Chapter 1

MATHEMATICS REVIEW

A.) Scalars:

1.) A *scalar* is a variable that has *magnitude* (i.e., size) but does not have *direction* associated with it.

a.) Example 1: Temperature measurements are not recorded as "25^o C *upward*." Why? Because temperature is a scalar, it has a magnitude but it does *not* have direction.

b.) Other examples of scalars: length, time, mass, and speed.

2.) *Scalars* add and subtract in the same way dollars and cents add and subtract.

a.) Example 2: If we know the temperatures $T_1 = 115^o$ and $T_2 = 98^o$, then $T_2 - T_1 = -17^o$.

B.) Vectors:

1.) A *vector* is a variable that has both *magnitude and direction* associated with it.

a.) Example: Force is a vector. Push a box up an incline and you will find that the *direction* of the force is as important as the *magnitude* of force. Specifically, the force required to move the box up the incline in Case 1 (see Figure 1.1) will be considerably greater than the force required in Case 2.

Both *magnitude* and *direction* are important with vectors.

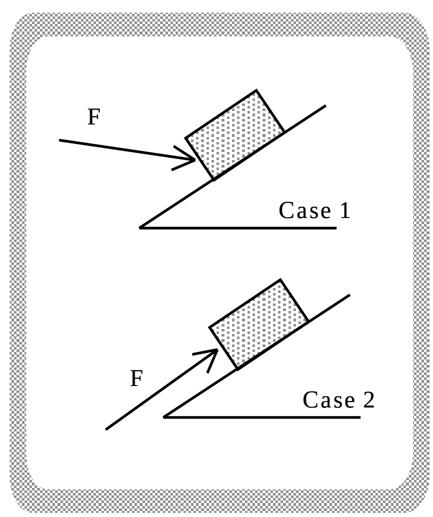


FIGURE 1.1

2.) Because *direction* makes a difference, vectors do *not* add or subtract like scalars.

a.) Example: A three newton force added to a four newton force will not necessarily sum to an equivalent force of seven newtons. Figure 1.2 depicts a situation in which the vector sum of those two forces generates an equivalent force of five newtons at an angle of 37^{o} with the horizontal.

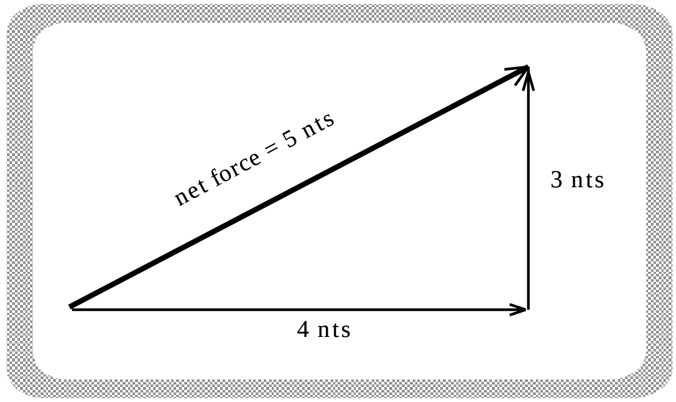


FIGURE 1.2

3.) A vector can be algebraically represented either by a *letter* with an arrow over it ($\vec{A}$) or by a letter in bold-face type (**A**). The former is most often used in classroom lectures and when doing problems longhand. The latter is used in texts as it is easier to represent on a computer. This book will use bold-face letters to represent vectors.

4.) There will be times when a vector's *magnitude* is all that is important. Notationally, the magnitude of the vector is represented as $|\mathbf{A}|$. On some occasions, this relatively formal approach is inconvenient. An alternate approach uses an unadorned letter like "A" (without the quotes) or a subscripted letter such as "A_x."

5.) The *graphical* representation of a vector is drawn as an arrow. The arrow's orientation depicts the vector's *direction* and the arrow's length is scaled to reflect the vector's *magnitude.*

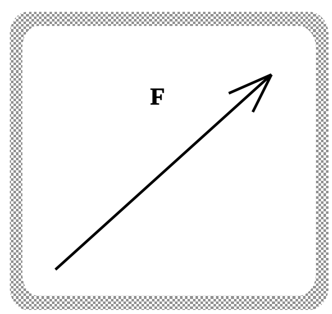


FIGURE 1.3

a.) Example: Figure 1.3 depicts a force vector ***F*** whose magnitude is 36 newtons and whose orientation is at 45^{o} with the horizontal. The scaling factor in the sketch is *one-half inch per 12 newtons.*

6.) Manipulating vectors, even for relatively mundane operations like addition or subtraction, requires special approaches. The two most commonly used are *graphical* manipulation and *algebraic* manipulation.

C.) Graphical Vector Manipulation:

1.) Consider the two velocity vectors ***A*** and ***B*** shown in Figure 1.4. To graphically add these, proceed as follows:

a.) Reproduce either vector (we'll use ***B***), drawing it to scale and keeping its orientation exactly as presented in the original sketch (Figure 1.5).

The scale used will be 50 *meters per second* per inch.

b.) Reproduce the second vector (vector ***A***) so that its *tail* is positioned at the *head* of the first vector (Figure 1.6). Again, make the drawing to scale and keep the vector's orientation intact.

c.) Draw the *resultant* vector ***C***=***A***+***B*** (Figure 1.7). This new vector will begin where the sketch began at the tail of ***B*** and end where the sketch ended at the head of ***A***.

d.) To determine the *magnitude* of the resultant vector ***C***, use a ruler or centimeter stick to measure ***C****'s* length, then multiply by the scaling factor. In this case we get approximately 87 newtons.

e.) To determine ***C****'s* direction *relative to the horizontal*, a protractor yields approximately 20 degrees.

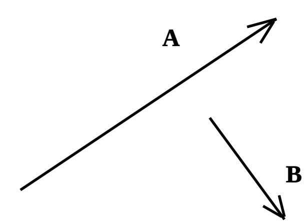


FIGURE 1.4

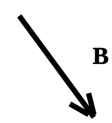


FIGURE 1.5

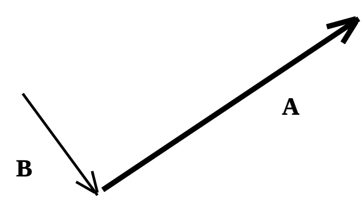


FIGURE 1.6

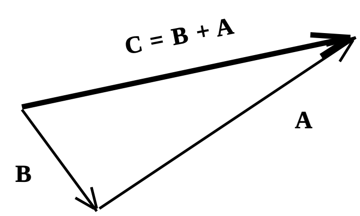


FIGURE 1.7

Note: Adding more than two vectors follows a similar pattern with the tail of each successive vector being placed at the head of the previous one until all the vectors are coupled (see Figure 1.8). Everything must be drawn to scale with relative directions kept intact. The *resultant* will be a vector that starts where you started and ends where you ended.

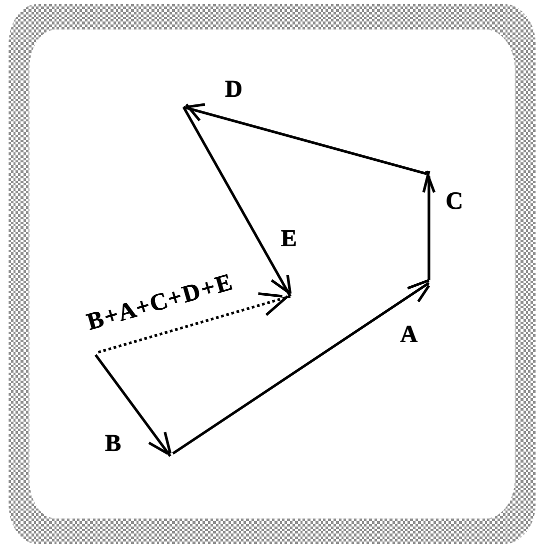


FIGURE 1.8

2.) Vectors can be multiplied by scalars as shown in Figures 1.9a through 1.9e below. Figure 1.9a depicts vector ***A***, then presents vectors *(1/2)**A**, 2**A**, -**A**,* and *-2**A*** respectively.

