM/(r+R)^2 = ma_c$$
$$= m(v^2/r).$$

Note: There will be times when the distance term in the centripetal force expression and the distance term in the gravitational force expression will, to a very good approximation, be the same. *This will not always be the case*, as the above situation shows.

iv.) We can determine *m's* velocity by noting that in the time required to orbit once (i.e., in a time interval equal to the planet's period *T*), the object moves a distance equal to the circumference of its path, or *2πr meters*. The ratio of *distance traveled* to *time* gives us the desired velocity magnitude. Doing so yields:

$$v = (2\pi r)/T.$$

v.) Substituting this into our force equation yields:

$$GmM/(r+R)^2 = m(v^2/r)$$
$$= m[(2\pi r/T)^2/r].$$

vi.) Dividing out the *m* terms and rearranging yields:

$$T^2 = [4\pi^2/GM]\ [r(r+R)^2].$$

vii.) The question? What happens if *M* is much larger than *m*? As *r>>R* (">>" means *much greater than*), our equation becomes:

$$T^2 = [4\pi^2/GM]r^3.$$

viii.) This is Kepler's Third Law: When dealing with planets (i.e., relatively small celestial objects) orbiting the sun (i.e., a much larger celestial object), the square of the period (T^2) will be proportional to the cube of the mean distance (r^3) between the sun and the planet (in the case of elliptical orbits, this distance *r* equals the length of the semi-major axis).

Note: Kepler's Third Law is the only one of his laws that is not exact. It is predicated on the assumption that one object is large while the other is small in comparison. In our solar system, the only planet that is even remotely close to the sun's mass is Jupiter . . . and it is only a few percent of the mass of the sun.

4.) The APPROACH outlined above is very powerful in generating useful equations when dealing with orbital motion. Knowing that $T^2 \propto r^3$ is nice, but your goal should be to understand how the result was acquired.

a.) The technique was simple: Use N.S.L. and the fact that the motion is *centripetal* to generate an expression for the velocity of the body in its orbit.

b.) Note that the *period* and *velocity* are related by $v = (2\pi r)/T$.

D.) Gravitation Inside a Massive Object:

1.) A seeming paradox arises when one tries to use Newton's gravitational force expression to determine the force on a mass m_1 when inside the earth (i.e., for $r < R_e$). Specifically, if the magnitude of that force is:

$$F_g = Gm_1m_e/r^2,$$

what happens when the mass is at the earth's center and $r = 0$? Mathematically, the above force expression explodes.

2.) The problem is solved by observing a peculiarity about inverse square functions.

a.) Assume a tunnel has been drilled completely through the earth. An object of mass m is placed in the tunnel a distance r from the earth's center. What is the gravitational force acting on the body?

b.) Before attempting to do this problem, draw a horizontal line through the object and rotate it to generate two sections, one below and one above the body. Once done, construct the sphere upon which the object sits (call its radius r). The set-up is shown in Figure 10.6 on the next page.

c.) What can we say about the gravitational force acting on the body due to the mass above the line (note that this mass is all *outside* the inner sphere)?

i.) The direction of the net force will be vertically upward.

ii.) The magnitude of the force will be the consequence of a relatively small amount of mass that is located, on average, relatively close to the object.

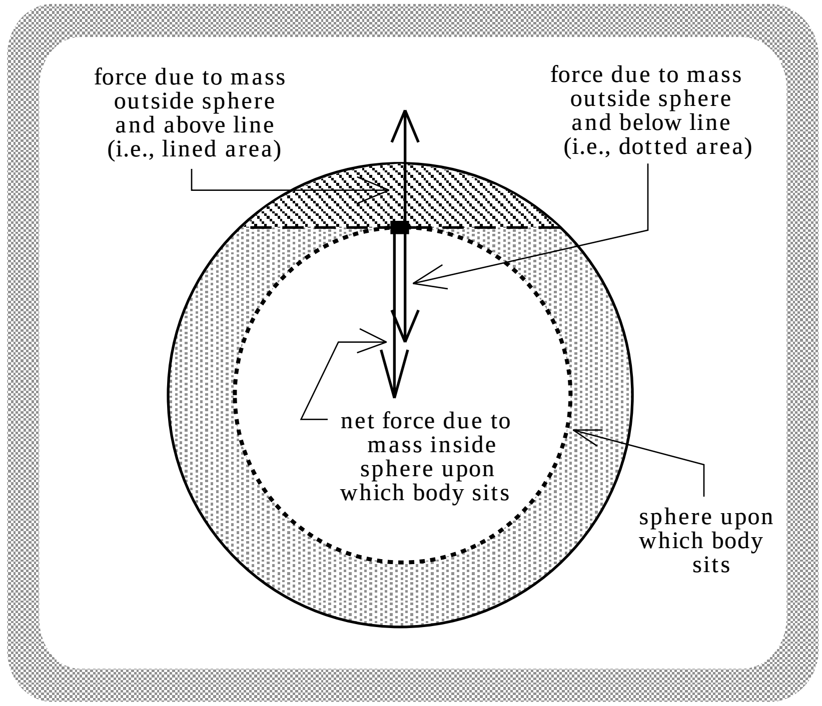

FIGURE 10.6

d.) What can we say about the gravitational force acting on the body due to the mass in the lower section and *outside the sphere*?

