Chapter 7

M O M E N T U M

A.) Momentum and Impulse:

1.) What parameters are important if you want to determine how relatively large an applied force must be to stop a moving body *in a given amount of time*?

Answer? The body's *resistance to motion-change* (i.e., its inertia . . . or mass) and *how fast* it's moving (i.e., its velocity).

When Newton developed his Second Law, these insights motivated him to multiply mass m and velocity $\boldsymbol{v}$ together to create a new vector. He called it *momentum*, he labeled it $\boldsymbol{p}$ (because he couldn't spell--he thought it was *pmomentum* . . . just kidding), and he defined it as:

$$\mathbf{p} = m\mathbf{v}.$$

He then observed that the *net force* $\boldsymbol{F}_{net}$ applied to a *mass* vectorially equals the *rate at which the momentum changes with time*, or:

$$\begin{aligned} \mathbf{F}_{net} &= d\mathbf{p}/dt \qquad \text{(Equation A)} \\ &= d(m\mathbf{v})/dt \\ &= m[d\mathbf{v}/dt] + [dm/dt]\mathbf{v}. \end{aligned}$$

Setting $d\boldsymbol{v}/dt = \boldsymbol{a}$, and assuming the mass does not change in the system (i.e., $dm/dt = 0$), this equation becomes $\boldsymbol{F}_{net} = m\boldsymbol{a}$--the form of *Newton's Second Law* you have come to know and love.

2.) Taking *Equation A* and multiplying both sides by *dt* yields:

$$\mathbf{F}_{net}dt = d\mathbf{p}.$$

The quantity $\boldsymbol{F}_{net}dt$ is called the *impulse* and is equal to the differential change of a body's momentum $d\boldsymbol{p}$ during the differential time period dt. Over a *long* period of time, assuming $\boldsymbol{F}_{net}$ is constant, this equation becomes $\boldsymbol{F}_{net}\Delta t = \Delta \boldsymbol{p}$.

3.) Example: A 2 kg sliding puck whose initial velocity-magnitude is $v_1 = 10\ m/s$ strikes a wall at a 30^o angle and bounces off (see Figure 7.1a). If it leaves the wall with velocity-magnitude $v_2 = 10$ *m/s* (that's right, we are assuming no mechanical energy is lost during the collision), and if the collision takes a total of .02 seconds to complete, what is the *average force* applied to the puck by the wall?

Note: This is clearly an *impulse* problem. The puck's *change of momentum* can be determined, and the *time* over which the change occurred is known. All that isn't known is the variable we are interested in--$\boldsymbol{F}_{net}$.

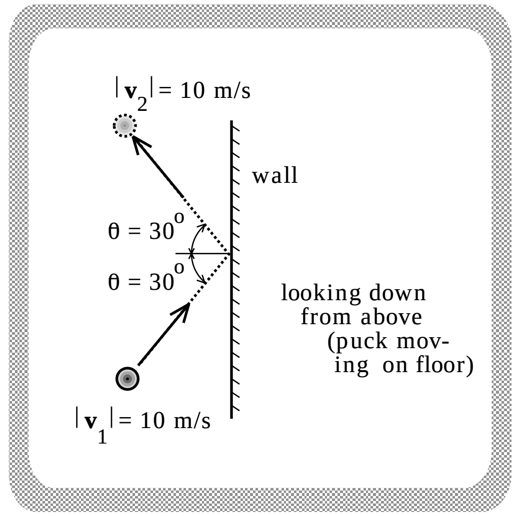


FIGURE 7.1a

a.) *Momentum* is a VECTOR. Whenever it is used in a problem, you have to treat it like a vector. Figure 7.1b shows the components of the *before-bounce* momentum $\boldsymbol{p}_1$ and the *after-bounce* momentum $\boldsymbol{p}_2$.

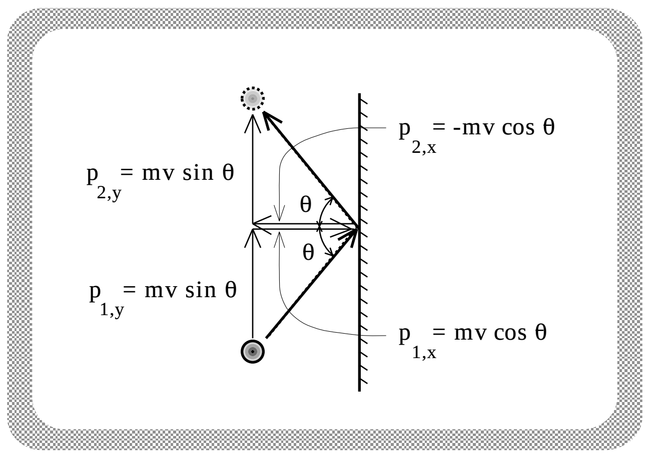


FIGURE 7.1b

b.) In the *x direction*, the incoming momentum $p_{1,x}$ is:

$$p_1 = mv_1 \cos\theta .$$

Remembering that *positive x velocities* are directed toward the *right* while *negative velocities* are directed toward the *left*, the *x component* of the outgoing momentum $p_{2,x}$ is:

$$p_2 = -\, mv_2 \cos\theta .$$

c.) The velocity-magnitudes v_1 and v_2 are equal in this problem. Calling both v for simplicity, the change of momentum in the *x direction* (hence, the impulse in the *x direction*) is:

$$\begin{aligned}\Delta(p_x) &= \quad p_{2,x} \quad - \quad p_{1,x}\\ &= (-mv\cos\theta) - (mv\cos\theta)\\ &= -2\,(mv\cos\theta)\\ &= -2\,(2\text{ kg})\,(10\text{ m/s})\cos 30^o\\ &= (-34.6\text{ kg m/s})\end{aligned}$$

$$\Rightarrow \quad F_{net,x} = \Delta p_x / \Delta t = (-34.6\text{ kg m/s}) / (.02\text{ sec}) = (-1730\text{ nts}).$$

d.) Following a similar process in the *y direction* yields $F_{net,y} = 0$ (try it).

Note: The temptation for many students when first confronted with a problem like this is to reason as follows: "The *initial momentum* is $mv_1 = 20$ *kg·m/s*. The *final momentum* is $mv_2 = 20$ *kg·m/s*. The difference between the two must, therefore, be zero. If that is true, the *net force* must be zero!"

This obviously doesn't make much sense--there has to be a force if the ball is to change its direction. The problem is that beginners often treat *momentum* quantities as though they are scalars. Don't be led astray. Momentum must be treated *as a vector*. SIGNS COUNT!

B.) Calculating the *Center of Mass* for a System of Discrete Masses:

1.) Until now, we have analyzed situations in which moving objects have been approximated as point masses (i.e., as structures that neither spin nor have pieces moving internally with different speeds). Even big objects like cars have been treated like point masses (if you know the speed of the car's radio, you know the speed of the car's transmission). The question arises, "What about *systems of objects* in which each part has its own velocity--two billiard balls colliding with one another, or a rocket that blows into a known number of pieces while in flight?"

In such cases, we need an approach that is not tied to the motion of a *specific* body but that deals with the *entire system*. What follows is the theoretical basis for doing just that.

2.) A qualitative description of the *center of mass* of a *single object* is "the *average position* of the mass in the body."

a.) A basketball, for instance, has its *center of mass* at the geometric center of the ball. No physical mass exists at that point--the basketball is, after all, hollow--but the *average* position of the mass making up the ball is at the ball's geometric center (we are assuming the ball is homogeneous--that it doesn't have a valve or any anomalies).

b.) The center of mass of an object does *not* have to be at the geometric center of the object. Anyone who has ever tried *lawn bowling* knows that when a lawn bowling ball rolls, it does not go straight.

The reason? The *center of mass* of the ball is off-set from the geometric center of the ball.

c.) Examples of a few other *center of mass* situations are shown in Figure 7.2.

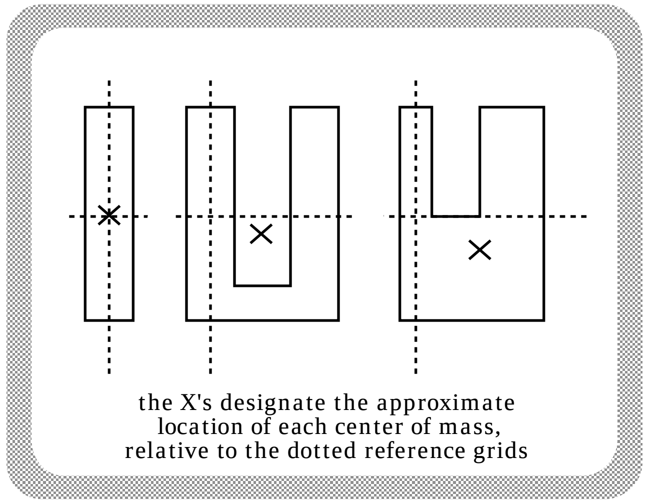


FIGURE 7.2

d.) An additional characteristic of a body's *center of mass*: when an object is thrown, its *center of mass* will follow the parabolic arc normally associated with a free-falling object. Other points on the object will not follow such an arc.

i.) Example: Throw a tomahawk end-over-end. The tip of the handle will follow a convoluted path; the *center of mass* will follow a parabolic arc. Both paths are simulated in Figure 7.3.

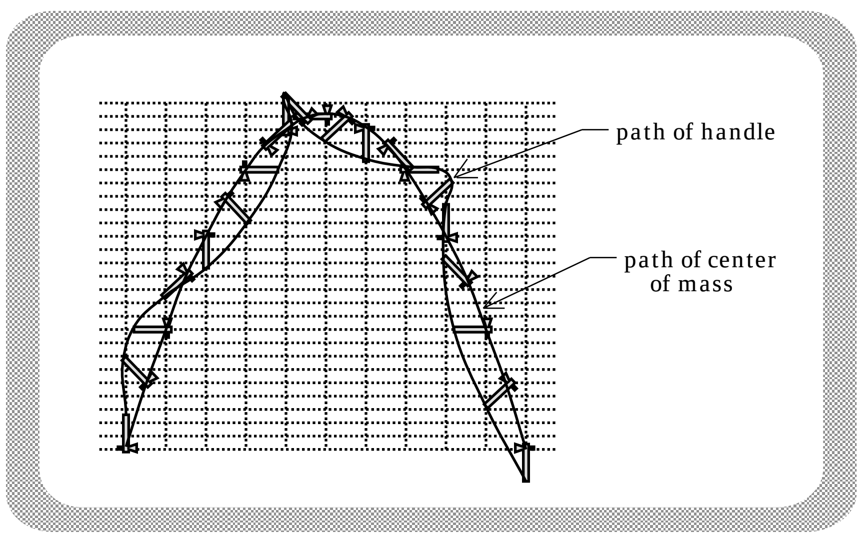


FIGURE 7.3

3.) A *group* of objects also has a *center of mass*. The *x coordinate* of a group's *center of mass* is determined by using the following equation:

$$x_{cm} = \Sigma\,(m_i x_i) \,/\, \Sigma\,(m_i)$$
$$= \Sigma\,(m_i x_i) \,/\, M,$$

where $\Sigma\ (m_i x_i)$ is the sum of the product of each mass and its *x coordinate* (i.e., $m_1 x_1 + m_2 x_2 + \ldots$ etc.) and $\Sigma\ (m_i)$ is the sum of all the mass in the system (i.e., the total mass M).

Similar equations define the *y* and *z coordinates* of a body's *center of mass*, relative to the coordinate axis used to define the body's position.

