CHAPTER 8 --ROTATIONAL MOTION I

8.1) Do not spend a lot of time on this problem.

a.) The integral $\int r^2 dm$ essentially tells you to identify all the mass (*dm*) a distance *r* units from the axis of interest (in this case $r = y$), multiply that *dm* by r^2, then sum up all possible $r^2 dm$ quantities using an integral.

In this case, we can define a differential strip of thickness *dy* and mass *dm* a distance *y* units from the *x axis* (i.e., the axis of rotation). Noting that the width of the strip at any particular *y* is *2x* (see Figure), that the Pythagorean relationship yields $x = (R^2 - y^2)^{1/2}$, and that the surface area density function is $\sigma = ky$, we can write:

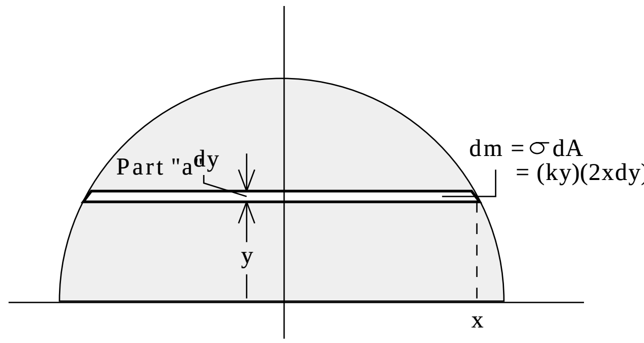


FIGURE I

$$\begin{aligned} dm &= \sigma dA \\ &= (ky)[(2x)dy] \\ &= (ky)[2(R^2 - y^2)^{1/2}dy]. \end{aligned}$$

Using this expression for *dm* in our *moment of inertia* expression, we get:

$$\begin{aligned} I &= \int y^2 dm \\ &= \int_{y=0}^{R} y^2\left[(ky)\left[2\left(R^2 - y^2\right)^{1/2} dy\right]\right] \\ &= 2k\int_{y=0}^{R} y^3\left(R^2 - y^2\right)^{1/2} dy \\ &= 2k\left[\left(\frac{\left(R^2 - y^2\right)^{5/2}}{5} - \frac{R^2\left(R^2 - y^2\right)^{3/2}}{3}\right)\right]_{y=0}^{R} \\ &= 2k\left[\left(\frac{\left(R^2 - R^2\right)^{5/2}}{5} - \frac{R^2\left(R^2 - R^2\right)^{3/2}}{3}\right) - \left(\frac{\left(R^2 - (0)^2\right)^{5/2}}{5} - \frac{R^2\left(R^2 - (0)^2\right)^{3/2}}{3}\right)\right] \\ &= .26kR^5. \end{aligned}$$

Note: How did we solve this integral? There is certainly a nifty way to do it using the approaches taught in your Calculus class, but not remembering the technique off hand, I simply looked it up in a *book* of integrals.

Bottom line: If you get a problem like this on a test, *setting up* the integral will be worth 95% of the problem.

b.) This geometry is more difficult than that in the previous problem because the given density is a function of y while the symmetry is radial. That is, if the body was homogenous we could define a differential half-hoop of radius r and thickness dr (note that the mass in the hoop is all the same distance from the *z axis*), determine the amount of mass dm in that section (see Figure II), and multiply that mass by the square of the distance r from the *z-axis*.

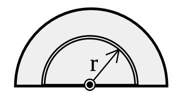


FIGURE II

Unfortunately, the mass distribution is not a function of r, it is a function of y.

To circumvent this problem, we must take an arbitrary *slice* of the hemisphere, *section* the slice into pieces, pick one piece whose height is defined as dr and whose width is $r\,d\theta$ (see Figure IIIa), determine the amount of mass in the section, sum over all of the possible sections in the slice, then sum over all of the slices.