i.) The magnitude of this force will be downward; and

ii.) Its magnitude will be the consequence of a relatively large amount of mass that is located, on average, relatively far from the object.

e.) Although it might not be obvious, the downward force due to the mass in the lower section *outside the sphere* will be equal in magnitude to the upward force due to the mass in the upper section *outside the sphere*.

f.) How can this be? It is a consequence of the inverse square law. Assuming symmetry, the net gravitational force on an object will be generated only by the mass found *inside the sphere upon which the object rests*. If the gravitational force function had not been of the form $1/r^2$, this would not be true (for those with nothing better to do, you might find it amusing to try to prove this).

Note: This holds true even when the object is on the earth's surface. In that case, the mass *inside the sphere upon which the object rests* is the total mass of the earth.

3.) With this observation, all we have to do is determine the amount of mass inside the sphere upon which the object rests, then use that mass in our gravitational expression:

$$F_{grav} = G(m_{\text{inside sphere}})m_{obj}/r^2.$$

4.) The easiest way to determine m_{inside} is to calculate the *proportion* of mass inside the sphere (i.e., $V_{inside}/V_{total} = (4/3)\pi r^3/(4/3)\pi r_e^{\,3}$), then multiply that by the total mass of the earth m_e. The solution is $m_{inside} = (r^3/r_e^{\,3})m_e$. The problem with this approach, at least from an instructional perspective, is that it only works for homogeneous bodies. The approach presented below works for any homogeneous or inhomogeneous body as long as there is both symmetry and a known *volume density function* for the system.

a.) Assume the earth is homogeneous with a constant volume density function ρ equal to:

$$\begin{aligned}\rho &= m_e/V_e \\ &= m_e/[(4/3)\pi r_e^{\,3}],\end{aligned}$$

where m_e is the earth's mass, r_e is the earth's radius, and $(4/3)\pi r_e^{\,3}$ is the earth's total volume.

b.) Consider the spherical shell of radius a and differential thickness da (see Figure 10.7). The differential volume dV of the shell equals the surface area of the shell $(4\pi a^2)$ times the shell's thickness da. Written mathematically, this is:

$$dV = (4\pi a^2)da.$$

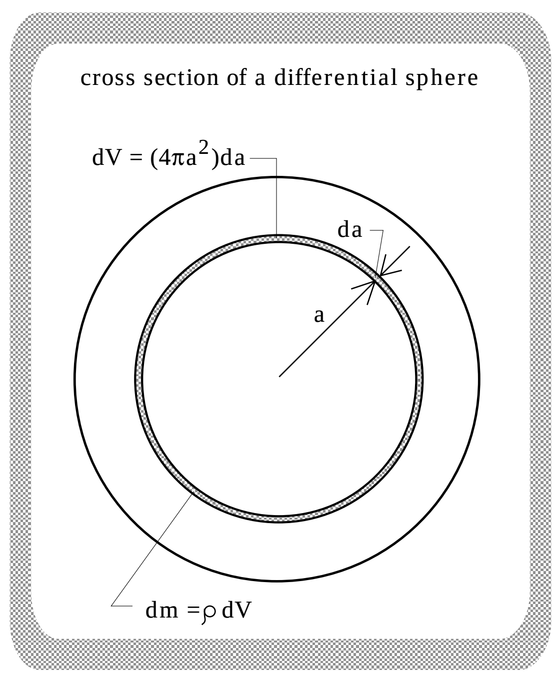

FIGURE 10.7

c.) The differential mass *dm* in the differential volume *dV* equals:

$$dm = \rho dV.$$

Substituting ρ and *dV* into the above expression yields:

$$\begin{aligned} dm &= [m_e/[(4/3)\pi r_e^3]]\ [(4\pi a^2)da] \\ &= (3m_e/r_e^3)a^2 da. \end{aligned}$$

d.) Summing all the *dm's* between $a = 0$ to $a = r$ will give us the total mass inside the sphere upon which the body sits.