Note: A vector multiplied by a positive *scalar* either increases or decreases the *magnitude* of the original vector but does not change its orientation. Multiplying by *-1* effectively flips the vector so that its orientation is opposite that of the original vector, and multiplying by a *negative scalar* changes both the *magnitude* and the *direction.*

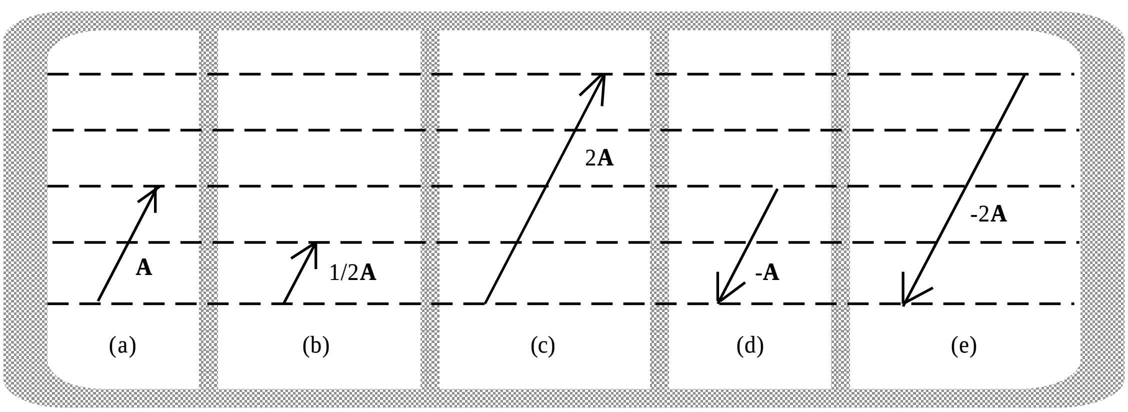


FIGURE 1.9

3.) Although it has its place, graphical vector manipulation is generally awkward. From a mathematical standpoint, it is usually more convenient to deal with vectors algebraically in the context of coordinate axes. We will make use of two such grid systems: one that uses Cartesian coordinates and one that uses polar coordinates.

D.) Algebraic Vector Manipulation-- Polar Notation:

1.) Polar notation defines a vector by designating the vector's *magnitude* $|\mathbf{A}|$ and *angle* θ relative to the + *x axis* (see Figure 1.10). Using that notation, the vector is written $\boldsymbol{A} = |A|\angle\theta$.

Note: Some math texts use an ordered paired vector notation like ($|\mathbf{A}|$, θ) to present polar information. As physicists do not usually use this notation, we will not use it in this book.

a.) Example 1: A force vector ***F*** with a *magnitude* of 12 newtons oriented at 210^o with the +*x axis* would be characterized as ***F*** = $12\angle 210^o$ (Figure 1.11).

b.) Example 2: A force vector ***F*** with a magnitude of 12 newtons oriented along the -*x axis* would be characterized as ***F*** = $12\angle 180^o$ (Figure 1.12).

2.) Students should be able to characterize a graphically presented vector in *polar notation*.

a.) Example: From the graph in Figure 1.13, characterize vectors ***C*** and ***D*** in *polar notation* (Answer: $\boldsymbol{C} = 2\angle 30^o$, $\boldsymbol{D} = 4\angle {-50}^o$).

Note: Negative angles are measured from the +*x axis* <u>*clockwise*</u>.

3.) Students should be able to graph a vector characterized in *polar notation*.

a.) Example: Graph $\boldsymbol{A} = 4\angle 150^o$ and $\boldsymbol{B} = 6\angle {-30}^o$ (the solution is shown in Figure 1.14 on next page).

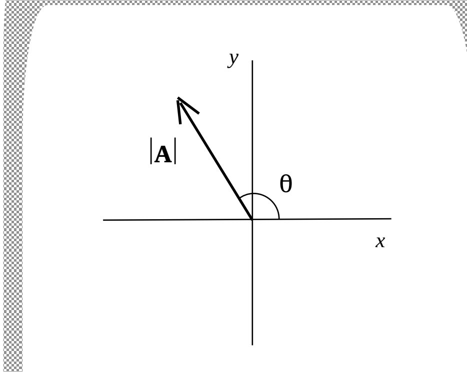


FIGURE 1.10

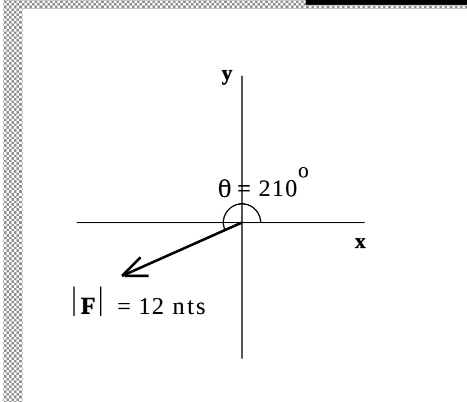


FIGURE 1.11

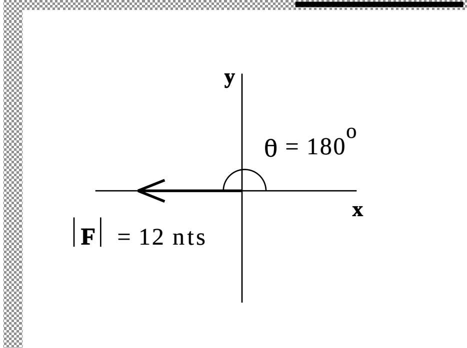


FIGURE 1.12

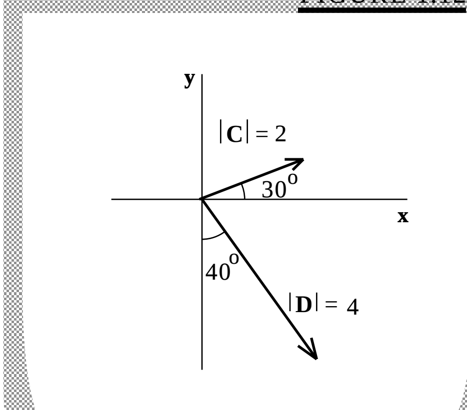


FIGURE 1.13

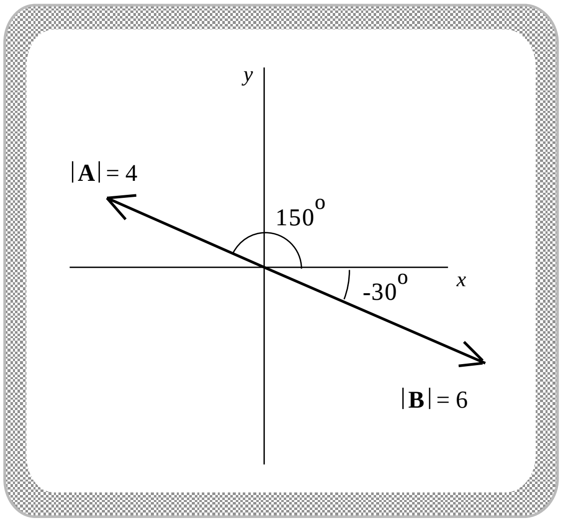


FIGURE 1.14

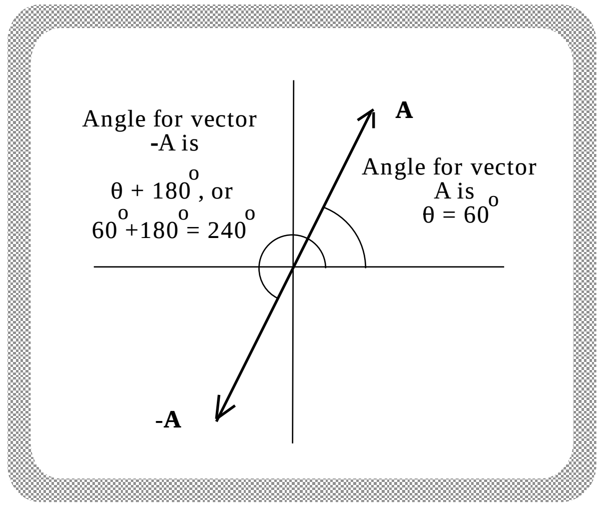


FIGURE 1.15

4.) Multiplying a vector by a *positive scalar* characterized in *polar notation* changes the *magnitude* of the vector but not the *direction* (i.e., the angle).

a.) Example: If **A** = 12 ∠ 210°, then 3**A** = 36 ∠210°.

5.) Multiplying a vector by (-1) does not change the vector's *magnitude*, but it does reverse its *direction* (see Figure 1.15). As direction in *polar notation* is denoted by the vector's angle, a reversal of direction effectively adds 180° to the angle.

a.) Example 1: If **D** = 4 ∠ 110°, then -**D** = 4 ∠ 290°.