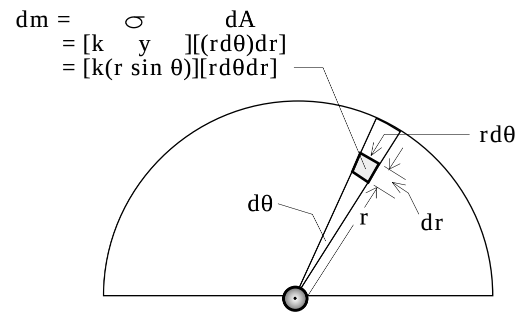


FIGURE IIIa

Noting that $y = r \sin\theta$ (see Figure IIIb) and using the expression for dm shown in the sketch, we can use our *moment of inertia* expression to write:

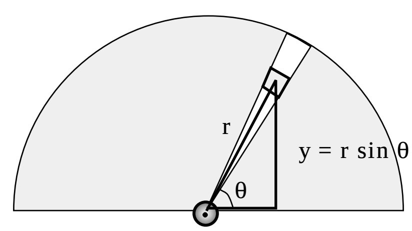


FIGURE IIIb

$$
\begin{aligned}
I &= \int r^2 dm \\
&= \int r^2 \left[k(r^2 \sin\theta)dr d\theta\right]. \\
&= \int k(r^4 \sin\theta)dr d\theta.
\end{aligned}
$$

This is a double integral. Fortunately, the radial variable r is not dependent upon the angular variable θ, so the integral can be re-written as shown below (in doing so, the evaluation becomes nowhere near as difficult as one might expect at first glance).

The integral, after variable separation, is:

$$I = k\int r^4 dr \int \sin\theta d\theta.$$

Doing the integral yields:

$$
\begin{aligned}
I &= k\int_{r=0}^{R} r^4 dr \int_{\theta=0}^{\pi} \sin\theta d\theta \\
&= \left[k\left[\frac{r^5}{5}\right]_{r=0}^{R}\right]\int_{\theta=0}^{\pi} \sin\theta d\theta \\
&= \frac{kR^5}{5}\left[-\cos\theta\right]_{\theta=0}^{\pi} \\
&= \frac{kR^5}{5}\left[(-\cos\pi)-(-\cos 0)\right] \\
&= \frac{kR^5}{5}\left[(-(-1))-(-1)\right] \\
&= \frac{2kR^5}{5}.
\end{aligned}
$$

The moral of the story? If you understand what the *moment of inertia* expression is asking you to do, using it in a problem like this isn't that hard. Setting up the integral may take a little thinking, but its underlying tenets are fairly transparent. Find all the mass a distance *r* from the axis of interest, call that mass *dm*, then multiply *dm* by that distance *r* squared.

8.2) This problem has been included to give you a look at the basic kinematic and rotation/translational relationships. Don't spend a lot of time on it; it's more to get you familiar with the ideas than anything else.

a.) Using dimensional analysis, going from *km/hr* to *m/s* requires converting kilometers to meters and hours to seconds. The best way to do this is by simply placing the units first, then by plugging in the numbers. That is, the relationship has the units of *hours* in the denominator. That means you want to multiply by a quantity that has *hours* in the numerator and the desired *seconds* in the denominator, or *hours/second.* In that way the *hour* terms cancel out and we are left with seconds in the denominator. Similar reasoning follows the *kilometer* variable in the numerator. Doing the calculation yields:

(km/hr)(hr/sec)(m/km).

Putting in the numbers, we get:

(54 km/hr)[(1 hr)/(3600 sec)][(1000 m)/(1 km)] = 15 m/s.

b.) The relationship between the velocity of the *center of mass* of a rolling object (i.e., one that both translates and rotates) is:

$$v_{cm} = R\omega.$$

Minor side point: The temptation is to assume that the *R* term is simply the radius of the round object. Although the value for *R* is equal to the radius of the object in this case, *R* in this expression is really telling us how many meters there are on the arc of a one radian angle (remember, we are relating a linear measure v_{cm} to an angular measure ω). Using the relationship $v_{cm} = R\omega$, rearranging, then plugging in the numbers, we get:

$$\begin{aligned}\omega &= v_{cm}/R \\ &= (15 \text{ m/s})/(.3 \text{ m/rad}) \\ &= 50 \text{ rad/sec.}\end{aligned}$$

Note: Because we have been careful with our units, we get the correct units for the *angular velocity*.