5.) Using all this in conjunction with Newton's universal gravitational equation yields:

$$\begin{aligned} F_{grav} &= -G\frac{m_1\left[m_{\text{inside sphere}}\right]}{r^2} \\ &= -G\frac{m_1\left[\int_{a=0}^{r} dm\right]}{r^2} \\ &= -G\frac{m_1\left[\left(\frac{3m_e}{r_e^3}\right)\int_{a=0}^{r} a^2 da\right]}{r^2} \\ &= -G\frac{m_1}{r^2}\left[\left(\frac{3m_e}{r_e^3}\right)\left[\frac{a^3}{3}\right]_{a=0}^{r}\right] \\ &= -\left(G\frac{m_1 m_e}{r_e^3}\right)r. \end{aligned}$$

6.) In other words, though the force between any two bits of the system is proportional to $1/r_{bit}^2$, where r_{bit} is the distance between the two bits, the expression that characterizes the overall gravitational force on a body situated somewhere *inside* a larger structure is a linear function of *r*, where *r* is the radius of the sphere upon which the body sits. Being linear, that function equals zero at $r = 0$, just as we would expect.

7.) This approach works even when a body is symmetric but inhomogeneous (the earth, for instance, is not a uniformly solid object). In such cases, the approach is exactly as presented above with the variable *dm* equaling $\rho\, dV$. The only alteration is in the fact that ρ is not constant for such problems--it must be defined as a function of *r*.

E.) Potential Energy and Gravitational Fields:

1.) Until now, our energy dealings with gravity have been solely from the perspective of a body near the earth's surface. That is, we have assumed a constant gravitational force field and have defined an appropriate gravitational *potential energy function* for the situation (i.e., a potential energy function that is linear--*mgy*--and that has a floating *zero potential energy level*).

In actuality, the earth's gravitational force field varies, diminishing as we get farther and farther away from the planet. The field is still conservative, but a linear *potential energy function* is no longer appropriate. In short, a different *potential energy function* must be derived.

2.) To determine a force field's *potential energy function*, we need to determine the amount of work the field does as a body moves from the *zero potential energy point* to some arbitrary point in the field. In the case of gravity, the *zero point* must be at infinity (that is where the gravitational force due to the field producing body is itself zero).

To make the calculation simple, assume that the field producing mass m_1 is at the origin of our coordinate axis and that m_2 moves in from infinity along the *x-axis* to a position *r* units from m_1 (this means that *r* in this equation is the *magnitude* of the distance between m_1*'s* center of mass and m_2*'s* center of mass). Mathematically, this operation looks like:

$$
\begin{aligned}
U(r) - U(\infty) &= -\int \mathbf{F} \bullet d\mathbf{r} \\
&= -\int_{x=\infty}^{r} \left(-G\frac{m_1 m_2}{x^2}\mathbf{i}\right) \bullet (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\
&= Gm_1 m_2 \int_{x=\infty}^{r} \left(\frac{1}{x^2}\right) dx \\
&= Gm_1 m_2 \left(\frac{-1}{x}\right)\Bigg|_{x=\infty}^{r} \\
&= -G\frac{m_1 m_2}{r}.
\end{aligned}
$$

3.) The negative sign should not be upsetting if you remember that potential energy functions make sense only when used in the context of $W_F = -\Delta U_F$. That is, only *changes* of potential energy have any meaning. As long as the *potential energy change* between two points yields the net work done on a body by the field as the body moves from the one point to the other point in the field, all is well. If there happens to be a negative sign in the expression that defines the potential field, so be it.

Big Note: The potential energy defined by this potential energy function is *not* zero at the earth's surface--it is zero at infinity. As such, a body resting on the earth's surface *has potential energy*. Don't forget this when using the *conservation of energy* equation.

F.) Potential Energy Wrapped up in a System of Bodies:

1.) To determine the two-body, gravitational potential energy function in the section above, we used the definition of *potential energy* to determine the amount of work gravity did on a body that was brought in from infinity to some arbitrarily chosen point *r*.

In its most general form, that calculation is denoted as:

$$U(r) - U(\infty) = -\int \mathbf{F}\cdot d\mathbf{r}.$$

Using the above form, the scalar expression derived for gravity was:

$$U(r) = -Gm_1m_2/r,$$

where *r* was effectively the distance between the *centers of mass* of the two bodies.

2.) The question arises, "How much potential energy is wrapped up in a system of masses?" To answer this question, we must extend our thinking just a bit. Specifically:

a.) Assume we want to assemble a three-mass system, the first mass of which is m_1.

b.) We have already derived an expression for the amount of *potential energy* the system will have after m_2 moves from infinity to a position a distance $r_{1,2}$ units from m_1. That expression is:

$$U_1(r_{1,2}) = -Gm_1m_2/r_{1,2}.$$

c.) Continuing in this vein, if a mass m_3 is brought in from infinity to a distance $r_{1,3}$ from mass m_1 and a distance $r_{2,3}$ from mass m_2, the ADDITIONAL gravitational potential energy provided to the system due to the presence of this new mass will be:

$$U_2(r_{1,3}) + U_3(r_{2,3}) = (-Gm_1m_3/r_{1,3}) + (-Gm_2m_3/r_{2,3}).$$

d.) The *total potential energy* of the system will therefore be:

$$U_{tot} = -Gm_1m_2/r_{1,2} - Gm_1m_3/r_{1,3} - Gm_2m_3/r_{2,3}.$$

Note 1: It doesn't matter which mass is placed first or which is brought in last, this expression will always be the same.