Note: -**D** is *not* -4 ∠ 110°.

b.) Example 2: If **D** = 4 ∠ 110°, then -(1/2)**D** = 2 ∠ 290°.

Note: Angles should never be greater than 360°. If ***A*** = *12∠ 220°, -**A** = 12 ∠ (220°+180°) = 12 ∠ 400° = 12 ∠ 40°.*

E.) Algebraic Manipulation in Cartesian Coordinates:

1.) An effective if not obscure system called *unit vector notation* is used to denote vectors presented within a Cartesian (x-y-z) coordinate system. Within that system, the vector shown in Figure 1.16 would be characterized as $\boldsymbol{A} = 4\mathbf{i} + 3\mathbf{j}$.

As this notation will be new to many students, the following will hopefully explain the rationale behind it:

a.) Consider Figure 1.17. Graphical addition suggests that the vector ***A*** is equal to the sum of "a vector in the *+x direction* whose magnitude is 4" and "a vector in the *+y direction* whose magnitude is 3."

b.) We now define a special vector $\hat{\mathbf{i}}$ whose magnitude is always *one* and whose direction is always in the *+x direction* (Figure 1.18). Such a vector is called a *unit vector* in the *+x direction*.

Note: Physics texts that use arrowheads over vector quantities replace those arrowheads with a *hat* when referring to *unit vectors*. To be technically complete, therefore, a *hat* has been included over the "**i**" term shown above in *Part b*. Although you must use a *hat* whenever writing unit vectors out longhand (you will notice that I always use a *hat* in class), this book will assume that all boldface **i**, **j**, and **k** vectors are UNIT VECTORS and will *not* subsequently include the hat in the text.

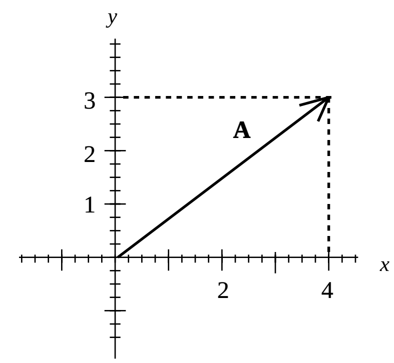


FIGURE 1.16

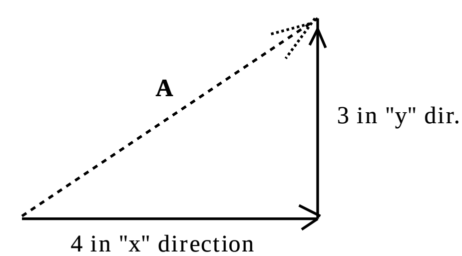


FIGURE 1.17

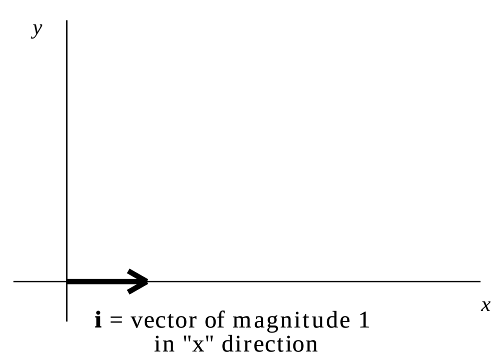


FIGURE 1.18

Put another way, when you see a boldface **j** in this text, know that it is being used to denote a *unit vector* in the *y direction* even though it has no hat above it.

c.) With ***i*** as defined above, notice that *4**i*** is a vector "whose magnitude is 4 and whose direction is along the +*x axis*" (Figure 1.19). This is half of the vector sum needed to carry out the vector addition required for the production of vector ***A*** as denoted in *Part a* above.

d.) Now define a *unit vector* directed along the +*y axis*; call it **j** (Figure 1.20).

e.) Notice that *3**j*** is a vector "whose magnitude is 3 and whose direction is along the +*y axis*" (Figure 1.21). This is the other half of the vector sum needed to carry out the addition required for the production of vector ***A*** as denoted in *Part a.*

f.) Having defined the idea of a unit vector, the unit vector characterization of vector ***A*** is ***A*** = *4**i*** + *3**j*** (Figure 1.22).

Note 1: For three dimensional situations, the +*z direction unit vector* is defined as **k**.

Note 2: Consider the vector ***A*** = -*4**k***. Technically, the *direction* should be associated with the *unit vector* ***k***. That is, the formally correct way of characterizing the vector would be as ***A*** = *4 (-**k**)*. Unfortunately, embedded signs can be overlooked, so it is

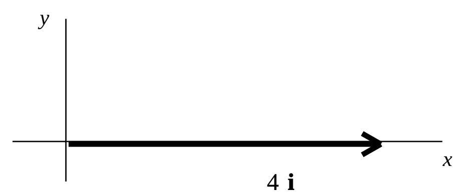


FIGURE 1.19

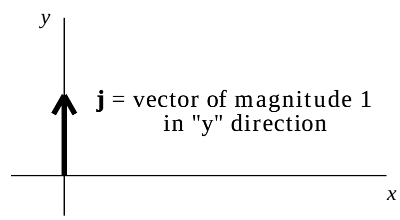


FIGURE 1.20

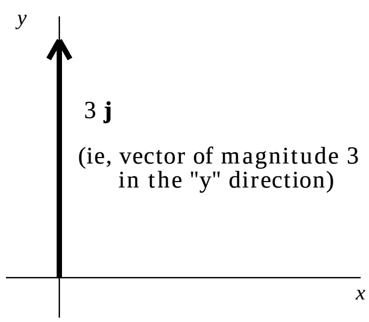


FIGURE 1.21

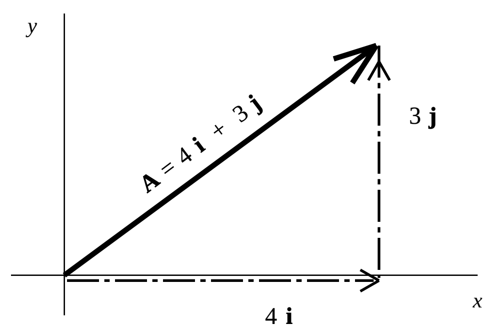


FIGURE 1.22

acceptable to put the negative sign in front of the expression as first depicted.

Note 3: In the vector $\boldsymbol{A} = -4\boldsymbol{i} + 3\boldsymbol{j}$, the "-4" part of the expression is called "the *x component* of ***A***," or A_x, and the "+3" term is called "the *y component* of ***A***," or A_y. *Components* are not really *magnitudes*--they can be negative.

2.) Students should be able to characterize a graphically presented vector, writing it out in *unit vector notation*.

a.) Example: From the graph in Figure 1.23, characterize vectors ***J*** and ***K*** in *unit vector notation* (Answer: $\boldsymbol{J} = 2\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{K} = 2\boldsymbol{i} - 5\boldsymbol{j}$).

3.) Students should be able to graph a vector characterized in *unit vector notation*.

a.) Example: Graph vectors $\boldsymbol{G} = -2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{H} = -3\boldsymbol{i} - 2\boldsymbol{j}$ (solution shown in Figure 1.24).

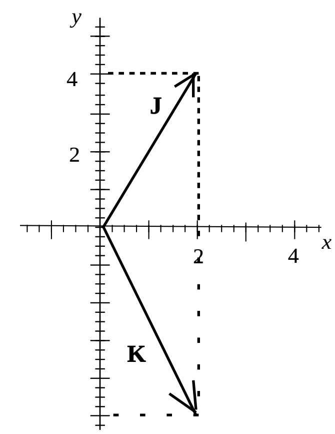


FIGURE 1.23

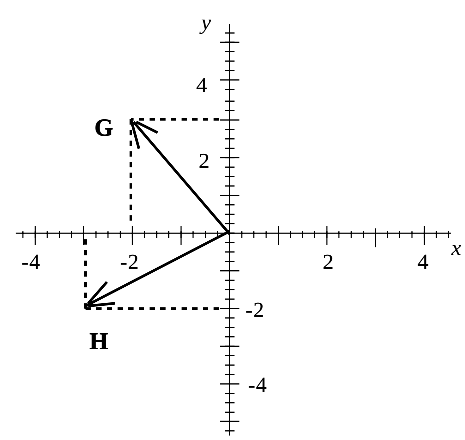


FIGURE 1.24

F.) Conversion: *Unit Vector* to *Polar Notation*:

1.) Consider the known vector $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j}$ shown in Figure 1.25. To characterize it in the *polar notation*, $\boldsymbol{A} = |\boldsymbol{A}|\angle\theta$.