4.) The best way to see how this works is to do an example:

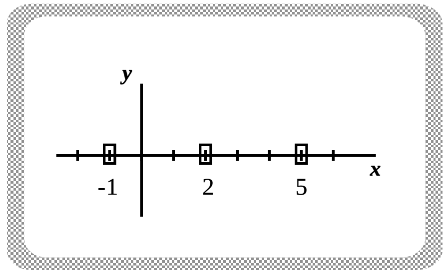


FIGURE 7.4a

a.) Consider the equal masses shown in Figure 7.4a. From the symmetry of the situation, it should be obvious that the *x coordinate* of the system's center of mass is at $x = 2$ *meters*. Checking this with our math, we get:

$$x_{cm} = \Sigma\,(m_i x_i) \,/\, \Sigma\,(m_i)$$
$$= (m_1 x_1 + m_2 x_2 + m_3 x_3)/(m_1 + m_2 + m_3)$$
$$= [(-1)(m) + (2)(m) + (5)(m)]/3m$$
$$= 2 \text{ meters.}$$

b.) If the bodies had differing *y coordinates*, a similar operation would have been used for the *y direction* and our final center of mass would have been some ordered pair (x_{cm}, y_{cm}).

C.) Calculating the *Center of Mass* for a Continuous Mass:

1.) The *center of mass* is defined by a set of coordinates that identifies the location of the *average position of all the mass* in a structure.

2.) For a discrete set of masses, the *x coordinate* of the *center of mass* is $x_{cm} = \Sigma\ (m_i x_i)/M$, where M is the total mass in the system.

3.) How do we deal with a situation in which the mass is not in lumps but, rather, is continuously distributed throughout the structure? To determine the *x coordinate* of the *center of mass* for such a case, we must identify all of the differential mass (*dm*) having the same *x coordinate*, multiply that *x* by *dm*, sum over all possible *x's* through integration, then divide by *M*. In other words, when dealing with an extended object:

$$x_{cm} = (1/M)\int x\,dm.$$

A similar expression holds for the *y* center of mass coordinate.

4.) An example will help. Determine the *x coordinate* of the center of mass for a rectangular solid of width *a*, height *b*, thickness *t*, and whose mass is *M* (see Figure 7.4b).

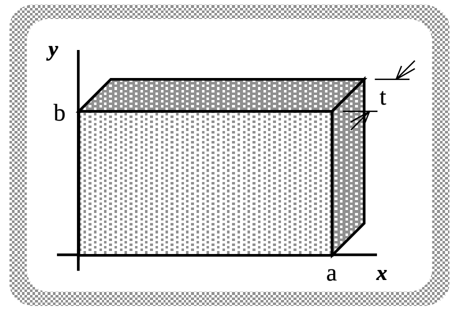


FIGURE 7.4b

a.) Consider Figure 7.4c. In that sketch, a slab of differential width *dx* and differential mass *dm* is drawn a distance *x* units from the *y axis* (note that *dm* is all the mass *x* units out!).

b.) To determine the *x* center of mass, we must multiply *x* by *dm*, then integrate. The problem? We need some way of relating *dm* to its *x position*.

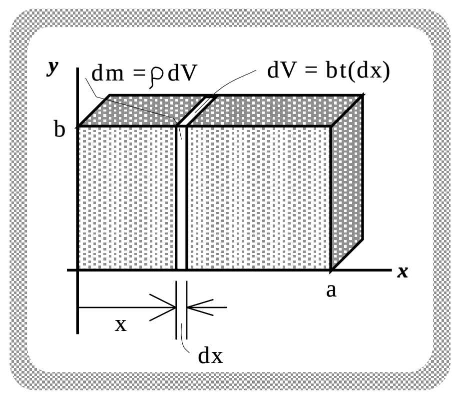


FIGURE 7.4c

c.) The trick here is to define a *volume-density function* ρ. Because the mass of the structure is homogenous (i.e., uniformly distributed throughout the structure), the definition of ρ allows us to write:

$$\rho = (\text{mass})/(\text{volume}).$$

This density function can be expressed in two ways:

i.) Looking at the entire structure macroscopically (i.e., as a whole), the *volume density* can be written as:

$$\rho = M/abt,$$

where *M* is the solid's total mass and *abt* is its total volume.

ii.) Looking microscopically, the *volume density* can be written as:

$$\rho = dm/dV,$$

where *dm* is the differential mass wrapped up in the differential slab and *dV* is the differential volume of the slab.

iii.) The differential volume is:

$$\begin{aligned} dV &= (\text{height})(\text{thickness})(\text{differential width}) \\ &= \quad (b) \qquad (t) \qquad (dx). \end{aligned}$$

This means that:

$$\begin{aligned} \rho &= (dm)/(dV) \\ &= (dm)/[bt(dx)] \,. \end{aligned}$$

iv.) Taking this last expression and solving for *dm*, we get:

$$\begin{aligned} dm &= \rho\, dV \qquad\qquad (\text{Equation A}) \\ &= \rho\,[bt(dx)]. \end{aligned}$$

This is the expression we will use to replace *dm* in the integral.

Note: The relationship presented in *Equation A* is ALWAYS true.

d.) We are now in a position to do the problem:

$$\begin{aligned} x_{cm} &= \left[\frac{1}{M}\right]\int (x)dm \\ &= \left[\frac{1}{M}\right]\int_{x=0}^{a} (x)\left[(\rho bt)dx\right] \\ &= \frac{\rho bt}{M}\int_{x=0}^{a} (x)dx. \end{aligned}$$

Substituting in $\rho = M/abt$, doing the integral, then canceling, we get:

$$\begin{aligned} x_{cm} &= \frac{\left[\frac{M}{abt}\right]bt}{M}\left[\frac{x^2}{2}\right]_{x=0}^{a} \\ &= \frac{1}{a}\left[\frac{a^2}{2}-\frac{0^2}{2}\right] \\ &= \frac{a}{2}. \end{aligned}$$

e.) Huzzah! This is exactly what we would have expected for the *x coordinate* of the *center of mass* of a rectangular solid.

Note: As the thickness variable t canceled out in the problem, we did not necessarily need to use a *volume* density function. Noting that the differential volume dV of our strip was equal to the differential surface-area dA (this equaled bdx) times the body's thickness t (i.e., $dV = b(dx)t$), we could have ignored the thickness term and instead defined an *area* density function σ (i.e., the *mass per unit area*). Using this approach:

$$dm = \sigma dA,$$

where dA is the differential face-area associated with the mass dm. This approach will be used in the next example.

5.) Next example: Determine the *y coordinate* of the center of mass of a solid, flat, thin, semi-circular plate of mass M and radius R.

a.) How much mass is y units up? Figure 7.4d shows a strip of differential mass dm with differential thickness dy drawn a distance y units from the *x axis*.

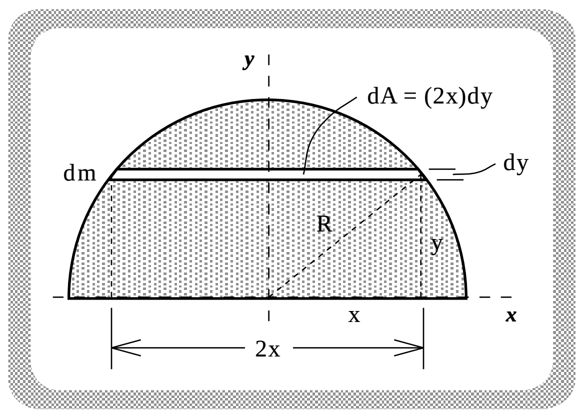


FIGURE 7.4d

b.) We can define σ in two ways: macroscopically and microscopically.

i.) Macroscopically:

$$\sigma = M/[(\pi R^2/2)]$$
$$= 2M/(\pi R^2),$$

where $\pi R^2/2$ is the area of the half-disk and M is its mass.

ii.) Microscopically:

$$\sigma = dm/dA,$$

where dm is the differential mass associated with the differential surface-area dA. This implies that:

$$dm = \sigma dA.$$

iii.) Using the variables provided in the sketch, we can write:

$$\begin{aligned} dA &= (\text{width})(\text{height}) \\ &= \quad (2x) \qquad dy. \end{aligned}$$

We need to relate x to y. To do so, notice that $x^2 + y^2 = R^2$. Manipulating, we can write:

$$dA = [2(R^2 - y^2)^{1/2} dy]$$

and

$$\begin{aligned} dm &= \sigma dA \\ &= \sigma[2(R^2 - y^2)^{1/2} dy]. \end{aligned}$$

c.) We are now ready to integrate:

$$\begin{aligned} y_{cm} &= \frac{1}{M}\int y\,dm \\ &= \frac{1}{M}\int_{y=0}^{R} y\left[\sigma\left[2\left(R^2 - y^2\right)^{1/2}\right]\right]dy \\ &= \frac{2\sigma}{M}\int_{y=0}^{R} y\left(R^2 - y^2\right)^{1/2} dy \\ &= -\frac{2\sigma}{3M}\left[\left(R^2 - y^2\right)^{3/2}\right]_{y=0}^{y=R} \\ &= -\frac{2\sigma}{3M}\left[\left(R^2 - R^2\right)^{3/2} - \left(R^2 - 0^2\right)^{3/2}\right] \\ &= \frac{2\sigma}{3M}R^3. \end{aligned}$$

Plugging in $\sigma = 2M/(\pi R^2)$, we get:

$$\begin{aligned} y_{cm} &= \frac{2\left[\frac{2M}{\pi R^2}\right]R^3}{3M} \\ &= \frac{4}{3\pi}R \\ &= .424R. \end{aligned}$$

D.) Systems of Bodies and Their Collective Motion:

1.) Although you won't be required to use the following material to solve problems, there are some interesting consequences that come from the definition of the *center of mass*. Specifically, if discrete bodies in a system are moving, the *center of mass* will also move. For a one-dimensional situation:

$$x_{cm} = \Sigma\ (m_i x_i) / M$$
$$\Rightarrow\quad Mx_{cm} = \Sigma\ (m_i x_i)$$
$$\Rightarrow\quad M\,(\Delta x_{cm}/\Delta t) = \Sigma\ [m_i\,(\Delta x_i/\Delta t)]$$
$$\Rightarrow\quad M\,(v_{cm}) = \Sigma\ [m_i(v_i)]$$
$$\Rightarrow\quad Mv_{cm} = m_1v_1 + m_2v_2 + m_3v_3 + \ldots$$

In other words, the *sum of the momenta* of all the pieces within the system will be equal to the *momentum* of a single particle whose mass *M* is the *total mass* of the system and whose *velocity* is the velocity of the system's *center of mass*. Similar expressions can be determined for *y* and *z momenta.*

2.) Taking this one step further, if the *velocities* of each of the various pieces of a system change due to forces acting on them, the *rate of change of the velocities* will be:

$$M\,(\Delta v_{cm}/\Delta t) = \Sigma\ [m_i\,(\Delta v_i/\Delta t)]$$
$$\Rightarrow\ M\,(a_{cm}) = \Sigma\ [m_i(a_i)]$$
$$\Rightarrow\ Ma_{cm} = m_1a_1 + m_2a_2 + m_3a_3 + \ldots$$

The *net force* F_1 on particle #1 (i.e., m_1a_1) added to the *net force* F_2 on particle #2, etc., will vectorially equal the total mass *M* of the system times the *acceleration of the system's center of mass.*

3.) The conclusion and importance of this section is simple: the motion of any rigid, "extended" object (a car, a cannonball, whatever) can be legitimately analyzed by treating it as though *all of its mass* were lumped together and positioned at the body's *center of mass*. This is the theoretical basis from which we have been allowed to treat, say, a car as though it were a point mass.