Note 2: If there were a fourth mass brought in, the expression would have *six* terms in it.

3.) It is interesting to note that the potential energy of this system is negative. What significance has this? Negative potential energy means the system is bound. Put another way, if you want to disassemble the system, you will have to provide to the system energy in the amount of $Gm_1m_2/r_{1,2} + Gm_1m_3/r_{1,3} + Gm_2m_3/r_{2,3}$.

This is called the *binding energy* of the system. It is defined as the amount of energy required to disassemble the system so that the parts no longer interact with one another. In the case of gravitational systems, it is the amount of energy required to push the masses infinitely far apart.

Note: The idea of *binding energy* is especially important in nuclear physics where there is an interest in the amount of energy required to spring a subatomic particle loose from its atom.

G.) Orbital Motion and Energy Considerations:

1.) It is interesting to see how energy distributes itself within an orbit-related system. Specifically, consider a small mass m (a satellite) orbiting in an approximately circular path at a distance r_1 units from the center of a much larger mass M (a planet). How much potential and kinetic energy is in the system, assuming that M is stationary?

a.) Mass *m's* potential energy is $U(r) = -G\frac{mM}{r}$.

Note: The *r* term in the potential energy function is defined as the distance from the center of the field-producing body (mass *M* in this case).

b.) Mass *m's* kinetic energy is $KE_r = (1/2)mv^2$.

i.) To finish this calculation, we need *v*. Consider Newton's Second Law and the fact that the motion is centripetal. Taking the direction of the gravitational force to be positive, we can write:

$\underline{\Sigma F_c:}$

$$G\frac{mM}{(r_{\text{between masses}})^2} = ma_c$$

$$= m\left(\frac{v^2}{(r_{\text{radius of circle}})}\right)$$

ii.) Note that the *r* terms on either side of this expression are the same. Why? The *r* term on the left side of N.S.L. is defined as the distance between *m's* center of mass and *M's* center of mass (if the bodies are far apart and considered point masses, this is *the distance between the two bodies*). The *r* term on the right side of N.S.L. is, according to the definition of centripetal acceleration, the radius of the circle upon which *m* travels. This is always measured from the SYSTEM'S *center of mass* (see Figure 10.8). Because *M* is so much more massive than *m* in this case, the distance between *m* and the system's center of mass is essentially the same as the distance between the two bodies, and the *r's* are equal.

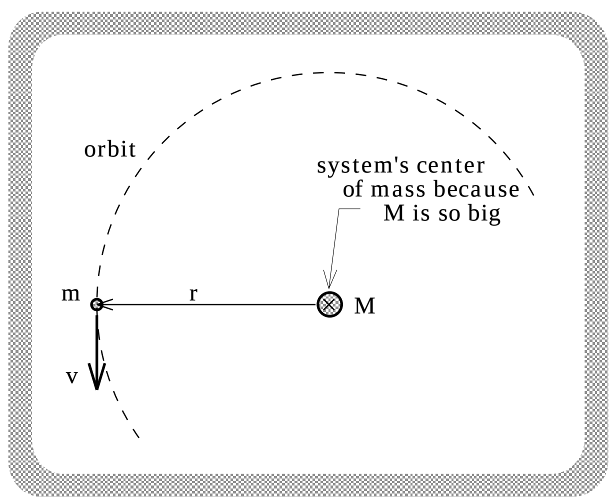

FIGURE 10.8

iii.) Solving for v, we get:

$$v^2 = \left(G\frac{M}{r} \right).$$

iv.) Finishing off the kinetic energy expression yields:

$$\begin{aligned} KE_r &= (1/2)mv^2 \\ &= \frac{1}{2}m\left(G\frac{M}{r} \right) \\ &= \frac{1}{2}G\frac{mM}{r}. \end{aligned}$$

c.) Putting it all together, we get the total energy (often referred to as *the mechanical energy*) E_{tot} of the system as:

$$\begin{aligned} E_{tot} &= \quad KE \quad + \quad U \\ &= \frac{1}{2}G\frac{mM}{r} - G\frac{mM}{r} \\ &= -\frac{1}{2}G\frac{mM}{r} \qquad \text{(this is usually written as } -G\frac{mM}{2r}\text{).} \end{aligned}$$

2.) How much energy is required to move a satellite from a circular orbit of radius r_1 to a circular orbit of radius r_2, assuming the change occurs slowly (i.e., the satellite moves in a slowly changing series of circular paths).

