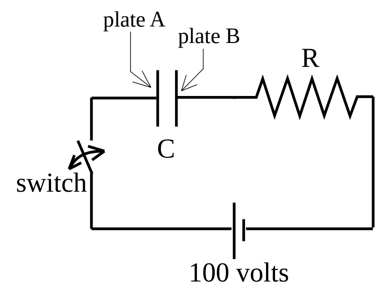
Capacitors -- Conceptual Solutions



1.) You have a power supply whose low voltage "ground" terminal is attached to a resistor whose resistance is $R = 10^4$ *ohms*. The resistor is attached to a plate (we'll call it *plate B*) which is next to, but not connected to, a second plate (we'll call it *plate A*). Reiterating, THERE IS NO CONNECTION between *plate A* and *plate B*. There is, additionally, no initial charge on either of the plates. Attached to *Plate A* is a switch. On the other side of the switch is the high voltage "hot" terminal of the power supply. A sketch of the situation is shown. At $t = 0$, the switch is closed.

a.) Current initially flows between the high voltage terminal and *Plate A*. Why? That is, what's going on?

Solution: There is a voltage difference between the high voltage terminal of the power supply and the chargeless plate. As such, an electric field will set itself up between the two points and current will flow.

b.) Current initially flows from *Plate B* through the resistor, and back to the ground of the power supply. Why? That is, what's going on?

Solution: The electric field between the plates forces charge carriers off *Plate B*. These move through the resistor and back to the ground side of the power supply. The net effect is that current appears to pass through the parallel plate device as though the plates were connected.

c.) What is the two-plate device called?

Solution: This device is called a *capacitor*.

d.) After a while, there is a voltage $V = 10$ *volts* across the plates. At that point in time, there is 10^{-10} coulombs of charge on *plate A*. The ratio of the *charge to voltage* is 10^{-9}.

i.) How much charge is on *plate B*?

Solution: The charge on *plate B* will be equal and opposite the charge on *plate A*, or -10^{-10} *coulombs*.

ii.) What is this ratio called?

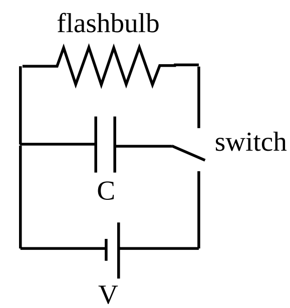
Solution: This *charge per volt* ratio is called *the capacitance* of the capacitor. It is the constant that identifies how large a capacitor is. (Note: You should be noticing a pattern here. The voltage V_R across a resistor is proportional to the current i through the resistor with the proportionality constant being the resistance R of the resistor. That is, $V_R = iR$. By the same token, the charge q

on one plate of a capacitor is proportional to the voltage V_c across the capacitor with the proportionality constant being the capacitance C of the capacitor. That is, $q = CV_c$.)

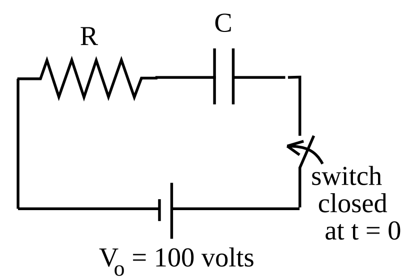
iii.) At some later point in time, the voltage across the plates is doubled. What is the ratio of *charge to voltage* in that case? Explain.

Solution: If the voltage doubles, the charge on one plate will double and the ratio will stay the same.

2.) What do capacitors (often referred to as *caps*) generally do in DC circuits? Give an example.



Solution: Capacitors store energy in the form of an electric field between the plates. The best example I can think of is a flashbulb circuit (a simple version of a flash circuit is shown to the right). When the flash is turned on, a switch puts the capacitor in series with a battery allowing the capacitor to charge up. When the photo is taken, the switch flips up putting the charged capacitor in series with the flash (this is shown as a resistor in the circuit). The capacitor discharges through the bulb motivating it to flash. Once discharged, the switch flips back to charging mode and the capacitor recharges (this is why it usually takes a few seconds before you can take the next flash picture).

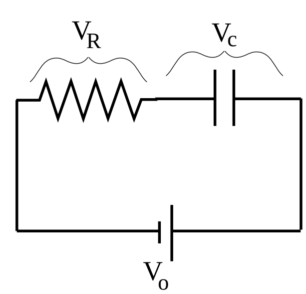


3.) A 10^{-6} *farad* capacitor is in series with a 10^4 *ohm* resistor, a battery whose voltage is $V_o = 100$ *volts*, and a switch. Assume the capacitor is initially uncharged and the switch is closed at $t = 0$.

a.) The capacitance value tells you something that is always true no matter what the voltage across the capacitor happens to be. What does it tell you?