E.) The Modified *Conservation of Momentum* Equation:

Note: With the exception of the "bottom line," you will *not* be held responsible for duplicating any of the material you are about to read. BUT, if you don't understand what has been done in getting to the "bottom line," the end result won't mean much and your ability to use it won't be worth a tinker's dam.

My suggestion: read the next section, not for memorization but for content. Follow each step as it comes *without* projecting ahead. When you finally get to the end result, take the time to reread the section to be sure you know the general trend and principles involved in the derivation.

1.) Begin by considering a collision between two free (unconstrained) masses. Mass m_1 moves in the *x direction* on a frictionless table. It strikes a second mass m_2 that is moving in the *-x direction* and that additionally has a rocket on its back that is constantly applying a force F_{ext} to it. Figures 7.5a and 7.5b directly below show the *before and after* set-up.

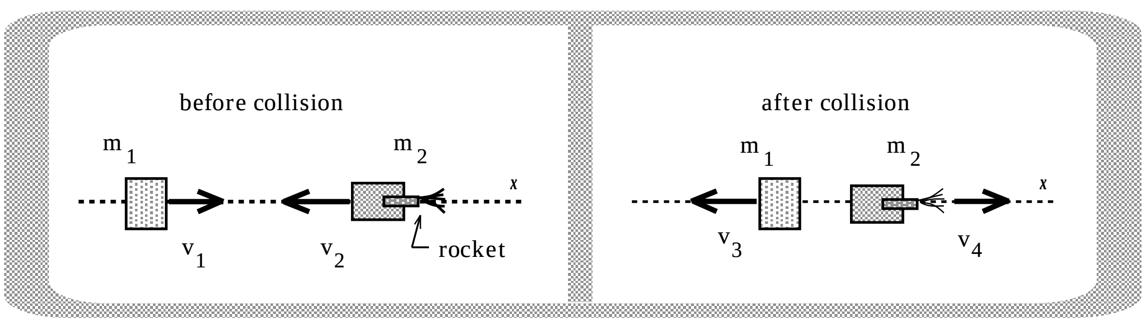


FIGURE 7.5a FIGURE 7.5b

All the motion is in the *x direction* so we need not worry about subscripting the *momentum magnitudes* with an *x* (we don't have to worry about confusing the momentum in the *x direction* for that in the *y direction*--there *is* none in the *y direction*). With that simplification, we can write the momenta of masses m_1 and m_2 just before and just after the collision as:

before:

$$\mathbf{p}_{1,\text{before}} = p_{1,\text{bef}}\ \mathbf{i}$$
$$\mathbf{p}_{2,\text{before}} = p_{2,\text{bef}}\ \mathbf{i}$$

after:

$$\mathbf{p}_{1,\text{after}} = p_{1,\text{aft}}\ \mathbf{i}$$
$$\mathbf{p}_{2,\text{after}} = P_{2,\text{aft}}\ \mathbf{i}.$$

where $p_{1,bef}$ stands for "the magnitude of the *momentum* of mass m_1 *before* the collision," etc.

2.) Consider just the motion of m_1 through the collision:

a.) The <u>*net force*</u> acting on m_1 in the *x direction* comes solely from the strike it takes from m_2 as the two collide (an f.b.d. on m_1 is shown in Figure 7.6). As the magnitude of the force varies (it starts out very small at the beginning of the hit, gets larger as the collision proceeds, then diminishes at the end), we will assume an *average* collision force F_i and go from there.

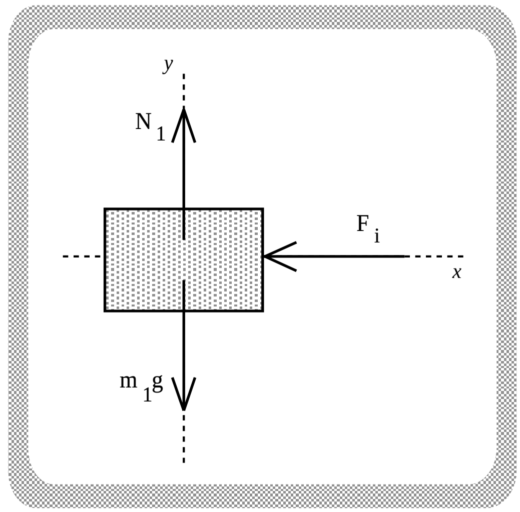


FIGURE 7.6

Note: Because the force F_i is generated from the interaction of m_1 with "other members of the system" (in this case, the *only* other member in the system--m_2), this kind of force is called an "*internal force*." To highlight this, the subscript "i" has been used to denote *internal*.

b.) Assuming the collision takes time Δt to be complete, the <u>*impulse equation*</u> ($F\Delta t = \Delta p$) written for m_1 during the time Δt yields:

$$-F_i \Delta t = (p_{1,aft} - p_{1,bef}) \qquad \text{(Equation A).}$$

Note: The force on m_1 due to its collision with m_2 is in the *negative* direction, hence the unembedded *negative* sign in front of F_i in the impulse equation. (Notice I haven't unembedded the momenta signs.)

3.) Consider just the motion of m_2 through the collision:

a.) The <u>*net force*</u> acting on m_2 in the *x direction* comes from two sources (see the f.b.d. shown in Figure 7.7):

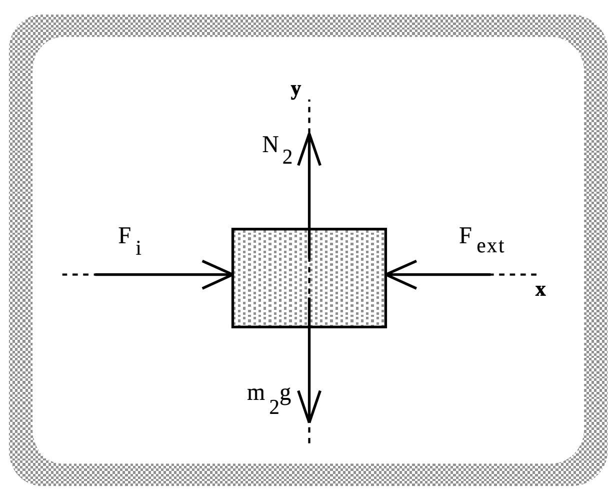


FIGURE 7.7

i.) The force $F_{ext,2}$ generated by the jet attached to m_2 (the force is subscripted "external" because it is provided by a source that is not *internal* to the two-body system); and

ii.) The internal force magnitude F_i generated by m_2's collision with m_1.

Note: The *internal force* on m_2 is the Newton's Third Law counterpart to the *internal force* acting on m_1 (for every force there is an *equal* and *opposite* reaction force, etc.). That means we can use the same symbol for that force as was used in Equation A, with the exception that the unembedded sign must be *positive*.

b.) As was the case with mass m_1, the collision takes time Δt to complete. The impulse equation $F\Delta t = \Delta p$ written for m_2 during the time Δt yields:

$$F_i \Delta t - F_{ext,2} \Delta t = (p_{2,aft} - p_{2,bef}) \qquad \text{(Equation B).}$$

4.) We would now like to vectorially add up all the *impulse* quantities acting on all the bodies within the system to get the system's *net impulse*. In this case, that is tantamount to adding the left-hand side of *Equation A* to the left-hand side of *Equation B*, etc.

a.) Doing the addition yields:

$$(- F_i \Delta t) + (F_i \Delta t - F_{ext,2} \Delta t) = (p_{1,aft} - p_{1,bef}) + (p_{2,aft} - p_{2,bef}).$$

b.) Noting that all the impulses provided by internal forces add to zero, the final equation can be rearranged to look like:

$$- F_{ext,2} \Delta t = (p_{1,aft} - p_{1,bef}) + (p_{2,aft} - p_{2,bef}).$$

c.) Putting all the *before* momenta on the left-hand side of the equation while leaving all the *after* momenta on the right-hand side, we get:

$$p_{1,bef} + p_{2,bef} - F_{ext,2} \Delta t = p_{1,aft} + p_{2,aft}.$$

d.) Written in a more general way:

$$\Sigma\ p_{bef} + \Sigma\ (F_{ext}\,\Delta t) = \Sigma\ p_{aft}.$$

e.) The general equation shown above is called the *MODIFIED CONSERVATION OF MOMENTUM* equation. It maintains that the *sum of the momentum* of all the parts of a system *before* a collision will equal the *sum of the momentum* of all the parts *after* a collision UNLESS there are *impulses* ($\Delta F_{ext}\Delta t$) due to *external forces* acting on the system during the time interval Δt.

Note 1: Everything done above has been for motion in the *x direction*. A similar equation holds true for both *y motion* and *z motion*. In other words, the most general form of this equation is:

$$\Sigma\ \mathbf{p}_{bef} + \Sigma\ (\mathbf{F}_{ext}\ \Delta t) = \Sigma\ \mathbf{p}_{aft}.$$

which, in turn, can be expressed as:

$$\Sigma\ p_{bef,x} + \Sigma\ (F_{ext,x}\,\Delta t) = \Sigma\ p_{aft,x},$$
$$\Sigma\ p_{bef,y} + \Sigma\ (F_{ext,y}\,\Delta t) = \Sigma\ p_{aft,y},$$
$$\Sigma\ p_{bef,z} + \Sigma\ (F_{ext,z}\,\Delta t) = \Sigma\ p_{aft,z}.$$

Note 2: Even though these equations were derived from a special situation--a collision--they are true in general. The *net momentum* of a system in a particular direction will remain the same throughout time--it will be CONSERVED--if there are no *external forces* acting in that direction.

That is not to say there cannot be change *within* the system. In fact, there may be *changes in the momentum* of *individual parts* of the system due to interaction-forces between those members, but the *net momentum* (as a vector) of the system will *not change with time* if there are no external forces acting on the system in the appointed direction.

5.) The BOTTOM LINE and a few conclusions:

a.) If there are no *external forces* acting in a particular direction, $F_{ext}\Delta t = 0$ and $\Sigma p_{bef} = \Sigma p_{aft}$ in that direction.

b.) Even if there are relatively small external impulses (the impulse gravity exerts, for instance), collisions and explosions characteristically take so little time that Δt is minuscule and $F\Delta t$ is tiny and the *total momentum* of the system in a particular direction *just before the collision* will equal the *total momentum* of the system in that direction *just after the collision*. In such cases, momentum is said to be conserved *through the collision* (even though this is technically a misuse of the word *conserved*).

Note that THIS IS TRUE ONLY if all pieces of the system are allowed to respond to collisions freely. If you take a ball and a wall as *the system*, the ball hitting the wall will not see momentum conserved because the wall is constrained--it is not free to respond to the impulse applied to it (on the other hand, if the system is just the ball, it will not see momentum conserved either--the wall will exert a very large external force to change the ball's momentum over a very small period of time).

Bottom line: Assuming external forces are small in a *free mass* system, you *can* assume that the *net momentum* of a system is conserved *through the collision*.

c.) Because the *x, y,* and *z directions* are independent, it is possible for momenta to be conserved in one direction and not in another. An example: a projectile in air. The momentum in the *y direction* will change over time because there is an *external force* acting in that direction (the force of gravity). If we ignore air friction, there are *no external forces* acting in the *x direction* so momentum in that direction is conserved.