a.) The temptation is to use the modified conservation of energy equation and write:

$$\begin{aligned} E_{tot,1} \qquad &+ W_{extraneous} = \qquad E_{tot,2} \\ \Rightarrow \left(\frac{\mathbf{1}}{\mathbf{2}}mv_1^{\ 2} - G\frac{mM}{r_1} \right) &+ \quad W_{ext} \quad = \left(\frac{\mathbf{1}}{\mathbf{2}}mv_2^{\ 2} - G\frac{mM}{r_2} \right) \end{aligned}$$

Note that this expression needs both the radii *and* the velocities required for the satellite to sustain each orbit.

b.) BUT, because the sum of the potential and kinetic energy for a body moving in a CIRCULAR ORBIT is $-G\frac{mM}{2r}$, we can instead write:

$$\begin{aligned} E_{tot,1} + W_{ext} &= E_{tot,2} \\ \Rightarrow \quad -G\frac{mM}{2r_1} + W_{ext} &= -G\frac{mM}{2r_2} \end{aligned}$$

c.) What's nice about this relationship is that you don't have to determine the velocity of the satellite in the final orbit.

Note: This amount of energy is *not* the same as the amount of energy required to take a satellite *sitting on the earth* and put it into orbit. In that case, you need to determine the required kinetic energy, given the orbit's radius, and add that to the *potential energy* difference (i.e., $\Delta U = (-Gm_s m_e/r_s) - (-Gm_s m_e/r_e)$) between the initial and final positions of the satellite. How so? $-\Delta U$ equals the amount of work *gravity* does on the body as it ascends; minus that amount (i.e., $+\Delta U$) equals the amount of work *you* must do to overcome gravity.

3.) In some cases, a more interesting way to look at energy considerations in the context of circular orbital motion is to see how the *kinetic energy*, *potential energy*, and *total energy* play off one another.

a.) To begin with, notice that for a particular circular orbit of radius r_1, there must be a given amount of potential energy ($-GmM/r_1$), kinetic energy ($GmM/2r_1$), and total energy ($-GmM/2r_1$) in the system if a body is to stay in that orbit.

Put another way, there is only one *kinetic energy/ potential energy/ total energy* combination, given *m* and *M*, that will allow a body to hold a particular orbit.

b.) To see how these energy quantities relate, consider the *Energy versus Orbital Radius* graph shown in Figure 10.9 to the right.

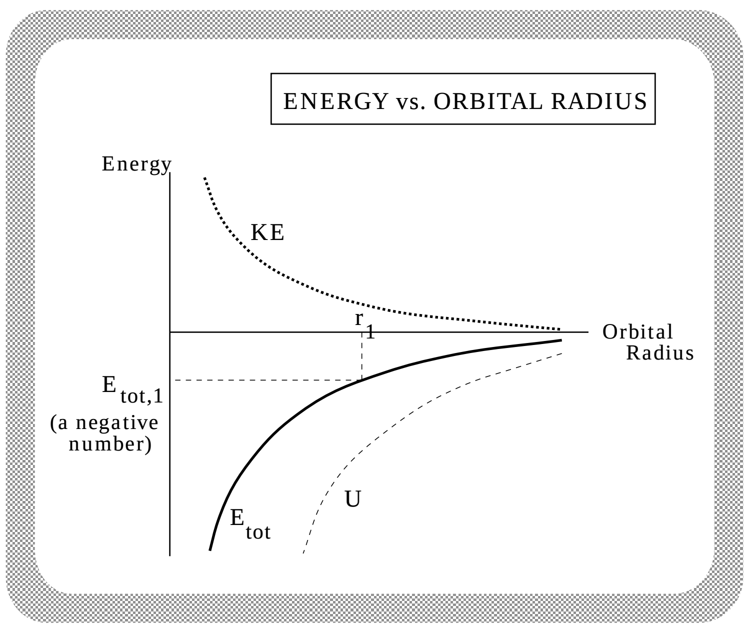

FIGURE 10.9

c.) Notice that as the radius of the motion gets larger (i.e., the bodies get further apart), the orbiting body's *kinetic energy* must decrease (i.e., it slows down) and its *potential energy* must increase (i.e., becomes less negative).

d.) This means the binding energy, as measured by the potential energy in the system, is decreasing (getting closer to zero). Put another way, as a satellite gets farther away from its orbital center, it takes less energy to release it from its orbital bonds.

4.) One last observation: If we have a satellite moving in a circular orbit around the earth, and we want to put it into a higher velocity orbit, how can we use the satellite's thrusters to make the adjustment?

a.) The temptation is to shoot the thrusters *backwards*, thus applying a forward force on the satellite. The problem here is that in doing so, we will force the satellite into elliptical motion--something we do not want to do. But even if we apply the thrusters slowly and gently, we won't get the result we want. Doing so will result in forcing the satellite into an orbit that is *farther away from the earth*. With less gravitational force being applied to the satellite (i.e., less centripetal force acting on the satellite), the satellite's motion will require a lower velocity to hold its orbit. In short, the satellite will ultimately have to slow down to hold orbit.