Solution: Capacitance tells you how many *coulombs per volt* the plates can accommodate. Unless you change the physical characteristics of the system, this will be a constant for a given capacitor.

b.) What is the initial current i_o in the circuit?



Solution: Notice that at any point in time, the voltage across the battery must equal the voltage V_c across the capacitor added to the voltage V_R across the resistor. Initially, though, the voltage across the capacitor is zero (there is no charge yet on its plates). That means the voltage across the power supply will initially equal the voltage across the resistor. According to Ohm's Law, this can be written as $V_o = i_oR$, or $i_o = V_o/R = (100\ volts)/(10^4\ \Omega) = .01\ amps$.

c.) What is the circuit's current after a long period of time?

Solution: After a long period of time, the accumulated charge on the capacitor's plates will produce a voltage across the capacitor that is equal to the voltage across the power supply. At that point, there will no longer be current in the circuit. (Interesting observation: No current through the resistor means no voltage drop across the resistor as $V_R = iR$ . . . it fits!)

d.) How much charge will the capacitor hold when fully charged?

Solution: The relationship between the charge q on the capacitor at any time and the voltage V_c across the capacitor at that time is $q = CV$. When the capacitor is fully charged, the voltage across the capacitor will equal the voltage across the power supply, and we can write $q = (10^{-6}\ f)(100\ volts) = 10^{-4}\ coulombs$.

e.) How much energy is wrapped up in the capacitor when fully charged?

Solution: The energy wrapped up in a capacitor is equal to $.5CV^2 = .5(10^{-6}\ f)(100\ volts)^2 = .005\ joules$.

f.) Where is the energy stored in the capacitor?

Solution: Energy in a capacitor is stored in the electric field found between the capacitor's charged plates.

g.) You are told that the time constant for the system is 10^{-2} seconds.

i.) What does that tell you about the system?

Solution: The *time constant* gives you a feel for how fast the cap in the capacitor/resistor combination will charge or discharge. Specifically, one *time constant* is the amount of time required for the capacitor to charge up to .63 of its maximum charge (that's 63%) or dump 63% of its charge through the resistor. Two time constants gives you 87% charge or discharge, and three time constants gives you 95%.

ii.) How much charge will be associated with the capacitor after one time constant?

Solution: The maximum charge on the capacitor will be 10^{-4} *coulombs*. Sixty-three percent of that is $.63x10^{-4}$ *coulombs*.

iii.) Where will the charge alluded to in *Part 3g-ii* be found?

Solution: The "charge on a capacitor" is, in fact, the amount of charge on *one plate* of the capacitor. The sum of the charge on BOTH plates is *zero*.

h.) After a very long time, the switch is opened. What happens to the capacitor? Will it hold its charge forever?

Solution: Opening the switch disconnects the capacitor from the battery. There will be a trickle of charge flow through the capacitor (the resistance of the insulator is not infinite--there will be some *ir* action internal to the capacitor with a very large *r* and a very small *i*). With time, in other words, the capacitor will lose its charge.

i.) At $t = 1$ *second*, the current is i_1. At $t = 2$ *seconds*, the current is i_2. At $t = 4$ *seconds*, the current is i_4, and at $t = 8$ *seconds*, the current is i_8. Is i_2/i_1 going to give you the same ratio as i_8/i_4?

Solution: Current in a charging circuit follows an exponential function. Its form is $i_1 = i_o e^{-t/RC}$. If time doubles, the current becomes $i_2 = i_o e^{-2t/RC}$. The ratio of $i_2/i_1 = i_o e^{-2t/RC} / i_o e^{-t/RC} = e^{-t/RC} = 1/e^{t/RC}$. From this, it becomes evident that the ratio depends upon the *time* that is being doubled. As *t* gets bigger, the ratio gets smaller.

4.) Can you have capacitance if you have only one plate?

Solution: This is obscure, but the answer is *yes*. By putting charge on a plate, you give it a voltage. If you take infinity to be the zero voltage point (i.e., the point where the electric field due to the plate's charge goes to zero), the voltage difference between the charged plate and a fictitious plate at infinity will simply be the voltage of the charged plate. Capacitance is defined as the ratio between the charge on one capacitor plate and the voltage difference between the plates. That is non-zero here, so the single plate does have capacitance. (Note: You will never have to use this--it is something university courses sometimes like to throw in, just for the sake of confusion.)

5.) You have a series combination of capacitors.

a.) What happens to the equivalent capacitance when you add another capacitor?