As for *4 m/s* corresponding to *13.33 rad/sec*, we can use $v_{cm} = R\omega$ to write:

$$\begin{aligned}&4 \text{ m/s} = (.3 \text{ m})\ \omega \\ \Rightarrow\ &\omega = 13.33 \text{ rad/sec.}\end{aligned}$$

c.) If the *angular displacement* had been one rotation--2π radians--the distance traveled by a point on the wheel's edge would have been the arclength of an angular displacement $\Delta\theta$ equal to 2π radians, or $\Delta s = 2\pi R$ meters (remember, the relationship between arclength Δs, radius-to-the-point-in-question *r*, and angular displacement $\Delta\theta$ is $\Delta s = r\,\Delta\theta$). If the *angular displacement* had been two rotations--4π radians--the distance traveled would have been $4\pi R$ meters

If we lay the arclength Δs out flat, we will get the *total linear distance* the wheel traveled. We know that distance. If we additionally know the radius of the wheel, we can write:

$$\begin{aligned}&\Delta s = R\Delta\theta \\ \Rightarrow\quad &\Delta\theta = \Delta s/R \\ &\quad\ = (50 \text{ m})/(.3 \text{ m/rad}) \\ &\quad\ = 166.7 \text{ radians.}\end{aligned}$$

d.) Angular acceleration:

$$(\omega_2)^2 = (\omega_1)^2 + 2\alpha(\theta_2 - \theta_1)$$
$$(13.33 \text{ rad/s})^2 = (50 \text{ rad/s})^2 + 2a(166.7 \text{ rad})$$
$$\Rightarrow \quad \alpha = -6.97 \text{ rad/s}^2.$$

e.) We know the relationship between *angular acceleration* of a body and the magnitude of the *translational acceleration* of a point a distance *r* units from the axis of rotation. Using it we get:

$$\begin{aligned} a &= r\alpha \\ &= (.3 \text{ rad/m})(-6.97 \text{ rad/s}^2) \\ &= -2.09 \text{ m/s}^2. \end{aligned}$$

f.) For elapsed time using $(\theta_2 - \theta_1) = \Delta\theta$, we get:

$$\Delta\theta = \omega_1 \Delta t + (1/2)\alpha(\Delta t)^2$$
$$(166.7 \text{ rad}) = (50 \text{ rad/s})t + .5(-6.97 \text{ rad/s}^2)t^2.$$

Using the quadratic formula, we get:

$$t = 5.27 \text{ seconds.}$$

g.) For elapsed time using ω_2 and not $\Delta\theta$:

$$\alpha = (\omega_2 - \omega_1)/\Delta t$$
$$\Rightarrow \quad \Delta t = [(13.33 \text{ rad/s} - 50 \text{ rad/s})]/(-6.97 \text{ rad/s}^2)$$
$$= 5.26 \text{ sec}$$ (yes, *Parts f* and *g* match).

h.) For the average angular velocity:

$$\begin{aligned} \omega_{avg} &= (\omega_2 + \omega_1) / 2 \\ &= [(13.33 \text{ rad/s}) + (50 \text{ rad/s})]/2 \\ &= 31.67 \text{ rad/s.} \end{aligned}$$

i.) Determine angular displacement $\Delta\theta$:

$$\Delta\theta = \omega_1 \Delta t + (1/2)\alpha(\Delta t)^2$$
$$= (50 \text{ rad/s})(.5 \text{ s}) + .5(-6.97 \text{ rad/s}^2)(.5 \text{ s})^2$$
$$= 24.13 \text{ rad.}$$

j.) *How far* translates as, "What was the arclength of the wheel's motion?"