a.) To determine $|\mathbf{A}|$: the right triangle shown in Figure 1.25 coupled with the Pythagorean relationship yields a vector magnitude:

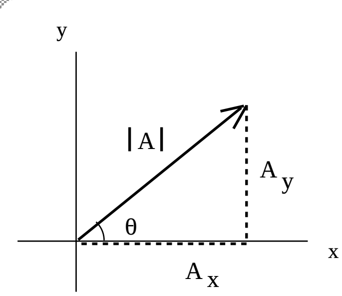


FIGURE 1.25

$$|\mathbf{A}| = [A_x^{\,2} + A_y^{\,2}]^{1/2}.$$

b.) To determine θ: The tangent of θ is defined as the ratio of the side opposite θ (i.e., A_y) divided by the side adjacent (A_x), or *tan* $\theta = A_y/A_x$. This implies that:

$$\theta = \tan^{-1}(A_y/A_x).$$

Note: This is expressed as θ, *the angle whose tangent is* (A_y/A_x).

c.) Example: The conversion of the velocity vector $\boldsymbol{A} = (-4\mathbf{i} + 3\mathbf{j})$ m/s to *polar notation* (see Figure 1.26):

$$\begin{aligned}\mathbf{A} &= |\mathbf{A}|\angle\theta \\ &= [A_x^{\,2} + A_y^{\,2}]^{1/2} \\ &\qquad \angle\,[\tan^{-1}(A_y/A_x)] \\ &= [(-4)^2 + (3)^2]^{1/2} \\ &\qquad \angle\,[\tan^{-1}(3/(-4))] \\ &= 5 \text{ m/s } \angle -36.9^{\circ}.\end{aligned}$$

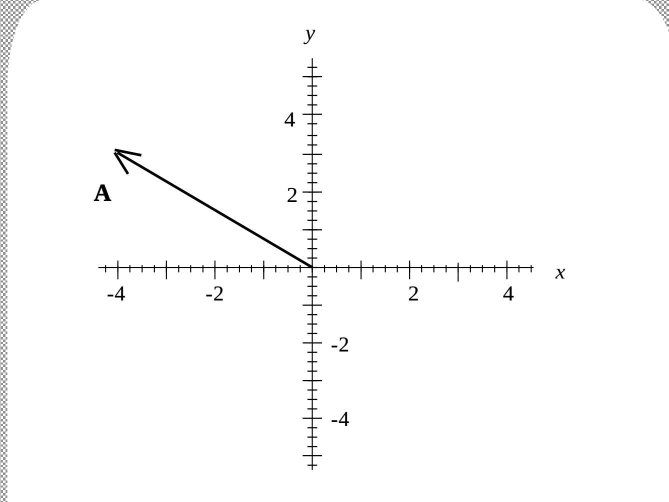


FIGURE 1.26

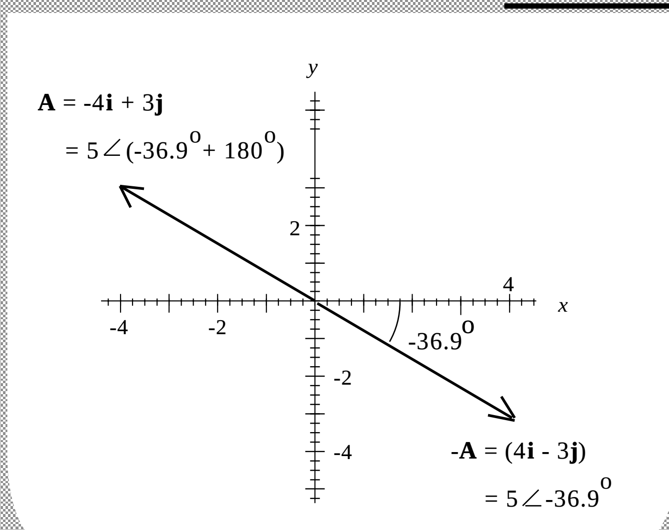


FIGURE 1.27

Note: There is SOMETHING WRONG here. ***A*** is a *second quadrant vector*, while the angle given by your calculator implies a fourth quadrant angle.

The problem lies in your calculator's inability to tell the difference between $tan^{-1}[3/(-4)]$--a second quadrant vector--and $tan^{-1}[(-3)/4]$--a fourth quadrant vector. As a consequence, all calculators assume they are dealing with *fourth quadrant values* whenever they are fed tangent arguments that are negative. In such cases, the correct *second quadrant angle* can be generated by adding 180^o to the calculator's value (see Figure 1.27 on previous page).

This means $\boldsymbol{A} = 5\ m/s\ \angle\ (-36.9^o+180^o) = 5\ m/s\ \angle\ 143.1^o$.

Note: A similar problem exists between first and third quadrant vectors. That is, *third quadrant* vectors have a tangent ratio of $(-A_y)/(-A_x)$, which is positive just as are *first quadrant* angles. All calculators assume that positive tangents are *first quadrant vectors*. Again, 180^o must be added to the calculator-generated angle if that is not the case.

G.) Conversion: *Polar* to *Unit Vector Notation*:

1.) There are two ways to do this. One is by memorizing a formula that always works; the other is by using your head (ooh, scary). As you'll undoubtedly forget the memorized equation sooner or later, understanding the *seat of your pants* approach is important. I'll outline both.

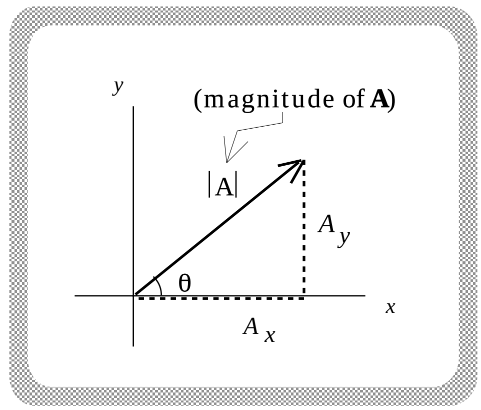


FIGURE 1.28

2.) Consider the known vector $\boldsymbol{A}= |\boldsymbol{A}|\angle\theta$ shown in Figure 1.28. To characterize it formally (i.e. in memorizable form) in *unit vector notation*:

a.) To determine A_x: *Cosine* $\angle\theta_1$ is defined as the ratio of the *side adjacent to* θ (i.e., A_x) divided by the *hypotenuse* ($|\boldsymbol{A}|$), or $\cos\ \theta = (A_x)/(|\boldsymbol{A}|)$. This implies that $A_x = |\mathbf{A}|\cos\ \theta$.

b.) Similar reasoning produces A_y as $A_y = |\mathbf{A}|\sin\ \theta$.

c.) These expressions will work for *any* angle (hence, if memorized, will never fail you . . . unless your memory fails you). As an example, the conversion of the velocity vector $\boldsymbol{A} = 7\ m/s\ \angle 130^o$ into *unit vector notation* (Figure 1.29) becomes:

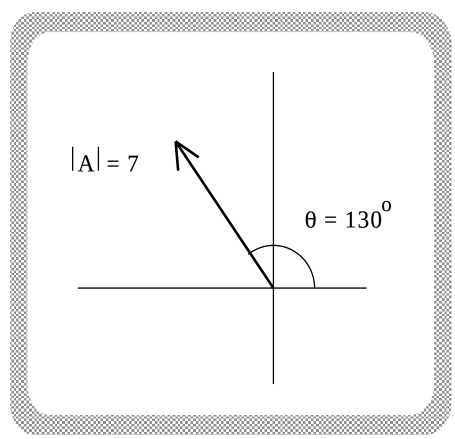


FIGURE 1.29

$$\begin{aligned}\mathbf{A} &= A_x\mathbf{i} + A_y\mathbf{j} \\ &= (|\mathbf{A}|\cos\theta)\,\mathbf{i} + (|\mathbf{A}|\sin\theta)\,\mathbf{j} \\ &= [(7\text{ m/s})\cos 130^o]\,\mathbf{i} + [(7\text{ m/s})\sin 130^o]\,\mathbf{j} \\ &= (-4.5\,\mathbf{i} + 5.4\,\mathbf{j})\text{ m/s.}\end{aligned}$$

Note: The *only* nice thing about the memorized approach is that it always gives each component's correct sign.