F.) Collision Examples:

1.) A one dimensional situation: A puck of mass m_1 moving with velocity magnitude v_1 strikes a second puck of mass m_2 moving in the opposite direction with velocity $2v_1$ head on (see Figure 7.8).

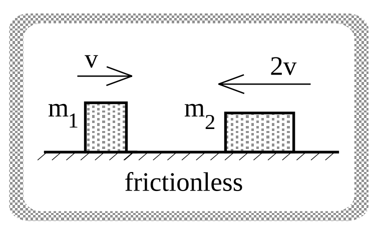


FIGURE 7.8

a.) What determines the direction each mass goes after the collision?

b.) What must the puck's mass ratio m_1/m_2 be if the two pucks are to leave the collision in opposite directions *with the same velocity magnitude v*?

c.) And, if a hard shelled bug happens to get squashed between the blocks during the collision, how will this change the final velocities relative to what would have happened if no bug were present during the collision (obscure? Yes . . . but informative . . .).

2.) And the answers are . . .

a.) The first query is more of a conceptual question than anything else. It asks, *what determines what will happen after the collision*? The answer to that depends upon the size of m_1 and m_2.

i.) If m_1 is a lot larger than m_2, its momentum will carry through the collision and the two will both proceed to the right with different velocities. In fact, if m_1 is a whole lot larger than m_2, its velocity will change very little. A determination of m_2*'s* velocity in that case is a little tricky. That kind of calculation will be treated at the end of the chapter.

ii.) If m_2 is a lot larger than m_1, its momentum will carry through the collision and the two will both proceed to the left with different velocities. Again, if m_2 is a whole lot larger than m_1, it's velocity will change very little as it proceeds toward the left.

iii.) If the two masses are comparable in size, they will reverse themselves at collision and leave with velocities *that conserved the total momentum* of the two-body system.

b.) This second query assumes that the collision occurs and the two masses respond by moving off in opposite directions but with the same velocity magnitude v. To determine the mass ratio that will motivate this to happen, we need equations.

i.) As both masses are free to respond to the impulse supplied by the other during the collision, and as there are no enormous external forces acting in the *x direction* during the collision, momentum will be conserved through the collision. Unembedding negative signs and writing out the *conservation of momentum* expression, we get:

$$\sum p_{before} = \sum p_{after}$$

$$m_1v_1 - m_2(2v_1) = -m_1v + m_2v \qquad \text{(Equation A).}$$

ii.) This gives us one relationship with four unknowns: m_1, m_2, v_1, and v. We clearly need more equations. At this juncture, all you can do is hope that some information is given about the *energy content* of the system.

iii.) Although it is highly unlikely that such a collision will ever occur, let's assume that the collision is elastic (mechanical energy conserved) and see where that leads. Observing that there are no potential energy changes anywhere in the system, we can write the *conservation of energy* expression as:

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2(2v_1)^2 = \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 \qquad \text{(Equation B).}$$

iv.) After canceling out the *1/2* terms in the energy expression and expanding, it is tempting to rewrite Equations A and B as:

$$(m_1 - 2m_2)v_1 = (-m_1 + m_2)v \qquad \text{(Equation A modified)}$$

and $$(m_1 + 4m_2)v_1^2 = (m_1 + m_2)v^2 \qquad \text{(Equation B modified),}$$

then try to solve by writing v_1 in terms of v from the momentum equation and substituting that back into the energy equation. The problem is that this leaves us with the very unpleasant expression:

$$(m_1 + 4m_2)\left(\frac{-m_1 + m_2}{m_1 - 2m_2}\right)^2 v^2 = (m_1 + m_2)v^2.$$

v.) Although this expression will allow us to solve for the mass ratio, it is kinda ungodly. Even more to the point, if we had been interested in solving for, say, unequal final velocities (should that have been the case), using that approach would have been even uglier. Fortunately, there is a clever way to shuffle the variables so as to create a third independent equation that is considerably easier to solve in conjunction with the momentum expression than was the original pair. That technique follows.

vi.) ESPECIALLY TRICKY APPROACH (ETA): For both conservation equations, group the m_1 terms on one side of the *equal sign* and the m_2 terms on the other side of the equal sign so that:

$$m_1v_1 - m_2(2v_1) = -m_1v + m_2v \qquad \text{(Equation A)}$$

becomes

$$\mathbf{m_1(v_1 + v) = m_2(v + 2v_1)} \qquad \text{(Equation C)}$$

and

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2(2v_1)^2 = \frac{1}{2}m_1v^2 + \frac{1}{2}m_2v^2 \qquad \text{(Equation B)}$$

becomes

$$\mathbf{(1/2)m_1(v_1^2 - v^2) = (1/2)m_2(v^2 - 4v_1^2)} \qquad \text{(Equation D).}$$

vii.) The *1/2's* cancel. Notice that $(v_1^2 - v^2)$ in the energy equation can be factored and equals $(v_1 + v)(v_1 - v)$, and that we can factor both sides of Equation D. We can also *divide* the left side of the factored version of Equation D by the left side of Equation C, and the right side by the right side, yielding:

$$\frac{m_1(v_1 + v)(v_1 - v)}{m_1(v_1 + v)} = \frac{m_2(v + 2v_1)(v - 2v_1)}{m_2(v + 2v_1)}.$$

Both mass terms and one of the factors on each side will cancel leaving:

$$v_1 - v = v - 2v_1,$$
$$\Rightarrow \qquad v = 3v_1/2.$$

Putting this into the original momentum equation (Equ. A) yields:

$$m_1v_1 - m_2(2v_1) = -m_1(3v_1/2) + m_2(3v_1/2)$$
$$\Rightarrow \qquad m_1/m_2 = 7/5.$$

viii.) In other words, this technique requires you to group velocities associated with each mass, factor the energy equation, divide one equation into the other, then use the resulting equation in conjunction with the *conservation of momentum* expression to solve for the required unknowns. It sounds complicated, but it's a lot easier than trying to deal with the squares in the original *conservation of energy* equation.

c.) So what's the deal with the squashed bug?

i.) To begin with, a problem similar to this was originally presented as a conceptual question on a quiz given to the freshman physics class at Caltech in 1999. I've included it because students who are learning physics for the first time don't generally have the time, experience, or in many instances, inclination, to think creatively and deeply about exotic problems. This problem could be done mathematically by power-grunting through it, or it could be taken apart conceptually by simply using your head. What you are about to read approaches the problem from that latter perspective. That is, the analysis you are about to read will not highlight an approach that will always work for you whenever you run into something exotic. What you are about to read is more *seat of the pants* thinking (versus *algorithmic* thinking). In short, you aren't going to be tested on this stuff, but do read it. If I put you on a desert island, gave you this problem, told you I'd be back in 24 hours with a machine gun . . . and that you'd better have the correct answer for me, I can guarantee you would sooner or later get past your belief that you couldn't possibly figure out what was going on, would get into the problem, and would begin to follow lines of thought that would be similar to the ones you are about to read. For you, for now, I'm doing the think-work gratis. Enjoy the luxury.

ii.) The first thing to notice is that work has to be done to squash the bug, so mechanical energy is no longer conserved (i.e., we are now dealing with an inelastic collision). That being the case, what do the velocities do? Well, at least one velocity must go down so that the total final mechanical energy will be less than it was for the conserved situation. But if one must go down, the other must go down also. Why? Because momentum must still be conserved.

Think about it. Let's assume that the total momentum of the system before the energy-conserved collision occurs is $2\ kg \cdot m/s$, and let's say m_1*'s* after-collision momentum in that case is $p_1 = 6\ kg \cdot m/s$ and m_2*'s* is $p_2 = -4\ kg \cdot m/s$ (note that the total after-collision momentum is still *2*

$kg{\cdot}m/s$ . . . $p_1 + p_2 = (6\ kg{\cdot}m/s) + (-4\ kg{\cdot}m/s) = 2\ kg{\cdot}m/s$). Now let's assume that the collision is not conserved so that at least one of the after-collision velocities, say that of m_1, decreases. That means the magnitude of its after-collision momentum will also decrease to, say, $5.8\ kg{\cdot}m/s$. The only way the sum of the momentum after the collision will equal the total momentum before the collision (remember, it was $2\ kg{\cdot}m/s$) is if m_2*'s* momentum magnitude also drops (in this case, to $-3.8\ kg{\cdot}m/s$ . . . now $p_{1,new} + p_{2,new} = 5.8\ kg{\cdot}m/s + (-3.8)\ kg{\cdot}m/s = 2\ kg{\cdot}m/s$). In other words, if one momentum magnitude decreases, so must the other. That means that both velocities must decrease.

iii.) But by how much? Though you probably wouldn't think to do so on your own, at least not without a fair amount of cogitating, the easiest way to determine this is to assign new final velocities (v_2 for m_2 and v_3 for m_1) and play with the momentum expression. Noting that the initial momentum is a constant (we'll call it c), and remembering that the mass ratio implies that $m_1 = (7/5)m_2$, we can write the *conservation of momentum* as:

$$\text{initial total momentum} = -\,m_1v_3 + m_2v_2$$
$$\Rightarrow \quad c = -\,(7m_2/5)\,v_3 + m_2v_2.$$

Dividing by v_3 yields:

$$c/v_3 = -\,(7m_2/5) + m_2(v_2/v_3).$$

iv.) Look at this expression and think. If v_3 decreases (relative to its earlier value), as we've decided it must, then the left side of the equation increases. The first term on the right is a constant, so that means that the second term on the right must go UP if the equal sign is to continue to hold. To do that, the denominator of that piece must be smaller than the numerator, which means that v_2 (the velocity associated with m_2) must be larger than v_3. In other words, both velocities will change, and the velocity associated with m_2 will change the least (if it's bigger than v_3, it's closer to the original v which means it hasn't changed as much as has v_3).

UNDERSTAND, this is all a consequence of the need for the system's momentum to be conserved coupled with the relationship the velocities have to satisfy with respect to energy. For a given situation, there is only one set of velocities that will do the trick. That's just the way nature works.

3.) Two dimensional situation: Looking down on a frictionless table, a puck of mass $m_1 = 4\ kg$ moves in the *x direction* with a velocity-magnitude of $v_o = 12\ m/s$. It strikes a second mass $m_2 = 6\ kg$ initially at rest (see Figure 7.8a). After the collision, the two move off at angles 25^o and 40^o respectively (see Figure 7.8b). What are the magnitudes of the after-collision velocities v_1 and v_2?

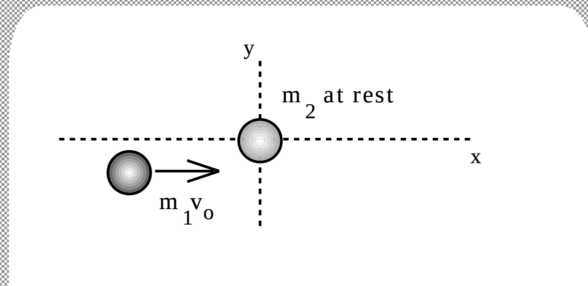


FIGURE 7.8a

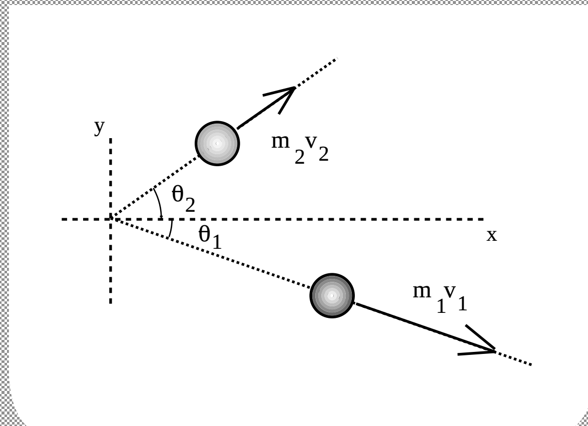


FIGURE 7.8b

Note: This problem will be solved in *Parts a* through *e*, complete with generalized algebra for ease of reading and explanation. In *Part f*, the problem is done again as it would be done on a test--that is, with math but *without* mountains of explanatory verbiage.

a.) The only forces acting in either the *x* or *y directions* during the collision are *internal forces*: the force m_2 applies to m_1 and the force m_1 applies to m_2. As such, momentum is conserved in both the *x* and *y directions* through the collision.

b.) The momenta are broken into their component parts in Figure 7.8c.