Solution: Capacitor combinations are the reverse of resistor combinations. That is, *parallel* resistor combinations (i.e., $1/R_{eq} = 1/R_1 + 1/R_2 + \ldots$) have the same equivalence form as do *series* capacitor combinations (i.e., $1/C_{equ} = 1/C_1 + 1/C_2 + \ldots$). As such, adding a capacitor to a series circuit will *decrease* the equivalent capacitance (just as adding a resistor to a parallel circuit decreases the equivalent resistance of that type of circuit).

b.) What is common to all the capacitors in the series combination?

Solution: Not only will the current through each capacitor be the same at a given point in time, the charge on each capacitor will also be the same at that time. This makes sense if you think about how charge passes from plate to plate. As charge accumulates on the first plate, it electrostatically repulses an equal amount of like charge off its other plate. Where does that removed charge go? It accumulates on the next plate down the line, repeating the process for each successive capacitor.

6.) You have a parallel combination of capacitors.

a.) What happens to the equivalent capacitance when you add another capacitor?

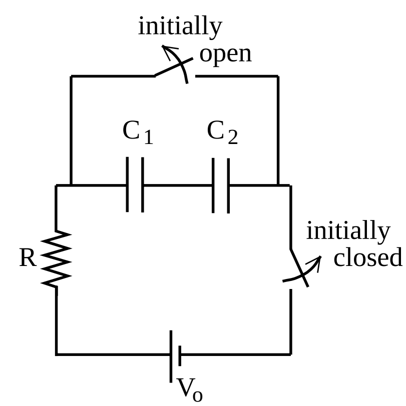
Solution: Again, capacitor combinations are the reverse of resistor combinations. Just as a *series* resistor combination (i.e., $R_{eq} = R_1 + R_2 + \ldots$) increases its equivalent resistance when a resistor is added, a *parallel* capacitance combination (i.e., $C_{equ} = C_1 + C_2 + \ldots$) increases its equivalent capacitance when a capacitor is added.

b.) What is common to all the capacitors in the parallel combination?

Solution: What is common to *all* parallel-type circuits is voltage. That is, each capacitor in a parallel combination will have the same voltage across its plates (this assumes there is only one capacitor per parallel branch--if there are multiple capacitors in a branch, the common voltage will be across the entire branch).

7.) You charge up two single capacitors that are in parallel. You disconnect the battery. What happens to the current in the system when you do this?

Solution: If the capacitors are in parallel, the voltage across each will be the same. That being the case, there will be no voltage difference between the high voltage sides of the caps, and no new current will flow.



8.) You charge up two unequal capacitors that are in series. You disconnect the battery, then reconnect the two capacitors by throwing *both* switches.

a.) What happens to the current in the system when you do this?

Solution: What is common in series combinations of capacitors is the *charge* on each capacitor. When the charged capacitors are reconnected, assuming the capacitances are different, there will be a voltage difference between the caps. Current will flow until that voltage difference disappears.

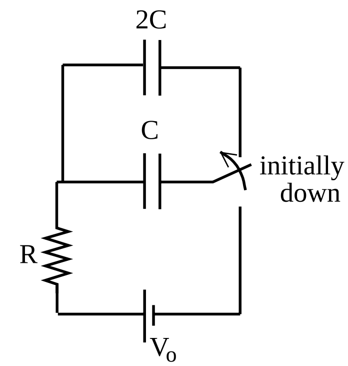
b.) Out of curiosity, why was the resistor included in the circuit?

Solution: There is always some resistance in a circuit. When you are dealing with a capacitor circuit, the resistance works with the capacitance to govern the rate at which the capacitor charges up. In other words, in this problem, the resistance information won't be used. To be kosher, though, a resistor needs to be included in all capacitor circuits.

c.) What kind of circuit do you have after both switches are thrown? That is, what kind of relationship will exist between the capacitors after the throw?

Solution: Though it isn't immediately evident, charge will flow until the voltage across each cap is the same. If you think about it, this is characteristic of a parallel combination. This would have been more evident if C_2 had been located right next to the upper switch. Nevertheless, that's the case.

9.) You use a battery whose voltage is V_o to charge up a capacitor C. When fully charged, there is q's worth of charge on the cap. You then disconnect the capacitor from the battery and reconnect it to a second uncharged capacitor whose capacitance is $2C$ (in the sketch, this disconnection, then reconnection is done with the switch). After the switch is thrown:



a.) What is the voltage across the second capacitor?

Solution: Charge will flow until there are no more voltage differences in the circuit. That is, half the voltage will be across the first cap and the other half across the second cap.

b.) How will the charge redistribute itself? That is, how much charge ends up on the second capacitor?