$$\Delta s = R\Delta\theta$$
$$= (.3 \text{ m/rad})(24.13 \text{ rad})$$
$$= 7.24 \text{ m.}$$

k.) Angular velocity calculated <u>without</u> using *time* variable:

$$(\omega_3)^2 = (\omega_1)^2 + 2\alpha\,\Delta\theta$$
$$= (50 \text{ rad/s})^2 + 2(-6.97 \text{ rad/s}^2)(24.13 \text{ rad})$$
$$= 46.51 \text{ rad/s.}$$

l.) Angular displacement $\Delta\theta$:

$$\Delta\theta = \omega_3 \Delta t + (1/2)\alpha(\Delta t)^2$$
$$= (46.51 \text{ rad/s})[(.7 \text{ s}) - (.5 \text{ s})] + .5(-6.97 \text{ rad/s}^2)(.2 \text{ s})^2$$
$$= 9.16 \text{ rad.}$$

m.) From the information given in *Part b*, we know that the angular velocity of our wheel when moving at 4 m/s is 13.33 rad/sec. We can solve for $\Delta s = R\Delta\theta$ if we know $\Delta\theta$ during the motion (R is the radius of the wheel, or .3 m). We can use $\Delta\theta = \omega_1 \Delta t + (1/2)\alpha(\Delta t)^2$ if we know α. We start:

$$\alpha = (\omega_4 - \omega_2) / \Delta t$$
$$= [(20 \text{ rad/s}) - (13.33 \text{ rad/s})]/(3 \text{ s})$$
$$= 2.22 \text{ rad/s}^2.$$

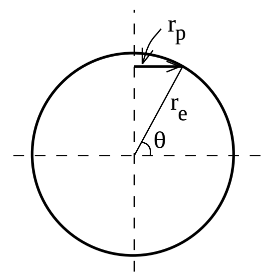
$$\Delta\theta = \omega_2 \Delta t + (1/2)\alpha(\Delta t)^2$$
$$= (13.33 \text{ rad/s})(3 \text{ s}) + .5(2.22 \text{ rad/s}^2)(3 \text{ s})^2$$
$$= 49.98 \text{ radians.}$$

$$\Delta s = R\Delta\theta$$
$$= (.3 \text{ m/rad})(49.98 \text{ rad})$$

= 14.99 meters.

8.3) We are given the earth's mass at $m_e = 5.98x10^{24}\ kg$, the earth's period $T = 24\ hours$ ($8.64x10^4$ seconds--use dimensional analysis to get this if you don't believe me), and its radius at $r_e = 6.37x10^6\ meters$.

a.) The earth rotates through 2π radians in 24 hours ($8.64x10^4$ seconds). Its angular displacement per unit time (i.e., its angular velocity) is, therefore:



$$\omega = \Delta\theta/\Delta t$$
$$= (2\pi \text{ rad})/(8.64x10^4 \text{ s})$$
$$= 7.27x10^{-5} \text{ rad/s}.$$

b.) The earth's equatorial velocity (magnitude) is equal to the linear distance a point on the equator travels per unit time. That is:

$$v_{eq} = (2\pi \text{ rad})(6.37x10^6 \text{ m/rad})/(8.64x10^4 \text{ s})$$
$$= 463.2 \text{ m/s} \qquad \text{(this is about 1000 mph)}.$$

Note: According to theory, v_{equ} should equal $R\omega_e$. Putting the numbers in yields:

$$R\omega_e = (6.37x10^6 \text{ m})(7.27x10^{-5} \text{ rad/s})$$
$$= 463.1 \text{m/s}.$$

Given round-off error, this is close enough for government work.

c.) Looking at the sketch above, the radius of the circle upon which that particle will be traveling will be:

$$r_p = r_e \cos 60^o$$
$$= (6.37x10^6 \text{ m})\ (.5).$$

It takes the same amount of time T for the particle at 60^o to travel through one rotation as it does for a particle on the equator, so:

$$v_p = 2\pi r_p/T$$
$$= (2\pi)(3.185\text{x}10^6 \text{ m})/(8.64\text{x}10^4 \text{ s})$$
$$= 231.6 \text{ m/s}.$$

Does this make sense? Sure it does. As you approach the geographic north pole the travel velocity should go to zero.

d.) Using the table at the end of Chapter 8 in your text, the *moment of inertia* of a solid sphere I_{ss} is:

$$I_{ss} = (2/5)MR^2$$
$$= .4(5.98\text{x}10^{24} \text{ kg})(6.37\text{x}10^6 \text{ m}))^2$$
$$= 9.7\text{x}10^{37} \text{ kg}\cdot\text{m}^2.$$

<u>Bottom line:</u> It's going to take a mighty stiff cosmic breeze to change the earth's rotational motion!