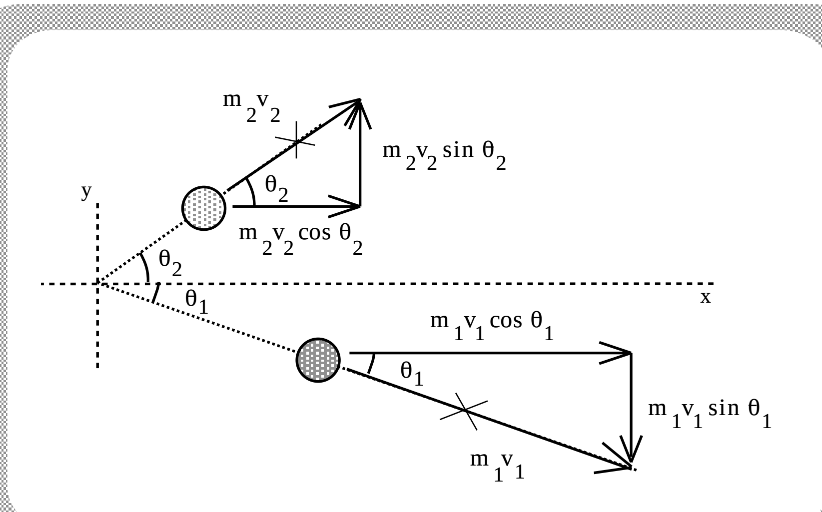


FIGURE 7.8c

c.) Using the *conservation of momentum* equation for motion in the *x direction* yields:

$$\underline{\sum p_{bef,x} = \sum p_{aft,x}:}$$

$$\Rightarrow \quad m_1v_o + m_2(0) = m_1v_1 \cos\theta_1 + m_2v_2 \cos\theta_2$$

$$\Rightarrow \quad v_1 = (m_1v_o - m_2v_2 \cos\theta_2)/(m_1 \cos\theta_1) \qquad \text{(Equ. A).}$$

d.) For the motion in the *y direction*, we get:

$$\sum p_{bef,y} = \sum p_{aft,y}:$$

$$\Rightarrow \quad m_1(0) + m_2(0) = -m_1v_1 \sin\theta_1 + m_2v_2 \sin\theta_2 \qquad \text{(Equ. B).}$$

Note: I've unembedded the sign of the after-collision velocity of m_1, leaving the v_1 term a *magnitude*.

e.) Plugging *Equation A* into *Equation B* yields:

$$0 = [-m_1 \left[(m_1v_o - m_2v_2 \cos\theta_2)/m_1 \cos\theta_1\right] \sin\theta_1] + m_2v_2 \sin\theta_2$$

$$\Rightarrow \quad v_2 = [m_1v_o \sin\theta_1/\cos\theta_1]/[m_2 \sin\theta_2 + (m_2 \cos\theta_2/\cos\theta_1) \sin\theta_1].$$

With some manipulation, this becomes:

$$\begin{aligned} v_2 &= [(m_1v_o / m_2)] (\tan\theta_1)/(\sin\theta_2 + \cos\theta_2 \tan\theta_1) \\ &= [(4 \text{ kg}) (12 \text{ m/s})/(6 \text{ kg})] (\tan 25^o)/(\sin 40^o + \cos 40^o \tan 25^o) \\ &= [(4 \text{ kg}) (12 \text{ m/s})/(6 \text{ kg})] (.47)/(.64 + .36) \\ &= 3.76 \text{ m/s}. \end{aligned}$$

Plugging back into *Equation A*:

$$\begin{aligned} v_1 &= (m_1v_o - m_2v_2 \cos\theta_2)/(m_1 \cos\theta_1) \\ &= [(4 \text{ kg})(12 \text{ m/s}) - (6 \text{ kg})(3.76 \text{ m/s})(\cos 40^o)]/(4 \text{ kg})(\cos 25^o) \\ &= 8.47 \text{ m/s}. \end{aligned}$$

f.) The problem solved *without verbiage*:

Write:

$\sum p_{bef,x} = \sum p_{aft,x}$:

$$\Rightarrow \quad m_1 v_o = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2$$
$$\Rightarrow \quad (4\text{ kg})(12\text{ m/s}) = (4\text{ kg})v_1 \cos(25^o) + (6\text{ kg})v_2 \cos(40^o)$$
$$\Rightarrow \quad v_1 = [48 - (4.6)\, v_2]/3.63$$
$$\Rightarrow \quad v_1 = 13.24 - 1.27 v_2 \qquad \text{(Equ. A).}$$

$\sum p_{bef,y} = \sum p_{aft,y}$:

$$\Rightarrow \quad 0 = -m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2$$
$$\Rightarrow \quad 0 = -(4\text{ kg})v_1 \sin(25^o) + (6\text{ kg})v_2 \sin(40^o)$$
$$\Rightarrow \quad 0 = -1.69\, v_1 + 3.86\, v_2 \qquad \text{(Equ. B).}$$

Substituting v_1 from *Equation A* into *Equation B* yields:

$$0 = -1.69\,(13.24 - 1.27 v_2) + 3.86\, v_2$$
$$\Rightarrow \quad v_2 = 3.73\text{ m/s.}$$

Plugging back into *Equation A* will yield:

$$v_1 = 13.24 - 1.27\,(3.73\text{ m/s})$$
$$= 8.5\text{ m/s.}$$

Note: These answers are not exactly as above (off by a few hundredths)--the discrepancy is due to round-off error.

4.) Example 2: A string of length L is attached at one end to the ceiling and at the other end to a ball of mass m_1. The ball is positioned as shown in Figure 7.9 and, from that position, allowed to freefall from rest. At the bottom of its arc, the ball strikes a second mass m_2 sitting at rest on a frictionless table. Assuming the ball loses three-quarters of its mechanical energy as it *inelastically* bounces off m_2, what will both the ball

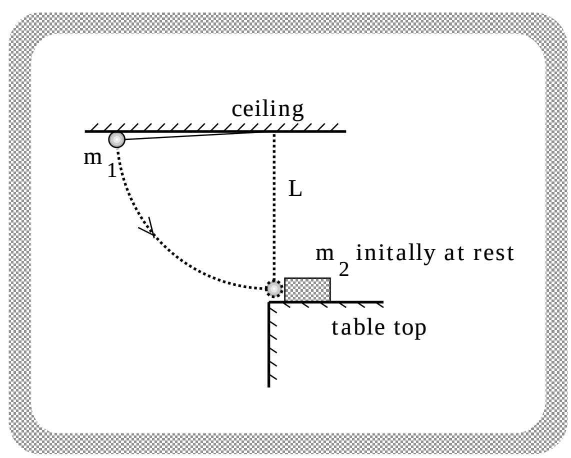


FIGURE 7.9

and block's after-collision velocities be (call them v_1 and v_2 respectively)?

Note 1: It is customary to term a collision *elastic*, *inelastic*, or *perfectly inelastic*. The definition of each follows:

Note 2a: An *elastic collision* is one in which *mechanical energy is conserved.* This situation never really occurs in nature (though the repulsion between like-charged electric particles comes very close), but is often assumed for the sake of mathematical simplification.

Minor Point: It should be noted that because collisions usually happen very quickly, the *potential energy* of the particles involved does not change through the collision (or if it does, it changes so slightly that it can be ignored). In other words, when *mechanical energy* is said to be *conserved through a collision*, we are really saying that the *sum of the kinetic energies of all the particles before the collision will equal the sum of the kinetic energies after the collision.* Due to this, many books simply define an *elastic collision* as one in which the *total* *kinetic* *energy* of the system is conserved *through the collision.*

Note 2b: An *inelastic collision* is one in which the kinetic energy of the system (i.e., the *kinetic energy* of all the pieces within the system added together) is *not* conserved. In most cases, energy is "lost" to heat, sound, or, in some cases, light. There can also be chemical changes internal to some part of the system (example: a piece of exploding dynamite).

Note 2c: A *perfectly inelastic collision* is one in which energy is not only *not* conserved, it's a collision in which the bodies additionally *stick together* upon contact. One obvious consequence of this situation: after the collision, everything is moving with the same *velocity*.

a.) Figures 7.10a and 7.10b show the ball at the bottom of the arc. It is moving solely in the *x direction* at that point. As momentum is always assumed to be

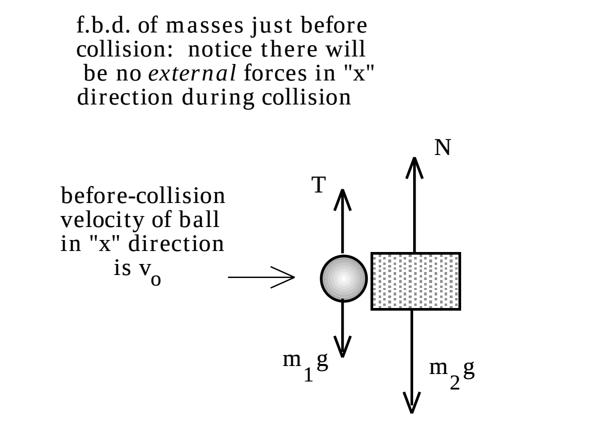


FIGURE 7.10a

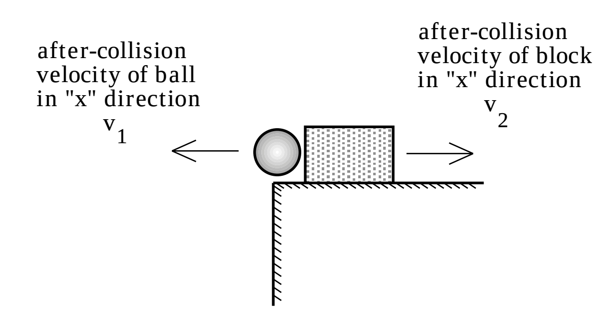


FIGURE 7.10b

conserved *through a collision*, we can equate the *x direction* momentum of the ball/block system before and after the strike. With m_1*'s* velocity at the bottom of the arc equal to v_o and m_2*'s* velocity equal to zero, we get:

$\underline{\sum p_{bef,x} = \sum p_{aft,x}:}$

$$\Rightarrow \quad m_1 v_o + m_2 (0) = - m_1 v_1 + m_2 v_2$$
$$\Rightarrow \quad m_1 v_o = - m_1 v_1 + m_2 v_2.$$

b.) There are *three* unknowns in this equation: v_o, v_1, and v_2. We know that mechanical energy is not conserved through the collision, but if we ignore air friction we know that mechanical energy will be conserved from the time the ball begins to fall until *just before* it strikes the block. Using *conservation of energy* for that part of the motion yields:

$$m_1 g h = (1/2) m_1 v_o^2$$
$$\Rightarrow \quad v_o = (2gh)^{1/2}.$$

c.) We now have two equations; we need a third. At this point, the only other help we can hope for is information about *energy loss* during the collision. If, for instance, we are told that *three-quarters* of the *before-collision* energy is lost, we know there will be *one-quarter* of the original kinetic energy left. In other words, *one-quarter* of the *before-collision* kinetic energy E_{bef} will equal the *after-collision* kinetic energy E_{aft} in the system. Mathematically, this gives us:

$$(1/4)\ E_{bef} = E_{aft}$$
$$\Rightarrow \quad (1/4)\ [(1/2) m_1 v_o^2 + (1/2) m_2 (0)^2] = (1/2)\ m_1 v_1^2 + (1/2)\ m_2 v_2^2.$$

This gives us our third equation which, though it may be a pain to solve, allows us to complete the problem.