Solution: If capacitance C tells you how much *charge per volt* the cap can hold, a capacitor that is twice as large ($2C$) will hold twice the charge. If the first cap gets Q's worth of charge, the second cap will get $2Q$'s worth of charge. With the total being $3Q$'s worth of charge, the first cap will get *1* part of the $3Q$ and the second cap will get *2* parts of the $3Q$. In short, C_2 gets two-thirds of the original charge on C_1.

10.) You charge up a parallel plate capacitor that has air between its plates. Once charged, you *disconnect it* from the battery, then insert a piece of plastic (an insulator) between the plates. The amount of charge on the capacitor does not change (being disconnected from the circuit, it has no place to go), but the voltage across the capacitor does change.

a.) What is the insulator usually called in these situations?

Solution: In such cases, the insulator is called *a dielectric*.

b.) How and why does the voltage change (up, down, what?)?

Solution: In the presence of the capacitor's charged plates, each of the insulator's plate-facing surfaces will take on the appearance of being charged. Electrons in the insulator will stay in their orbitals (remember, valence electrons in insulators can't wander about the way valence electrons in metallically bonded structures can), but the plate charge will motivate them to spend most of their time close to the positive plate. The consequence of this polarization (it is a Van der Waal phenomenon) is that the polarized charge will set up a weak, reverse electric field through the insulator and between the plates. That field will subtract from the electric field set up by the capacitor's plates. With an effectively diminished electric field across its plates, the plate voltage *decreases*.

c.) What happens to the capacitance of the capacitor?

Solution: Capacitance is the ratio of *charge on one plate* to *the voltage across the plates*. The charge on one plate hasn't changed, but the voltage has gone down, so the new capacitance gets larger ($C = q/V$ . . . when V gets smaller, C gets larger).

d.) What happens to the energy content of the capacitor? If it goes up, from whence did the new energy come? If it goes down, where did the energy go?

Solution: Energy in a capacitor is calculated using $.5CV^2$. In this case, the voltage has decreased while the capacitance has gone up. As voltage is squared in the relationship, though, it looks like the energy content has diminished (don't believe me? $C = q/V$, so $.5CV^2 = .5(q/V)V^2 = .5qV$ . . . q is constant here while V decreases--the energy goes down). Where does the energy go? When the dielectric is inserted, it is actually pulled into the region between the plates. The work done to do that pulling is where the energy goes.

11.) You have a parallel plate capacitor with air between its plates hooked up to a power supply whose voltage is V_o. *Without disconnecting the battery*, you carefully insert a piece of plastic between the plates.

a.) What happens to the voltage across the capacitor?

Solution: This is an interesting question. When you slip the insulator between the capacitor's plates, you will get an induced charge polarization on the insulator's surfaces. That will drop the effective voltage across the plates which, in turn, will create a voltage difference between the high voltage plate of the capacitor and the high voltage plate of the power supply. Remembering that current ceases in capacitor circuits only when the voltage across the capacitor is the same as the voltage across the power supply, this voltage difference creates an electric field in the wire that motivates current to flow. The short-lived current puts more charge on the plates thereby bringing the plate voltage back up to V_o. In short, the voltage across the capacitor won't change at all.

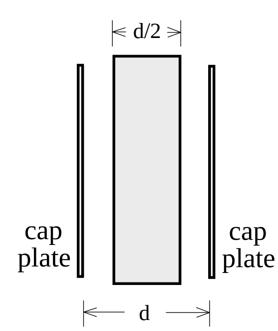
b.) What happens to the capacitor's capacitance?

Solution: From above, we've decided that the plate voltage stays the same but the *charge on each plate* increases. That means the capacitance ratio q/V increases. Note that this is just what you would expect. Whenever a dielectric is placed between the plates of a capacitor, the capacitance always goes up.

c.) What happens to the charge on the capacitor's plates?

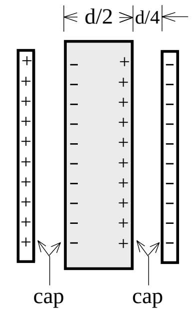
Solution: As was pointed out in *Part 11a*, the charge on the plates increases.

12.) You have two parallel plate capacitors that have air between their plates. Between the plates of the first cap, you insert a dielectric whose thickness is half the plate separation. Between the plates of the second cap, you insert a piece of metal whose thickness is also half the plate separation. The sketch shows a side view of both situations.



a.) Which modified capacitor will end up with the greater capacitance? Justify.