3.) Assuming your memory is as miserable as mine, the more intelligent way to do the conversion outlined above is by *the seat of your pants* approach. That is:

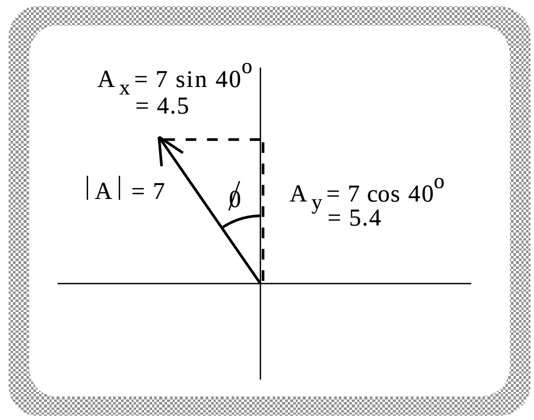


FIGURE 1.30

a.) Create a convenient *right triangle* like the one shown in Figure 1.30.

b.) Use the appropriate trig functions to determine the vector's components.

c.) Once you have the magnitude of the components, add signs and you're done (i.e., write the *x component* as negative). Doing so yields $\boldsymbol{A} = (-4.5\,\boldsymbol{i} + 5.4\,\boldsymbol{j})\ m/s$.

H.) *Dot Product* In Polar Notation:

1.) In polar notation, consider the two vectors: $\boldsymbol{A} = |\boldsymbol{A}|\angle\theta_1$ and $\boldsymbol{B} = |\boldsymbol{B}|\angle\theta_2$ (see Figure 1.31). The *dot product* between ***A*** and ***B*** produces a scalar quantity. The *magnitude* of the *scalar product* is defined as:

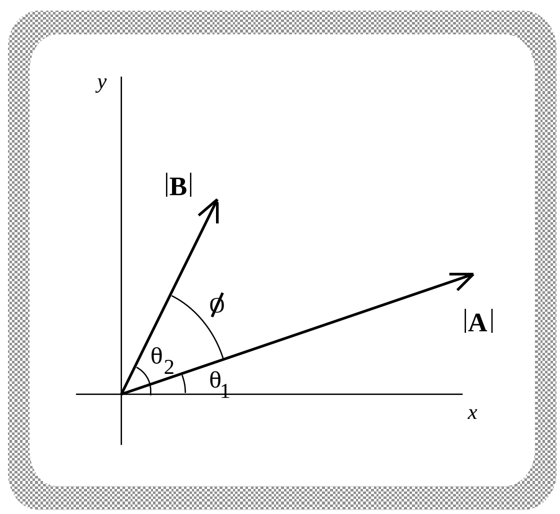


FIGURE 1.31

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\,\phi,$$

where ϕ is the *net angle* *between the line of the two vectors*.

2.) Example: Let $\boldsymbol{A} = 5\ nt\angle 30^o$ and $\boldsymbol{B} = 12\ m\angle 180^o$ (see Figure 1.32 on next page). What is $\boldsymbol{A}\cdot\boldsymbol{B}$?

a.) Following the definition of the *dot product*:

$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\,\theta,$
$= (5\text{ nt})\,(12\text{ m})\cos\,(150^o)$
$= -\,52\text{ nt}\cdot\text{m}.$

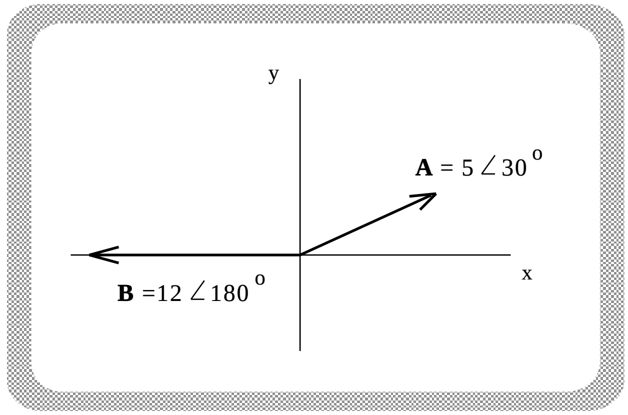


FIGURE 1.32

I.) *Dot Product* In Unit Vector Notation:

1.) In unit vector notation, consider the vectors: $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$ and $\boldsymbol{B} = B_x\boldsymbol{i} + B_y\boldsymbol{j} + B_z\boldsymbol{k}$. The *dot product* between $\boldsymbol{A}$ and $\boldsymbol{B}$ produces a scalar quantity that is mathematically equal to:

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z.$$

Note that this is easily derived:

a.) Beginning with $\boldsymbol{A}\cdot\boldsymbol{B} = (A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k})\cdot(B_x\boldsymbol{i} + B_y\boldsymbol{j} + B_z\boldsymbol{k}).$

b.) Calling upon the distributive nature of *dot products*, we get a whole string of mini *dot products*:

$$\mathbf{A}\cdot\mathbf{B} = [(A_x\mathbf{i})\cdot(B_x\mathbf{i})] + [(A_x\mathbf{i})\cdot(B_y\mathbf{j})] + \ldots$$

c.) As the angle between two vectors in the $\boldsymbol{i}$ direction is zero degrees, the first mini *dot product* shown above is equal to $(A_x)(B_x)(\cos 0^o) = A_xB_x$. The implication is that *like-termed* products will *not* be zero (assuming neither A_x nor B_x are zero).

d.) As the angle between the $\boldsymbol{i}$ direction and the $\boldsymbol{j}$ direction is 90^o, the second mini *dot product* is equal to $(A_x)(B_y)(\cos 90^o) = 0$. The implication is that all *cross-termed* products *will* be zero.

e.) Bottom line: In *unit vector notation*, $\boldsymbol{A}\cdot\boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z$.

2.) Example: Let ***A*** = *(3**i** - 4**j** - 5**k**) newtons* and ***B*** = *(2**i** + 7**j** +3**k**) meters.* What is $\boldsymbol{A} \cdot \boldsymbol{B}$? Following the derived expression for the *dot product* expressed in unit vector notation:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= A_x B_x + A_y B_y + A_z B_z \\ &= (3 \text{ nt})(2 \text{ m}) + (-4 \text{ nt})(7 \text{ m}) + (-5 \text{ nt})(3 \text{ m}) \\ &= -37 \text{ nt}\cdot\text{m}. \end{aligned}$$

J.) *Dot Product* In General:

1.) To understand the physical significance of the *dot product*, consider Figure 1.33. In it, the vector ***B*** has been split into two components--one parallel to the *line of **A*** and one perpendicular to the *line of **A***. Notice that the component parallel to the line of ***A*** has a magnitude of $|\boldsymbol{A}||\boldsymbol{B}| \cos \phi$.

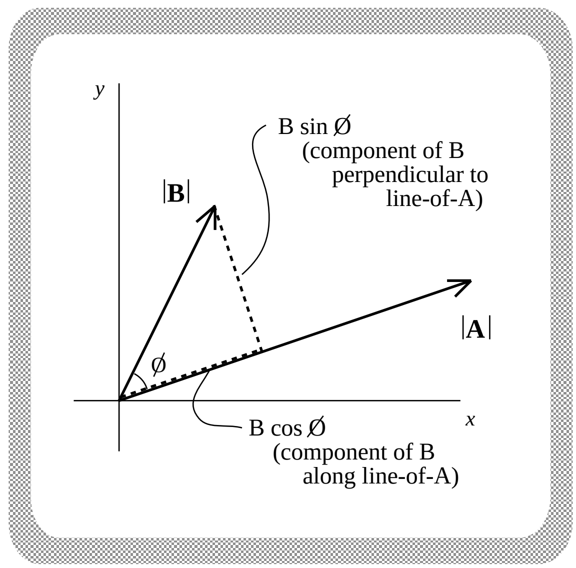


FIGURE 1.33

a.) Conclusion: When the *dot product* is taken between two vectors, it generates a quantity equal to the product of:

i.) The *magnitude* of one vector ($|\boldsymbol{A}|$ in this case) and,

ii.) The *magnitude* of the second vector's component that runs *parallel* to the first vector (i.e., $|\boldsymbol{B}| \cos \phi$).

Note 1: It doesn't matter whether you take $(|\boldsymbol{A}|)(|\boldsymbol{B}| \cos \phi)$ or $(|\boldsymbol{A}| \cos \phi)(|\boldsymbol{B}|)$. Both will work.