5.) Parting shot: *Momentum* is a *vector*. Its change ($\Delta \mathbf{p}$) is equal to the impulse ($\mathbf{F}\Delta t$) received by the body in question. In cases where a number of bodies interact with each other providing "internal" impulses to one another, the momentum of individual pieces might change but the *total momentum* of all the pieces in a given direction will always sum to the same amount. In short, the *total momentum* in that direction will be *conserved*.

Under some circumstances, *energy consideration* will come in handy in providing one-more-solvable-equation, but one needs to be careful that *energy conservation* is truly justified before using the approach (Translation: know the *criteria* for the use of both the *conservation of energy* and the *conservation of momentum*--they are two very different conservation theorems).

G.) Confusion Be Gone--What is Conserved When?

1.) In a collision situation, when is mechanical energy conserved?

a.) Almost never (only when you are told that the collision is *elastic*).

i.) This is generally a contrived situation. It is usually assumed when physics authors want the student to have access to the *conservation of energy* equation (most books don't include the modified conservation of energy equation, so energy considerations are not generally possible if mechanical energy isn't assumed to be conserved). About the closest we can come to a truly elastic collision is one involving the interaction of magnets.

2.) In a collision, when is mechanical energy conserved (yes, I'm repeating!)?

a.) In real life, NEVER! In physics class, *only* when you are told that the collision is elastic (remember, an elastic collision is a contrived circumstance that exists in the world of physics only to simplify a situation and make the math doable).

3.) In a scenario in which a collision has occurred, when *might* mechanical energy be conserved?

a.) Before the actual collision (i.e., if potential energy has been converted into kinetic energy, or vice versa).

i.) Example: Before a collision, if a block slides down an incline or swings down from a higher position to a lower position pendulum style, mechanical energy will be conserved *before* the collision (assuming there is no friction or other non-conservative forces).

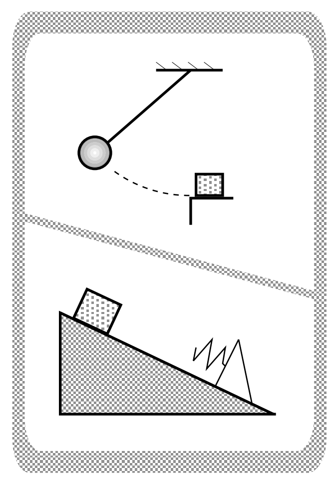


FIGURE 7.10c

b.) After the actual collision (i.e., if kinetic energy has been converted into potential energy, or vice versa).

i.) Example: A block strikes a spring that is attached to a body that is free to move (versus being pushed up against a rigid wall). Energy will

NOT be conserved through the collision (there is sound, heat, and possible molecular deformation during the hit), but just after the collision, the total mechanical energy of the whole system will stay conserved (i.e., kinetic energy will be lost as the spring compresses and spring potential energy increases) if there is no friction or other non-conservative forces acting.

ii.) How does MOMENTUM fit into the above example? Momentum will be conserved *through the entire spring collision* as long as the spring is attached to something (a block) that is *free to move* during the collision. In that case, the force the originally sliding block applies to the spring/block combination will be internal and will equal the force the spring/block combination applies to the originally sliding block. If, on the other hand, the spring/block combination is pushed up against a rigid structure like a wall, then a large, external, wall force will be applied to the system that will stop all motion dead, and momentum will not be conserved even through the collision.

4.) When is momentum generally *not* conserved?

a.) When an external force is applied in a particular direction.

i.) Example: For a ball thrown into the air, assuming we ignore friction, gravity is the only external force acting, and it is in the *y direction*. That means that momentum in that direction will change with time. In the *x direction*, there are no external forces and, hence, momentum will be conserved.

5.) When is momentum *not* conserved *through a collision*? That is, when is the total momentum of a system *not* the same from the time *just before* to the time *just after* a collision?

a.) When a BIG external force (hence big impulse $F_{ext} \Delta t$) is applied.

i.) Example: A ball hits a wall. The time during which the collision takes place is very small, so Δt is small, but the force generated during the collision is large enough to stop the ball dead.

6.) When is momentum conserved through a collision?

a.) It is conserved when (1) the only forces acting are internal (i.e., are the consequence of the interaction of the pieces of the system), or (2) when the external forces acting are small so that $F_{ext} \Delta t$ over the short time period of the collision is small.

i.) Example for 1: Whenever unrestrained bodies collide.

ii.) Example for 2: Gravity acting during a .01 second collision.

H.) A Bit of A.P. Nastiness:

1.) In an effort to confuse the hell out of anyone taking an AP test, the writers of those little gems have concocted an interesting kind of question designed to see just how completely you understand energy and momentum conservation. The following example will illuminate this bit of cheer.

2.) A bullet of known mass m_b is fired into a block of known mass m_k that sits on a frictionless table. The block is, itself, jammed up against (but not rigidly connected to) an ideal spring which, in turn, is placed against a wall. The spring *is* attached to the wall (see Figure 7.10d). The bullet embeds itself in the block. The block and bullet push the spring in to some known maximum displacement d before recoiling back outward. Assuming the unknown initial velocity of the bullet is v_o and the unknown final velocity of the bullet and block after separation from the recoil is v_r, determine the velocity of the bullet and block just after the bullet embeds itself but before the spring compresses appreciably.

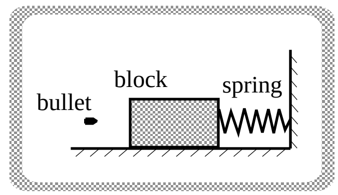


FIGURE 7.10d

a.) This is a nasty question. Why? When bright people in high school or at the university level start out dealing with physics, they have three things to worry about. They have to learn the principles, they have to understand the conceptual side of the principles (i.e., the consequences of the principles), and they have to cope with the math.

One of many intelligent ways to deal with the math is to become familiar with the characteristics that are often present when certain types of problem arise. If you are looking for the velocity of an object that is "falling" through a force field for which you happen to know the potential energy function, try the *conservation of energy*. If you are looking for the velocity of a couple of objects that have undergone a collision, try *conservation of momentum*.

This is a collision problem. You are trying to determine the velocity of the bullet/block system just after the collision. Energy is NOT conserved through the collision--a characteristic that is always true unless you are told otherwise (i.e., unless you are told it is an *elastic* collision), so what approach should you use? You'd think *conservation of momentum*.

As the problem is stated, this assumption is a bad one.

b.) If you did use the *conservation of momentum* on this problem, the resulting equation would look like:

$$m_b v_o = (m_b + m_k)v_{just\ after}.$$

The problem? There are two unknowns in this expression. You are looking for $v_{just\ after}$, but you don't know v_o.

c.) So what to do? As this is your first real experience with momentum and energy, and because the most superficially obvious approach isn't going to do the job, you're up the proverbial creek . . . unless you have the presence of mind to look at the information that has actually been given in the problem.

d.) What is that information? You don't know anything about the *before collision* situation, but you do know something about the *after* collision situation. Specifically, you know that the spring is deformed a maximum distance *d*. Maybe that information might help . . .

In fact, what this question is really testing is whether you are insightful enough to use what you know about the system *after the collision has taken place* to solve the problem.

e.) So let's try it. Just after the collision has taken place (but before the spring has depressed appreciably), the bullet and block have become one, have velocity $v_{just\ after}$ and have kinetic energy:

$$(1/2)(m_b+m_k)(v_{just\ after})^2.$$

f.) As the spring depresses after the collision, the bullet and block slow down. The amount of kinetic energy lost by the bullet and block during this part of the motion is absorbed by the spring in the form of *spring potential energy*.

g.) At the spring's maximum deflection, all of the mechanical energy in the system has converted to spring potential energy. And because the spring is ideal, *no energy was lost* while the spring is being compressed. That is, mechanical energy is conserved *after the collision*. Remembering that the spring potential energy function is $(1/2)kx^2$ or, in the case of maximum deflection, $(1/2)kd^2$, we can use this bit of insight to write out the AFTER-COLLISION *conservation of energy* expression as:

$$(1/2)(m_b+m_k)(v_{justafter})^2 = (1/2)kd^2.$$

This is nice as we know all the variables except $v_{just\ after}$.

h.) So what is the moral of the story? To do the problem *with the information provided*, we have to ignore the tried and true road sign that suggests that this is a *conservation of momentum* problem and look into the *middle* of the scenario to find the key that will unlock the door. If you've never run into a problem that requires you to think in this way, a problem like this can be a real killer. If you are prepared for this kind of twist, it is still a killer . . . but it's doable.

3.) For the chuckle of it, the next part of a problem like this would undoubtedly ask, "What is the initial velocity v_o of the bullet?" Given that we now know $v_{just\ after}$, this is where the *conservation of momentum* comes in handy. In fact, the expression that does the job is found in *Part H-2b*.

4.) And for complete amusement, the final question would undoubtedly be, "What is the velocity of the bullet and block after recoil (i.e., after leaving the spring)?" Interestingly enough, this problem requires no calculations. The spring will lose no mechanical energy as it compresses, then expands outward, so the mechanical energy of the bullet and block as they leave the spring should be the same as the mechanical energy of the bullet and block just after the collision but before the spring is compressed. (Remember, energy is conserved starting just after the collision.) That means the *velocity* of the bullet and block after recoil will be the same as the velocity of the bullet and block just after the collision. That velocity was the object of the original problem (i.e., the one done in *Part H-2g*).

I.) A Little More A.P. Nastiness . . .

1.) A bullet moving horizontally with velocity v_o embeds itself into an immovable block of wood. As the bullet moves into the block, it experiences a retarding force that is proportional to velocity (that is, $F = -bv$, where b is a constant). Assuming very little mechanical energy is lost due to sound or heat during the impact, how deep will the bullet penetrate before coming to rest?

2.) The temptation is to try the work/energy theorem on this problem (after all, energy *was* mentioned in the problem and little was lost due to the collision).

a.) The work/energy expression yields:

$$\begin{aligned}\frac{1}{2}mv_2^{\ 2} - \frac{1}{2}mv_1^{\ 2} &= \int \mathbf{F} \bullet d\mathbf{r} \\ &= \int(-bv\mathbf{i}) \bullet (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= -\int(bv)dx.\end{aligned}$$

b.) Unfortunately, even if we try to substitute $v = dx/dt$ into the integral, it still isn't going to fall out. So what next?