From a little different perspective, firing the thrusters backwards will do *positive* work on the system. Putting energy *into* the system makes the *total energy* move closer to zero (remember, the total energy of an orbiting system is *negative*--adding energy moves the total energy from a large negative number to a smaller negative number). Looking at our graph, we see that when the total energy moves toward zero, the orbital radius increases (that is what we concluded above) and the kinetic energy (hence velocity) *decreases*.

b.) We know that higher velocity is associated with higher kinetic energy. That means we need to do whatever is required to increase the body's kinetic energy. Looking at our *energy versus orbital radius* graph (Figure 10.9), we see that putting the satellite into a smaller radius orbit will increase the satellite's kinetic energy and associated velocity.

c.) The trick, therefore, is to shoot the thrusters *very gently* in the *forward* direction, thereby doing negative work. As the satellite slows, gravity will pull the satellite in closer to the earth, but it will do it uniformly throughout the satellite's motion and as a consequence, the satellite will continue in nearly circular motion.

Following on, the satellite's gravitational potential energy will decrease (i.e., increase negatively) as the satellite gets closer to the earth. Although part of that potential energy loss will be caused by the satellite's absorption of energy due to the negative work done by the thrusters, part of the potential energy loss will be due to a conversion of potential energy into kinetic energy. That is, the kinetic energy of the satellite will increase (and so will its velocity).

The observed increase of kinetic energy will *not* throw the satellite into a bigger orbit as would have been the case if we had fired the thrusters backwards. Why? Because firing backwards would have *increased* the total energy of the system whereas firing the thrusters forward *decreases* the total energy of the system. *The increase of velocity with diminished orbital radius is due to potential energy converting itself into kinetic energy. It is not due to extra energy being put into the system.*

From an *energy perspective*, firing the thruster forwards does negative work on the system. When added to the negative total energy, this *increases* the system's negative total energy. Looking at our graph to see what happens when the total energy becomes more negative, we find the orbital radius decreasing and the kinetic energy (and velocity) increasing.

H.) Energy Symmetry:

1.) In the section above, we observed a kind of symmetry between a body's orbital *kinetic* and *potential energy* due to the fact that the total energy of the motion had to be conserved. This idea of energy symmetry can be very useful in other ways.

2.) As an example, consider a stationary body poised a distance r_o units above the earth's center. Release the body and it begins to freefall (this is obviously *not* an orbital problem--the body is freefalling vertically). What does its *Kinetic Energy vs. Position* graph look like?

a.) We don't have a function for the kinetic energy of a freefalling body as its position changes, but we do have an expression for a body's *potential energy* as a function of position. That function was used to obtain the graph presented in Figure 10.10. This graph is *not* a function of *time*. It is designed to show how energy is distributed when the body is at various positions above the earth's surface. Do not be put off by the fact that the *beginning* position r_o (the word *beginning* being a time-related description) is found to the right of the so-called *final* position r_e.

b.) As there is no initial *kinetic energy*, the *total energy* must equal the initial *potential energy* in the system. That is, $E_{tot} = U_o = -GmM/r_o$.

c.) This is where the symmetry comes in. We have a *potential energy versus position* graph, and we know that the system's *potential energy* and *kinetic energy* must always add to the same number--the *constant total energy*. From symmetry, then, the *kinetic energy versus position* graph must be constructed so that for every *potential energy* decrease, a corresponding *kinetic energy* increase exists.

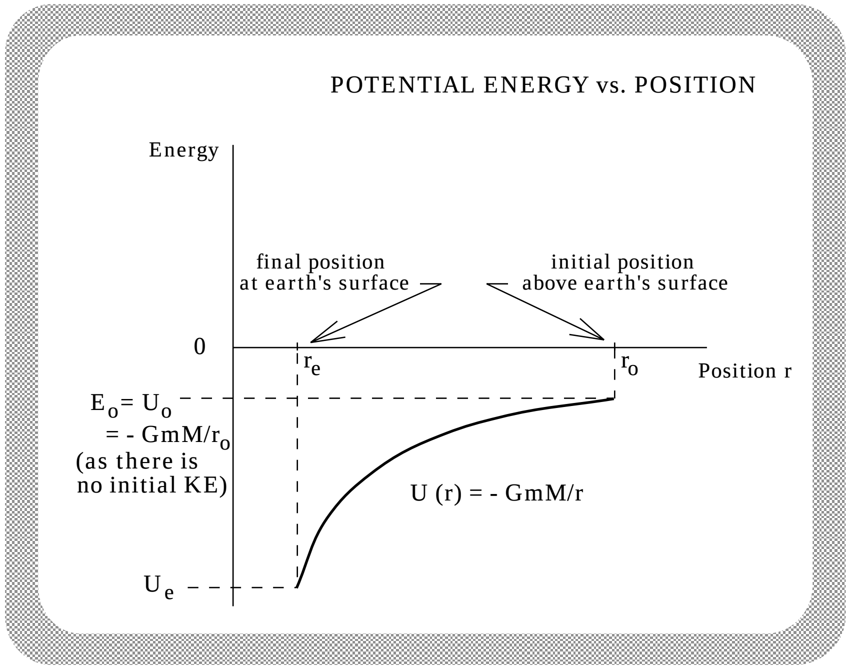

FIGURE 10.10

In other words, the general outline of our desired kinetic energy graph must be a *mirror image* of our potential energy graph.