Note 2: For the sake of visualization, if $\boldsymbol{A} = 5 \text{ nt} \angle 30^o$ and $\boldsymbol{B} = 12 \text{ m} \angle 180^o$, the component of **A** along the *line of **B*** is shown in Figure 1.34 and the

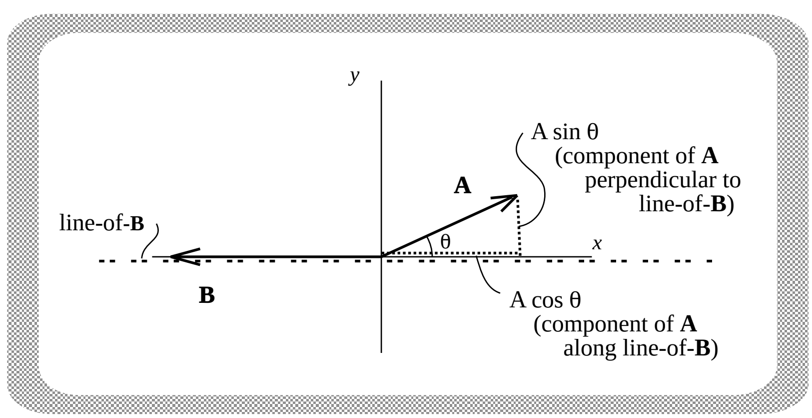


FIGURE 1.34

component of ***B*** along the *line of **A*** is shown in Figure 1.35.

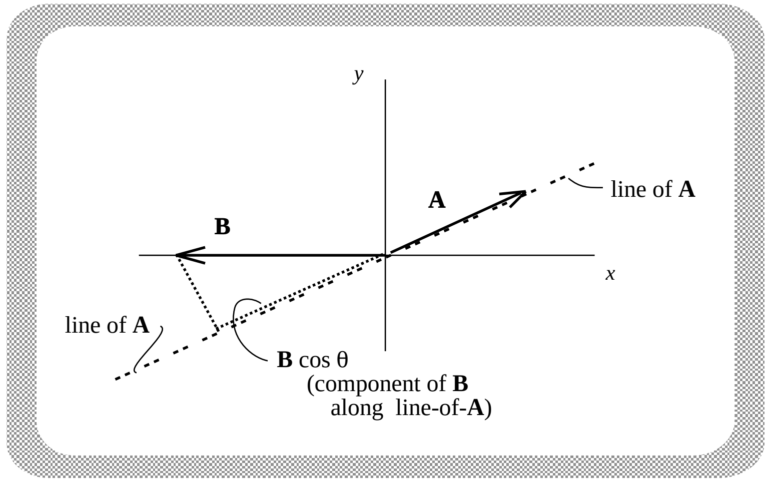


FIGURE 1.35

2.) Example of a situation in which the *dot product* comes in handy: Consider Fig-ure 1.36.

a.) A force ***F*** is applied to a moving object as it traverses over a frictionless surface for a distance ***d***. Notice that the force will increase the object's speed. Notice also that the force component that makes the object increase its speed is *the component of **F** along the line of **d***, or $|\mathbf{F}|\cos\theta$.

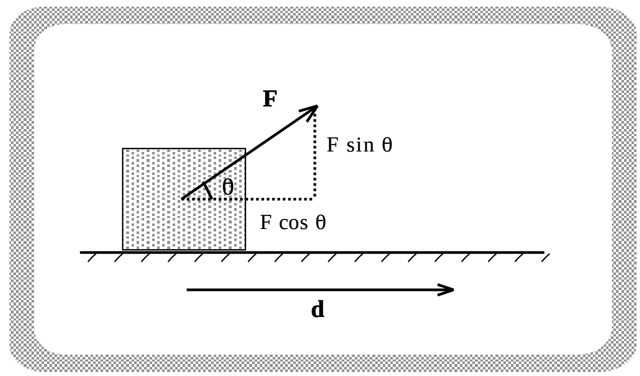


FIGURE 1.36

b.) As the amount of speed the object picks up is dependent only upon the distance over which the force acts (i.e., the *magnitude* of the displace-ment vector ***d***) and the component of ***F*** along the line of ***d*** (i.e., $|\boldsymbol{F}|\ \cos\ \theta$), the product of those two quantities is deemed important enough to be given a special name--*WORK.*

c.) In short, the *work* done by the force ***F*** acting on an object whose dis-placement is defined by a distance ***d*** is mathematically defined as $W_F = \boldsymbol{F}\cdot\boldsymbol{d}$.

K.) *Cross Product* In Polar Notation:

1.) Consider two vectors $\boldsymbol{A} = |\boldsymbol{A}|\angle\theta_1$ and $\boldsymbol{B} = |\boldsymbol{B}|\angle\theta_2$. The *cross product* between ***A*** and ***B*** produces a <u>vector</u> whose *magnitude* is mathematically defined as:

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\ \phi,$$

where ϕ is the angle between the two vectors.

2.) Example: Let $\boldsymbol{A} = 5\ newtons\ \angle\ 30^o$ and $\boldsymbol{B} = 12\ meters\ \angle\ 180^o$ (look back at Figure 1.32). What is $|\boldsymbol{B}x\boldsymbol{A}|$?

a.) Following the definition of the *magnitude* of the *cross product*:

$$
\begin{aligned}
|\mathbf{B}x\mathbf{A}| &= |\mathbf{B}|\ |\mathbf{A}|\ \sin\phi, \\
&= (12\text{ m})(5\text{ nt})\sin(150^o) \\
&= 30\text{ nt}\cdot\text{m}.
\end{aligned}
$$

b.) When dealing with vectors in *polar notation*, the *direction* of a cross product vector can be found using the right-hand rule. This rule is outlined below:

i.) The <u>line of both vectors</u> should be extended until they intersect (they already intersect in the problem we are examining);

ii.) With the wrist of the <u>right hand</u> placed at the intersection of the two vectors, the straightened fingers of the open right hand should be positioned so that they are parallel to the direction of the first vector (vector ***B*** in this case);

iii.) The fingers of the right hand should then be curled (waved) in the direction of the second vector (vector ***A*** in this case). *Note that you may have to flip your hand over to do this.*

iv.) If the thumb of the right hand is held out at a right angle to the fingers, the *direction* of the thumb will point in the direction of the cross product vector.

Note: Both vectors ***A*** and ***B*** are in the plane of the paper, whereas the direction of the right thumb alluded to above will be *out of* or *into* that plane. As peculiar as this may seem now, the direction of a *cross product* is always perpendicular to the plane defined by the vectors being crossed.

c.) In the problem above, your thumb should end up pointing downward into the page. Assuming the *+**i** direction* is to the right and the *+**j** direction* is upward toward the top of the page, the cross product's direction will be in the *negative z* direction characterized by a -***k*** in unit vector notation.

Note: Assuming we are not dealing with an angle of 180^o, there are two angles between any given vectors--one less than 180^o and one more than 180^o. The "wave" should always be through the angle *less than* 180^o.

d.) Putting everything together, we get ***B***x***A*** = *30 nt·m (-**k**)*. Or, if you do not like having the negative sign embedded in the middle of an expression, the cross product could be written ***B***x***A*** = *-30 nt·m (**k**)*.

Note 1: There will be times when the two vectors being crossed will not have a common starting point. When that occurs, it is very important that you extend the line of the two vectors until they intersect before trying to execute the right-hand rule.

Note 2: An alternate approach to determining *cross product* directions: Position your open right hand so that your thumb is in the direction of the first vector (***B*** in this case) and your straightened fingers are in the direction of the second vector (you may have to flip your hand over to accomplish this). The direction the palm of your right hand faces (i.e., along a vector coming out of the palm) will be the direction of the cross product. Try it for the above situation (Figure 1.37) and you will find that your palm faces downward.

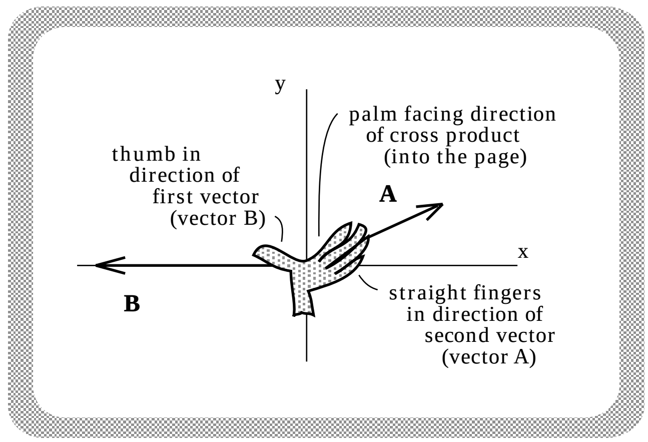


FIGURE 1.37

Note 3: If you are wondering why anyone would want an operation that takes two *x-y plane* vectors and produces a third vector whose direction is *perpendicular* to the *x-y plane*, an example is coming in the section CROSS PRODUCT IN GENERAL.