3.) Consider the impulse expression. You know the initial and final momentum, so if you could determine the net impulse the bullet experiences between the time it hits the block and the time it stops, you'd at least have an expression you might be able to solve.

a.) Noting that the final momentum is zero (the bullet is at rest at the end) and the one-dimensional force is equal to $-bv$, the impulse equation for this situation is:

$$p_o + \int F dt = p_f$$

$$\Rightarrow mv_o + \int F dt = 0$$

$$\Rightarrow mv_o = -\int (-bv) dt$$

b.) Substitute dx/dt for v, this becomes:

$$mv_o = -\int (-b\frac{dx}{dt}) dt$$

$$= b\int_{x=0}^{x=L} dx$$

$$\Rightarrow \ L = \frac{mv_o}{b}.$$

Tricky, eh?

J.) Exotica--The *Center of Mass* Reference Frame:

1.) There are times when *creative problem solving* can make life a lot easier. Most physics texts present one such approach using what is called *the center of mass frame-of-reference.*

2.) Even though there may be considerable motion and interaction within a system during an experiment, you and I normally observe the action from a frame of reference that is *stationary* relative to the room in which the experiment is being performed. That frame, our normal observation-frame, is called *the laboratory frame-of-reference.*

3.) It is possible to consider problems from a frame of reference other than the *lab frame*. For instance, you could look at an interaction from the perspective of the *center of mass*, almost as though you were sitting at the center of mass watching the motion of the system's particles from that vantage point. Doing so elicits a number of observations:

a.) Observation 1: Although the *center of mass* may move with velocity v_{cm} relative to the *laboratory frame-of-reference*, the *center of mass* is STATIONARY in the *center of mass frame-of-reference*. In other words, examining a problem from the *center of mass's point of view* assumes the *center of mass* is motionless with all else moving around it.

b.) Observation 2: We know that $Mv_{cm} = m_1v_1 + m_2v_2 + m_3v_3 + \ldots$, where v_{cm} is the *velocity* of the *center of mass* relative to the frame of reference in which one is observing the motion. As the velocity of the *center of mass* in the *center of mass frame* is zero, the sum of all the *momenta* of the system as viewed in the *center of mass frame* will also be zero (this fact sometimes makes problem solving from the perspective of the *center of mass frame* easy).

c.) Observation 3: Consider two colliding billiard balls: Figure 7.11a shows the *lab frame* positions of the balls at a number of points-in-time before and after the collision. It also shows the position of the *center of mass*, relative to the lab-frame, at those times.

Figure 7.11b shows the ball-positions from the *center of mass frame-of-reference*. Notice that in the *center of mass frame*, the ball's motion is

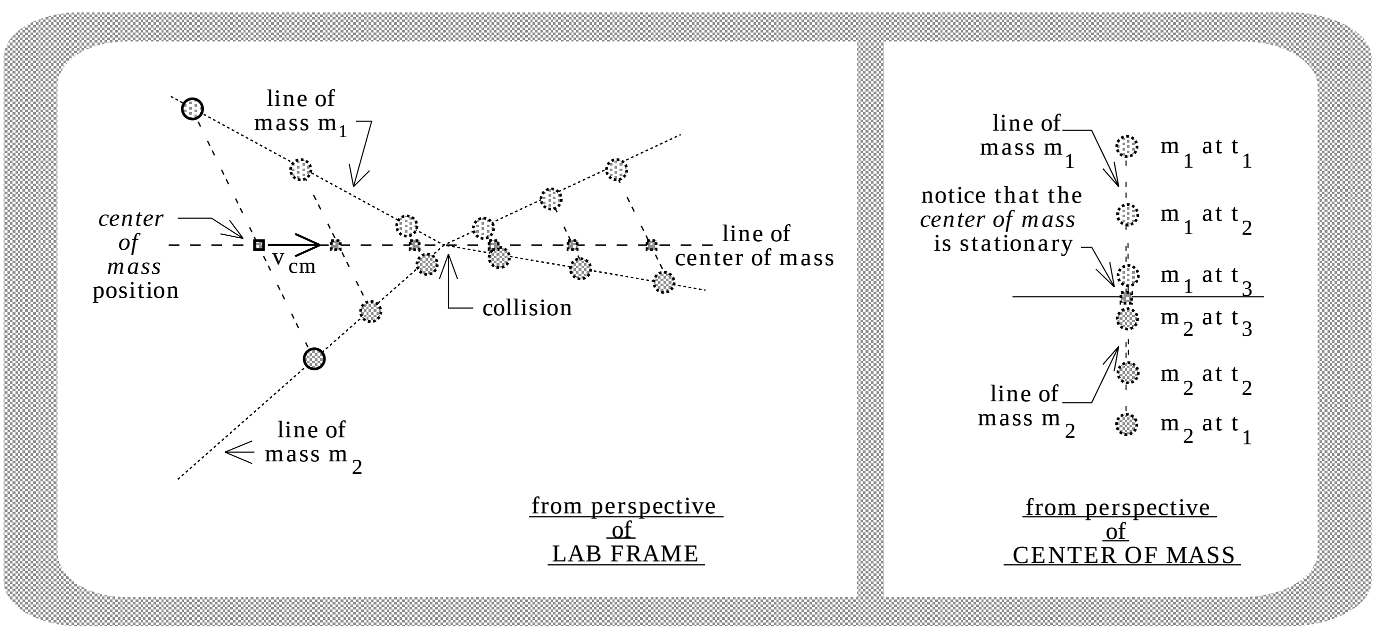


FIGURE 7.11a FIGURE 7.11b

linear--another simplification that can make problem solving in the *center of mass* frame easier.

d.) Observation 4: If we know the *center of mass's* velocity in the *lab frame* (call this $\boldsymbol{v}_{cm}$) and the velocity of a particle as viewed from the *center of mass frame* (call this $\boldsymbol{v}_{part.rel.\ to\ cm}$), the velocity of the particle in the *lab frame* (call this $\boldsymbol{v}_{lab}$) will be:

$$\mathbf{v}_{\text{lab}} = \mathbf{v}_{\text{part. rel. to cm}} + \mathbf{v}_{\text{cm}}.$$

e.) Observation 5: It should be obvious from the sketch that the velocities observed in the *lab frame* are greater than the velocities observed in the *center of mass frame*. One consequence: the mechanical energy of the system as determined in the *lab frame* will be *greater* than the mechanical energy of the system as determined in the *center of mass frame*.

Note: It turns out that the *total kinetic energy* in the lab frame equals the *total kinetic energy* of the pieces as measured in the *center of mass frame*, plus the kinetic energy of a particle whose mass is the total mass of the system and whose velocity is the *lab frame's* center of mass velocity. Mathematically, this is:

$$\text{KE}_{\text{lab}} = \text{KE}_{\text{pieces in cm frame}} + \text{KE}_{\text{cm}}.$$

f.) Observation 6: The *kinetic energy expression* stated above is not particularly important here, but its consequence is. You cannot assume that the energy calculated in the *lab frame* is the same as the mechanical energy determined in the *center of mass frame*.

g.) With all this information, the problem-solving technique is simple. If a velocity problem looks difficult in the lab frame:

i.) Transform to the *center of mass frame* (that is, look at the problem from the point of view of the system's *center of mass*);

ii.) Determine the velocity of the particle-of-interest as seen in the *center of mass frame*;

iii.) Take the particle's velocity as determined in the *center of mass frame* and transform it back into the *lab frame* using $\boldsymbol{v}_{lab} = \boldsymbol{v}_{part.\ rel.\ to\ cm} + \boldsymbol{v}_{cm}$.

iv.) Depending upon the problem, the technique can work easily or with great pain.

4.) An Example: Consider the following one dimensional problem: A ball of mass m moving horizontally with velocity v_o strikes a stationary, massive wall (i.e., the wall is, relative to the ball, assumed to be *infinitely* massive). Upon collision, the ball rebounds in the horizontal with velocity v_o (i.e., mechanical energy is conserved)--see Figure 7.12a and 7.12b.

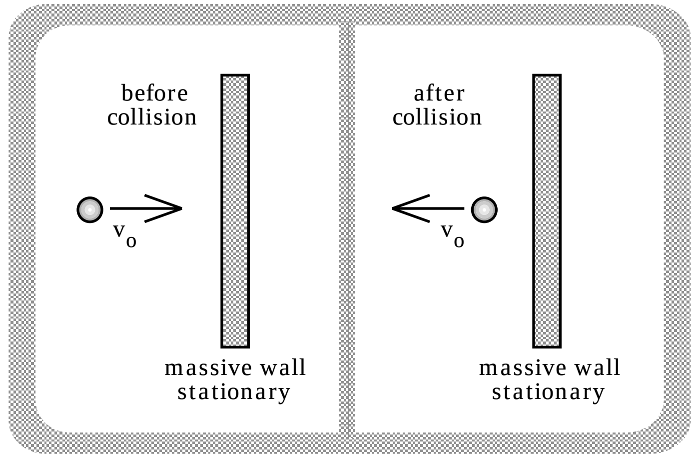


FIGURE 7.12a FIGURE 7.12b

The same wall is then made to move horizontally with velocity v_o. The same ball is projected horizontally at the wall with velocity v_o (see Figures 7.13). What is the ball's velocity after striking and bouncing off of the wall?

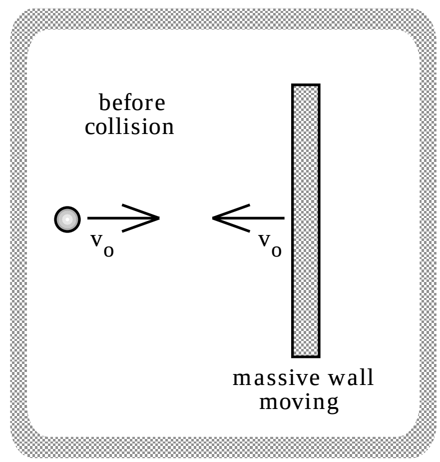


FIGURE 7.13

a.) The temptation is to say $2v_o$, but that would be a mistake.

b.) Notice:

i.) Because the wall is infinitely massive, the *center of mass* of the system *is* the wall.

ii.) When the wall is stationary, the *lab frame of reference* and the *center of mass frame of reference* are the same. That is not true when the wall is moving (i.e., when the wall moves, the c. of m. moves with it while you in the *lab frame* stay stationary).

iii.) As the *center of mass frame* is the frame in which the wall *is not moving*, and as that situation is the only one we know anything about as far as energy goes (i.e., in the frame in which the wall is not moving, the ball's mechanical energy is *conserved*), and as energy in the *lab frame* and energy in the *center of mass frame* are different, we *have* to deal with the problem from the *center of mass* point of view.

c.) From the *center of mass* perspective, the wall is stationary and the mass m is moving toward the wall with velocity $2v_o$. As mechanical energy is conserved in this frame of reference, the ball will bounce off the wall and leave with velocity $2v_o$ *relative to the wall.*

d.) But the wall, itself, is moving with velocity v_o, relative to the *lab frame*. That means that relative to the *lab frame*, the BALL is moving with velocity $2v_o$ (relative to the wall) plus v_o (the wall's velocity relative to the lab frame). In other words, transforming our solution in the *center of mass frame* ($2v_o$) back into the *laboratory frame* requires us to use:

$$\mathbf{v}_{\text{lab}} = \mathbf{v}_{\text{rel. to cm}} + \mathbf{v}_{\text{cm}}.$$

Noting that this is a one-dimensional situation, we can write:

$$\text{v}_{\text{lab}} = 2\text{v}_\text{o} + \text{v}_\text{o}.$$

e.) <u>Bottom line</u>: The ball leaves the moving wall with velocity $3v_o$, relative to the *lab frame*.

f.) As improbable as this solution might appear, there is an easy way to experimentally check it. Consider a fairly massive ball (a tennis ball or a large superball will do) freefalling from a height h toward a hardwood floor. As it does, a much lighter ball (a ping pong ball) follows just above it. Assume the tennis ball strikes the floor with velocity v_o. If we further assume an *elastic collision*, the tennis ball will bounce off the floor and return with an initial upward velocity of v_o. Meanwhile, the ping pong ball will have dropped from approximately the same height and will have a downward velocity that is close to v_o as it nears the floor. Because the ping pong ball is trailing the tennis ball, and because the tennis ball will execute its bounce before colliding with the ping pong ball, we now have a situation that is close to our "small ball moving with velocity v_o hits massive wall moving with velocity $-v_o$" problem.