d.) In summary, even though *kinetic energy* is a function of v^2 and the potential energy is a function of $1/r$, the two functions are symmetric as plotted. Why? Because the sum of the two evaluated at a particular point must be a constant.

e.) A graph of the desired function is shown in Figure 10.11.

f.) Having said all that, what would the body's *Potential Energy vs. Velocity* graph look like? Think about it if you have the time!

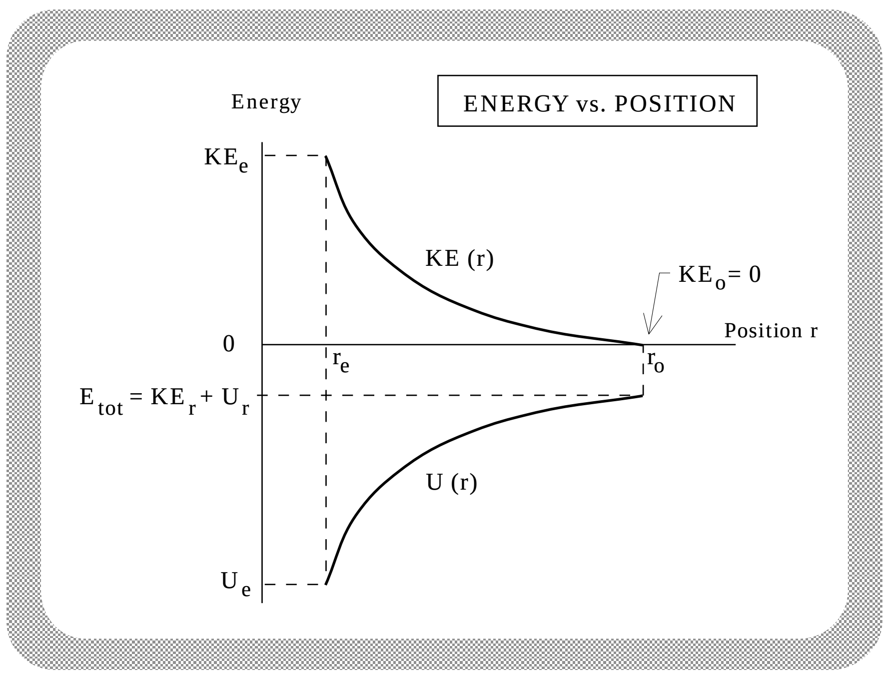

FIGURE 10.11

QUESTIONS

10.1) You and a friend sit 5 meters apart on an ice pond assumed to be a frictionless surface. Your mass is 80 kg and your friend's mass is 60 kg. If Newton was correct, you will exert a gravitational force on your friend, and vice versa (that's right, he maintained that gravitational forces act between *all* massive bodies, even little bodies like you and me). For the situation outlined above, determine:

a.) The gravitational force you exert on your friend; and

b.) The resulting acceleration of your friend.

10.2) Three bodies, each of mass m, orbit in such a way as to create an equilateral triangle (see Figure I to the right). If the distance between each body is d, determine the magnitude of the velocity of each body.

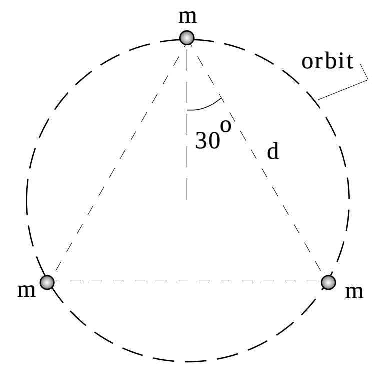

FIGURE I

10.3) The moon (mass 7.36×10^{22} kg and radius 1.74×10^{6} meters) is approximately 3.82×10^{8} meters from the earth (mass 5.98×10^{24} kg and radius of 6.37×10^{6} meters). If we could fix the earth, stop the moon, then allow the moon to freefall toward the earth:

a.) What would its initial acceleration be?

b.) What would its acceleration be just before striking the earth?

c.) How fast would it be moving when it reached the earth, assuming the earth remained stationary during the freefall?

d.) How would the problem have changed if the earth had not been assumed to be stationary?