L.) ***Cross Product*** **In Unit Vector Notation:**

1.) In unit vector notation, consider the vectors $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ and $\mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$. The *cross product* between ***B*** and ***A*** produces a vector the *magnitude* and *direction* of which can be determined by evaluating the following matrix:

$$\mathbf{B} \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix} = \begin{aligned} &\mathbf{i}\,[(B_y)(A_z) - (B_z)(A_y)] \\ &+ \mathbf{j}\,[(B_z)(A_x) - (B_x)(A_z)] \\ &+ \mathbf{k}\,[(B_x)(A_y) - (B_y)(A_x)]. \end{aligned}$$

Note: Because there are unit vectors directly placed within the matrix, its evaluation will automatically give you both the magnitude *and direction* of the cross product. You do not have to mess with the right-hand rule when evaluating cross products in u.v.n.

2.) To begin with, memorizing the end-result of the *cross product* matrix is a bit of a waste of time. It is much better to simply learn how to evaluate such a matrix (you will need to know how to do this later when we get to *circuit analysis*). The obvious question is, *how do you do that?*

3.) Consider the following example. Let $\boldsymbol{A} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{B} = -4\boldsymbol{i} - 5\boldsymbol{j} - 6\boldsymbol{k}$. What is $\boldsymbol{B} \times \boldsymbol{A}$?

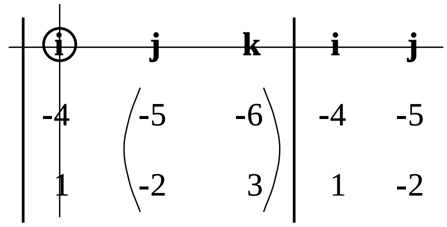
a.) The matrix requires that you array the unit vectors across the top row as shown. The first vector (***B*** in this case) in the cross product has its *components* (signs included) placed in the second row of the matrix, and the second vector has its components placed in the third row. So:

$$\mathbf{B} \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -5 & -6 \\ 1 & -2 & 3 \end{vmatrix}$$

b.) There are two ways to proceed from here. I prefer the approach that is, in my opinion, the simplest. To follow that technique, the first two columns must be reproduced to the right of the matrix as shown below (the reason for doing this will become evident shortly).

$$\left|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -5 & -6 \\ 1 & -2 & 3 \end{matrix}\right| \begin{matrix} \mathbf{i} & \mathbf{j} \\ -4 & -5 \\ 1 & -2 \end{matrix}$$

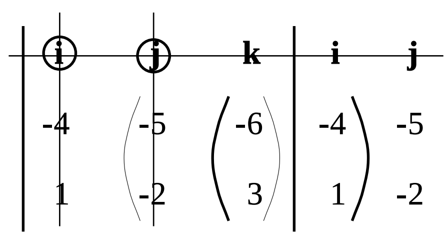
c.) Once set up, circle the first ***i*** unit vector, then cross out the row and column in which it resides.



d.) The ***x*** component of the cross product will be ***i*** (i.e., the unit vector that is circled) times the evaluation of the 2x2 matrix shown in the sketch. Note that the upper left-hand piece of that 2x2 matrix is *one column over, one row down* from the circled ***i***).

e.) To evaluate a 2x2 matrix, take the upper left-hand piece times the bottom right-hand piece MINUS the upper right-hand piece times the lower left-hand piece. In our example, that will be (-5)(3) - (-6)(-2) = -27. Once you multiply that by ***i*** to get *-27**i***, you have the *x component* of the cross product.

f.) The exact same thing is done with the ***j*** column and row (this is usually overlaid on top of the previous set-up--it isn't likely that you are going to want to recreate the entire matrix to do the next step). That is shown below.



g.) For this part, multiply the circled ***j*** times the evaluation of the 2x2 matrix shown in the sketch above (the upper left-hand piece is *-6*). The evaluation for this part will be (-6)(1) - (-4)(3) = +6. Multiplying by the ***j*** yields *+6**j***, and you have the *y component* of the cross product.

h.) The *z component* is taken care of similarly (try it) yielding a final cross product, unit vectors and all, of:

B x**A** = **i** [(-5)(3) - (-6)(-2)] + **j** [(-6)(1) - (-4)(3)] + **k** [(-4)(-2)-(-5)(1)]
= -27**i** + 6**j** + 13**k**.

M.) *Cross Product* In General:

1.) To understand the physical significance of the cross product, reconsider Figure 1.33.

a.) Vector ***B*** has been split into two components--one *parallel* to the line of ***A*** and one *perpendicular* to the line of ***A***. Notice that the component perpendicular to the line of ***A*** has a magnitude of $|\mathbf{B}|\ sin\ \phi$.

b.) Evidently, cross products generate a vector whose *magnitude* is equal to the product of:

i.) The *magnitude* of one of the vectors ($|\mathbf{A}|$ in Fig 1.33), and

ii.) The *magnitude* of the second-vector's-component that runs *perpendicular* to the first vector ($|\mathbf{B}|\ sin\ \phi$).

c.) An example of such a situation follows in *Part 3* of this section.

Note: Again, it makes no difference whether the magnitude is obtained by determining $(|\mathbf{A}|)(|\mathbf{B}|\ sin\ \phi)$ or $(|\mathbf{A}|\ sin\ \phi)(|\mathbf{B}|)$. A sketch of the information required to determine the former is shown in Figure 1.33, whereas the information required to determine the latter is shown in Figure 1.38.

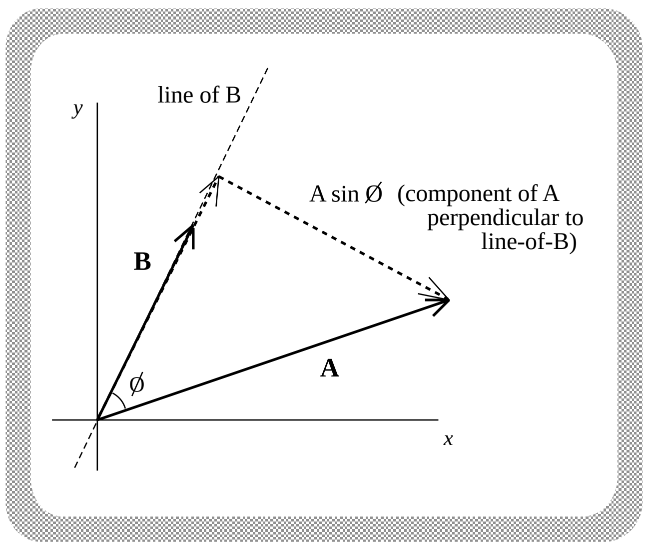


FIGURE 1.38

2.) The significance of a cross product's *direction* depends upon the situation in which the *cross product* is used. For instance, a charged particle moving with velocity ***v*** in a magnetic field ***B*** will feel a magnetic force ***F*** that is proportional to ***vxB***. In this case, the *direction* of the *cross product* is the *direction* of the magnetic force as it is applied to the particle (the apparently odd fact that the *direction* of a magnetic force on a moving charged particle is always *perpendicular* to the plane defined by the *velocity* and *magnetic field* vectors was deduced experimentally).

Note: Concerning this "perpendicular-to-the-plane-of-the-two-vectors" characteristic of *cross-product directions*, notice that if the vectors being crossed are in the *x-y plane*--a situation that will *always* be the case when working in *polar notation*--the *cross product* direction will be either in the +***k*** or -***k*** *direction*.

3.) Example of a situation in which the *cross product* comes in handy: Consider Figure 1.39.