In short, if the theory is correct the ping pong ball should collide with the massive ball, bounce off the massive ball, and leave with a velocity close to $3v_o$. Does it? Go home and try it. With three times its pre-collision velocity, the ping pong ball should have *nine* times its pre-collision kinetic energy (remember, kinetic energy is a function of velocity *squared*). That, in turn, means the ping pong ball should fly *nine times higher* than its initial height above the floor.

Note: You will not be tested on the frame of reference changes discussed in this section. It was included because it has educational value and because approaching such problems can be fun. It is also a technique that is often used by physicists in analyzing problems. I hope you found it amusing!

QUESTIONS

7.1) Just before striking a ceiling, a .4 kilogram ball is moving 11 m/s. It strikes the ceiling at a 30° angle relative to the vertical (see Figure I). Assuming only a small frictional force acting at the ceiling:

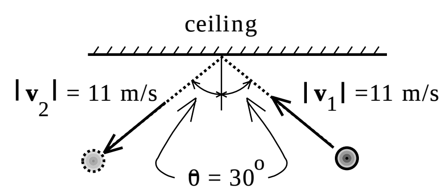


FIGURE I

a.) Is momentum conserved *through the collision*:

i.) In the *y direction*? Explain.

ii.) In the *x direction*? Explain.

b.) What *kind* of collision is it (i.e., elastic, inelastic, perfectly inelastic . . . what?)?

c.) What is the *impulse* applied to the ceiling by the ball?

d.) If the *average force* applied to the ceiling during the collision is 3200 newtons, over what *period of time* does the collision occur?

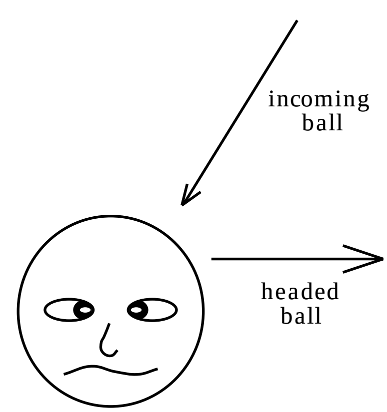
7.2) You have just made the football team. You are told to get into one of two tackling lines when you realize you have a choice: you can either be tackled by a 60 kilogram player who runs at 10 m/s or by a 120 kilogram player who runs at 5 m/s.

a.) Which player has the greater momentum?

b.) Which player has the greater kinetic energy?

c.) By whom would you prefer to be hit? Explain.

7.3) A .5 kg soccer ball comes down at a 30° angle relative to the vertical moving at 25 m/s (this is the same as having a ball freefall approximately 100 feet). It is headed by a player, leaving the player's head at an angle of 90° relative to the vertical moving with a velocity magnitude of 18 m/s.



a.) What was the ball's *momentum-change* off the player's head?

b.) If the collision takes .08 seconds, what is the *force* applied to the player's head (note that this is opposite the force applied to the ball)?

7.4) Determine the *y coordinate* of the center of mass of the inhomogeneous solid (i.e., a solid whose mass is *not* uniformly distributed throughout the form) shown in Figure II. Assume that the body's *surface density function* is *ky*, where *k* is a constant whose magnitude is *1* and whose units are appropriate to the situation.

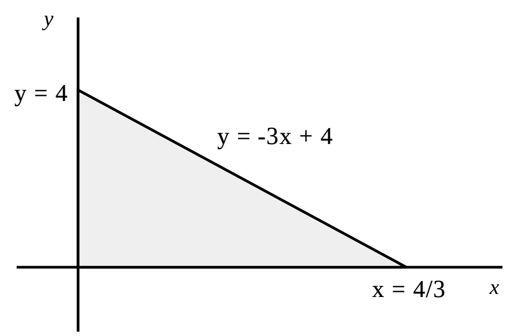


FIGURE II

7.5) A 60 kilogram bum stands on an 800 kilogram flatcar that is moving 15 m/s. The bum begins to run, finally reaching a speed of 5 m/s relative to the flat-car. How fast does the flat-car end up moving if:

a.) The bum runs in the same direction as that of the flat-car?

b.) The bum runs opposite the direction of motion of the flat car?

c.) The bum runs in a direction perpendicular to the direction of motion of the flat-car?

d.) For the situation outlined in *Part a*, compare the system's total *kinetic energy* before and after the bum began to run.

7.6) A cannon shell moving 240 m/s at an angle of 60^o with the horizontal explodes into two pieces. The first piece, comprised of two-thirds of the shell's mass, leaves the explosion with initial velocity of 260 m/s in the horizontal (i.e., at angle 0^o).

a.) What is the *velocity* of the second piece?

b.) Assuming the shell's mass is 30 kilograms, how much chemical energy was converted to kinetic energy in the explosion?

7.7) An 880 kilogram car rear-ends a stationary 1000 kilogram car whose brakes are locked. The collision is perfectly inelastic. If the coefficient of friction between each car's wheels and the pavement is .6, and if the cars slide-with-friction a distance of 1.2 meters after the collision, what must the first car's *velocity magnitude* have been just before the accident?

7.8) A man of mass 90 kilograms is ice skating out of control. While moving at 8 m/s in the *x direction*, he collides with his 55 kilogram girlfriend who is moving with velocity 10 m/s at an angle of 120^o with the *x axis* at the time. If the collision is perfectly inelastic, determine the *velocity* (as a vector) of the couple after the collision.

7.9) A spring-gun of mass m_g = *2 kilograms* uses an ideal spring with a spring-constant k = *120 nt/m* to shoot a ball of mass m_b = *.04 kilograms* out of its barrel. At a particular point in time, the cocked gun and ball are moving backwards over a *frictionless table* with velocity v_o = *5 m/s* (the word "backwards" is intended to mean that when the gun is fired the bullet will leave opposite the direction of the gun's motion). Relative to the table, what will the gun's velocity (v_g) and the ball's velocity (v_b) be just after firing? Assume the spring's *compression-distance* is x = *.15 meters* when the gun is cocked.

7.10) Jane (mass 40 kg) has entered a soap-box derby. The course ends by passing through a mountain via a tunnel, over a bridge, then to the finish line (a sketch--Figure III--is provided to the right).

Unfortunately, the bridge has collapsed. Tarzan, love-sick and definitely not-too-bright, sees the danger from above. Quickly, he makes a few calculations and realizes that if he can run fast enough, he can use the vine Cheetah is playing with to swing down from above and meet Jane's cart as it comes out of the tunnel. With the right momentum, he could stop the cart by colliding with it (again, see Figure III).

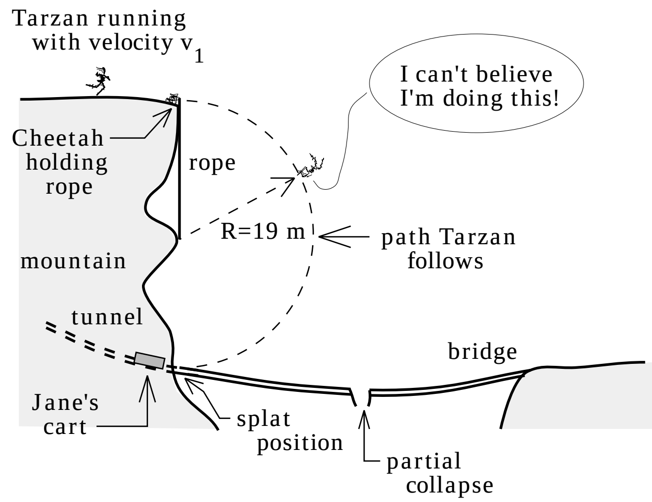


FIGURE III

a.) Assume the vine is 19 meters long. If Jane's cart has a mass of 190 kgs (without Jane) and a speed of 11 m/s out of the tunnel, and if Tarzan's mass is 90 kg, how fast must Tarzan run "up top" to effect a dead stop of Jane's cart just as it comes out of the tunnel? Assume Tarzan does not bounce off the cart upon collision, that everything stops dead when he hits the cart, and that the collision occurs directly under the point at which Tarzan begins his swing.

b.) For the amusement of it, what is the tension in the vine just before Tarzan and the cart collide?

7.11) NOTE: This question is for anyone interested in trying a problem using a *center of mass frame of reference*. There will *not* be a problem like this one on your next test.

A satellite approaching a planet at just the right angle will pick up kinetic energy from the planet as it whips around the planet and exits in the opposite direction (see Figure IV). This interaction can be approximated as an elastic collision (i.e., a collision in which the bodies within the system change their motion *while mechanical energy is conserved* within the system).

A satellite is observed to be moving with velocity 7 km/s at a particular distance *d* from a planet that is, itself, moving with velocity 12 km/s. The satellite sling-shots around the planet. What is the satellite's velocity v_s once it again reaches a distance *d* from the planet?

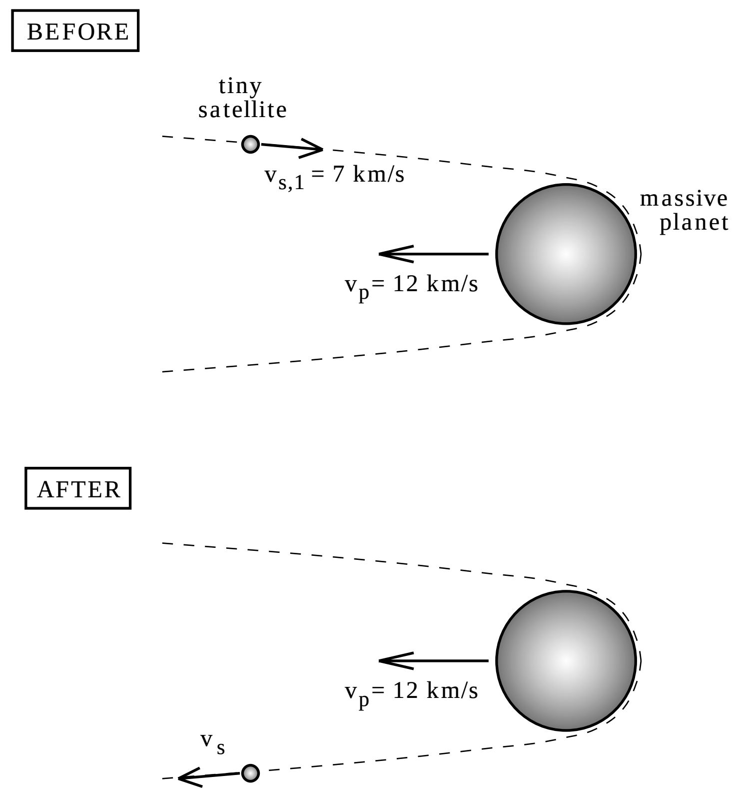


FIGURE IV