10.4) A planet has a volume density distribution of $\rho = (m_p/r_p^{\,4})r$, where r_p and m_p are respectively the radius and mass of the planet and r is the distance from the planet's center to a point of interest. Assume a tunnel is dug through the planet.

a.) Derive an expression for the gravitational force applied to a mass m_1 when at a distance r units from the planet's center (remember, $r < r_p$).

b.) Derive a *potential energy function* for the force determined in *Part a.*

10.5) A satellite travels in a circular orbit a distance 1.3×10^6 meters above the earth's surface ($m_e = 5.98 \times 10^{24}\ kg$ and $r_e = 6.3 \times 10^6\ m$). If the satellite's mass is 400 kg;

a.) How fast is it moving?

b.) What is its period?

c.) How much energy was required to put the satellite in orbit, assuming it started from rest at the earth's surface (that is, assuming we can ignore the earth's rotational speed).

d.) How would the calculation in *Part c* have changed if we hadn't ignored the earth's rotation?

e.) For whatever reason, the satellite loses energy at a rate of 2×10^5 joules per complete orbit. Assuming its radius of motion diminishes slowly, how far *from the surface* will it be after 1800 revolutions?

f.) For the situation outlined in *Part e* above, what is the magnitude of the average retarding force acting during the 1800 orbits?

g.) For the situation outlined in *Part e*, angular momentum is *not* conserved as there is an external torque acting on the system (it is due to the drag force acting on the satellite). Modify the *conservation of angular momentum* equation to determine the amount of *time* required for the 1800 revolutions (think about how the *conservation of linear momentum* equation was modified to accommodate external forces acting over a time interval Δt).

10.6) A planet moving in an elliptical orbit has 2.5×10^{33} joules of *kinetic energy* when at its closest point in its path around its star (this is approximately the kinetic energy of the earth as it orbits the sun). If we call this closest point r_{min}:

a.) What quantities are conserved for the planet's motion?

b.) The planet has a certain amount of *potential energy* due to its proximity to the star. Figure II on the next page shows the graph of the planet's *Potential Energy versus Distance from the star*. On the same axis, graph both the planet's *Total* and *Kinetic Energy versus Distance* functions.

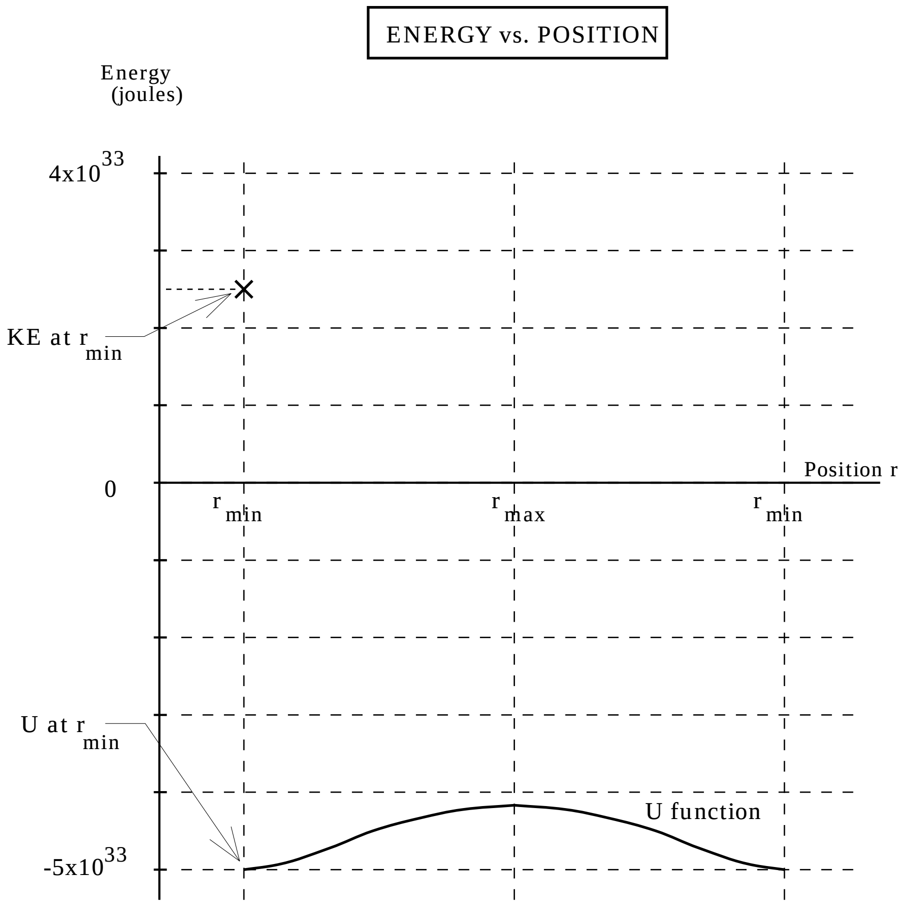
ENERGY vs. POSITION
Energy
(joules)
4x10^33
KE at r min
0
Position r
r min
r max
r min
U at r min
U function
-5x10^33

FIGURE II