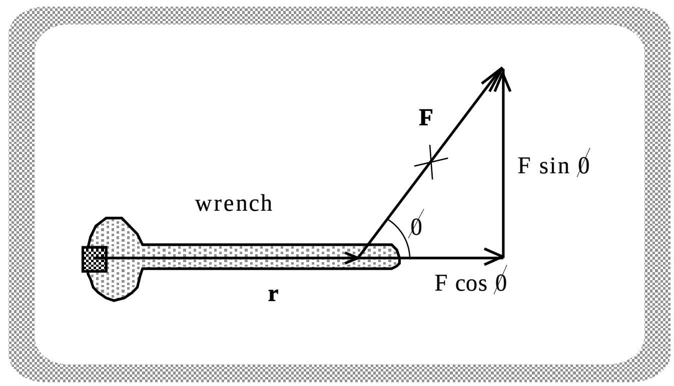


FIGURE 1.39

a.) A force ***F*** is applied to a wrench at a distance ***r*** units from the axis of rotation. Note that:

i.) The greater $|\boldsymbol{r}|$, the less difficult it will be to rotate the bolt;

ii.) The greater $|\boldsymbol{F}|$, the less difficult it will be to rotate the bolt; and

iii.) The force component that will make the bolt rotate will be the component *perpendicular* to the line of ***r*** (i.e., $|\boldsymbol{F}| \sin \phi$).

b.) As ease of rotation is related to $|\boldsymbol{r}|$ and $(|\boldsymbol{F}| \sin \phi)$, the product of those two variables is deemed important enough to be given a special name--*torque* ($\boldsymbol{\Gamma}$). In short, the magnitude of the torque applied by ***F*** about the axis of rotation will be $|\boldsymbol{\Gamma}| = |\boldsymbol{r} x \boldsymbol{F}|$.

c.) Assuming ***r*** and ***F*** are in the x-y plane, the direction of the *cross product* will either be in the +***k*** or -***k*** *direction* (using the right-hand rule outlined above, it turns out to be the +***k*** *direction*).

Note 1: For straight-line motion, the use of the ***i, j,*** and ***k*** unit vectors to denote, say, a velocity's direction, is easy to decipher. In such a case, they quite literally tell you the *direction* in which the body is moving. Unfortunately, when dealing with planar rotational motion and the *cross products* that define them, the idea of *direction* is a little more complicated.

Note 2: If *Part d* (below) does not make perfect sense now, don't worry about it. All you really need to be able to do at this point is to calculate a *cross product* (both magnitude and direction). The significance of the calculation will become much more evident when we begin using the math in specific situations.

d.) Direction of a *cross product*:

i.) To begin with, notice that the motion of a spinning object (we'll assume it is spinning in the *x-y plane*) does not really *go* anywhere--it just sits there spinning. That means that the *direction* of, say, the *rotational velocity* of the object can have nothing to do with *linear displacement*.

ii.) In fact, direction as related to spinning objects addresses one question only: Is the rotation *clockwise* or *counterclockwise*?

iii.) As there are no *unit vectors* defined in the clockwise direction, a little fancy footwork is required to convey the *sense of rotation*, using the unit vector notation we already have at our disposal.

iv.) The vectors used to identify the torque that started the object spinning in the first place (i.e., ***F*** and ***r***) are in the *x-y plane*, but notice that the object rotates about a line parallel to the z-axis.

v.) Physicists and mathematicians traditionally define the sense-of-rotation by denoting the axis about which the object rotates.

vi.) If, therefore, a rotation in the *x-y plane* is clockwise (this kind of rotation would screw a bolt INTO the *x-y plane*), the axis of rotation is along the *z axis* (inward) and the unit vector assigned to denote that fact would be -***k***.

By the same token, if a rotation in the *x-y plane* is counterclockwise (i.e., screwing the bolt OUT), the axis of rotation is along the *z axis* (outward) and the unit vector assigned to denote that fact would be +***k***.

vii.) Using that designation, torques that produce *counterclockwise* motion unscrew bolts *out* of the page and are designated as having a *+**k** direction*. Torques that produce clockwise motion screw a bolt *into* the page and are defined as having a *-**k** direction*.

viii.) In light of all this, it shouldn't be surprising to find that a torque that makes a bolt rotate, say, clockwise, will have a *cross product* whose direction is along the ***-k** axis*. Put another way, when a torque calculation readings ***r*x*F*** = *(3 nt·m)(-**k**)*, the -***k*** part of the cross product tells you that the torque in question has a *clockwise* sense.

QUESTIONS

Note: If it is now at least three days into the course, I would suggest you begin with Question #1.3. Do Questions #1.1 and #1.2 only if you have time--they are *not* terribly important concepts.

1.1) With some urgency, a newlywed couple on their honeymoon hires a not-so-bright boatman to row them across a 100 meter wide river to their honeymoon hotel. In calm water, the boatman (we'll call him Jack *the idiot*, for reasons that will become obvious shortly) can row his dinghy 5 miles per hour. The river moves at 2 miles per hour.

When sitting directly across from the hotel, the boatman points his bow at the hotel and proceeds to row like mad without a second glance. In blissful ignorance, he, the newlyweds, and the boat do not move directly across the river but instead move across and down the river--Jack has not compensated for the current. (Landing nowhere close to the hotel, the couple is less than delighted with Jack's performance, hence tagging him "the idiot.")

a.) Using *graphical manipulation*, determine the boat's actual velocity (as a *vector*) relative to dry land.

b.) (This one is a stinker--don't spend a lot of time on it). Changing the problem slightly, let's assume the boat starts 30 meters down river from the original starting position (that is, 30 meters below a line drawn directly across the river to the hotel--see Figure I). If Jack is clever, he can point the boat upstream to compensate for the moving water and make the boat's net movement travel directly toward the hotel. How would he have to orient the boat to do so and, as a consequence, how fast would the boat move relative to dry land? Again, use graphical means to determine this!

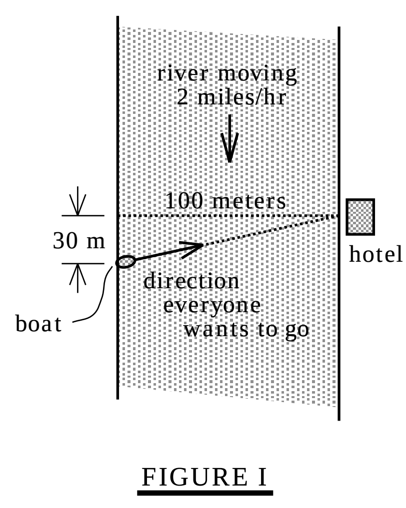


FIGURE I

1.2) A plane flies 80 miles north, then makes a north-westerly turn (versus south-westerly turn) at an unknown angle and flies an additional 60 miles. Upon landing, the pilot realizes she is exactly 30^o west of north, *relative to her starting point*.

a.) Use graphical manipulation to determine the "unknown angle" of her westerly turn.

b.) What was her *net* distance traveled (i.e., her net displacement)?

__Note:__ Graphical manipulation is only rarely used in physics, but it is something that is usually included in physics classes for the sake of completeness.

1.3) Graph the vectors $\boldsymbol{T} = 8\boldsymbol{i} - 12\boldsymbol{j}$ and $\boldsymbol{P} = 7\angle(-60^o)$.

1.4) From the graph in Figure II, characterize vectors ***A*** and ***C*** in *unit vector notation* and vectors ***B*** and ***D*** in *polar notation*.

1.5) On the graph shown in Figure II, vectors ***B*** and ***D*** are almost the same length even though their magnitudes are different. Why, and what does the goof do to the angles?

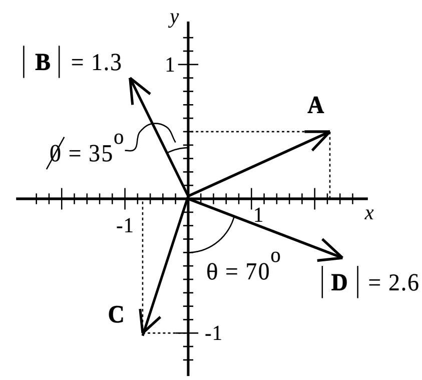


FIGURE II

1.6) Assume:

$\mathbf{A} = -8\mathbf{i} + 12\mathbf{j}$
$\mathbf{B} = -4\mathbf{i} - 3\mathbf{j}$
$\mathbf{C} = 5\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}$
$\mathbf{D} = 7 \angle (-60^o)$
$\mathbf{E} = 12 \angle 225^o$
$\mathbf{F} = 2 \angle 105^o$.

In the problems below, use the above vectors *as they are __denoted__ above* (that is, if asked to do ***A***x***B***, do so in unit vector notation if ***A*** and ***B*** are *given* in u.v.n.). With that in mind, determine:

a.) -(1/3)**A**;
b.) -6**E**;
c.) **A** + **B** - **C**;
d.) **E** converted to unit vector notation;
e.) **F** converted to u.v.n.
f.) **A** converted to polar
g.) **B** converted to polar
h.) **A**·**C**;
i.) **D**·**E**;
j.) **A**x**B**;
k.) **C**x**B**;
l.) **D**x**E**.

1.7) What does ***A***·***B*** really tell you?
1.8) What does ***A***x***B*** really tell you?