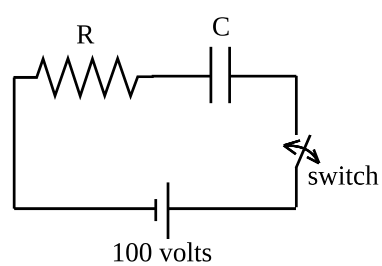
Solution: We have already established that putting a dielectric between the plates of a capacitor will increase the capacitance. We don't know by how much because we don't know the dielectric constant. Nevertheless, if the capacitance goes down with the insertion of the conductor, we won't need the dielectric constant. In short, what we need to determine before doing anything else is what a conductor will do to the capacitance when placed between the plates. The most conceptually elegant way to do this is to note how the conductor will respond to the presence of charged plates in its vicinity. In fact, electrons in the conductor will be attracted to the capacitor's positive plate *in the amount equal to the charge on the capacitor's plate*. That will leave an equal amount of positive charge on the other side of the conducting plate. What you will end up with will look a lot like two identical capacitors in series with one another. As you know, the equivalent capacitance of series combinations are always *smaller* than the smallest capacitor in the combination (this is similar to parallel resistor combinations). In fact, with all else held constant, the equivalent capacitance of two equal capacitors in series will be $C/2$. The problem is that we have done more than simply make one cap into two. We have also cut down on the distance between the plates. In fact, each plate is effectively $d/4$ units apart. According to the capacitance relationship for a parallel plate cap, this should increase the capacitance of each cap by a factor of 4. The net effect of these two changes means the new capacitance is $2C$. The only way the capacitance of the dielectric-filled cap will be greater is if its dielectric constant is greater than 2.



b.) What is the ratio of the two capacitances?

SOLUTION: Again, it depends on how big the dielectric constant is.

13.) You have a capacitor in series with a switch, a resistor, and a power supply. At $t = 0$, you throw the switch and current begins to flow.



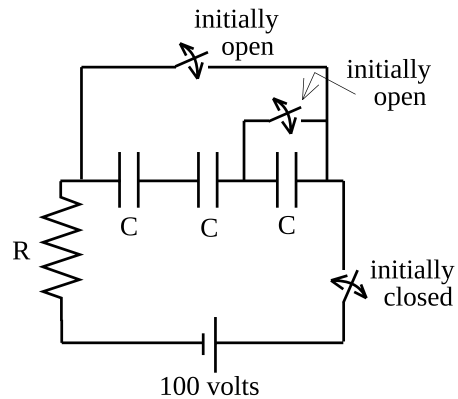
a.) For the amusement of it, draw the circuit.

Solution: See sketch.

b.) If the capacitor had been half as big, how would the current have flowed? That is, would the cap have charged faster or slower? Justify your response.

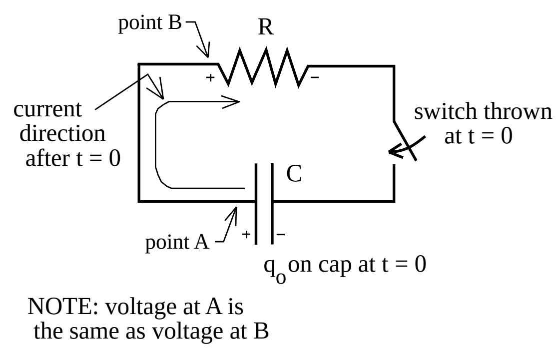
Solution: The time constant RC tells you how fast the cap in a capacitor/resistor circuit will charge. If the capacitance is halved, the time constant will halve and the capacitor will take *less time* to charge to 63%. In other words, the current will be greater with the smaller cap.

14.) All the capacitors in the circuit have capacitance C. Switch #1 is closed at $t = 0$. Switch #2 is open at $t = 0$. Switch #3 is open at $t = 0$. After the capacitors are charged, all the switches are thrown simultaneously. In terms of the battery voltage and C, how much charge is on each cap after all is said and done?



Solution: The equivalent capacitance of the two original series capacitors is $C/2$. That means the charge is $q = C_{eq}V_o = (C/2)V_o = CV_o/2$. When the switches are thrown, the battery is cut out of the circuit and a third capacitor is plated in series with the other two caps. Because the capacitors are all the same size, you would expect the charge to distribute itself evenly between the three. That means the charge on one cap will be $(1/3)(CV_o/2) = CV_o/6$. This wasn't that hard, but it hopefully made you think about the way capacitors act in circuits.

15) A capacitor of known capacitance C is charged to q_o. The cap is hooked across a resistor R and an open switch. The switch is thrown at $t = 0$.



A student is interested in deriving the time-varying expression for the charge on the capacitor during its discharge. She draws the circuit shown to the right. She knows that the sum of the voltage-changes around the closed loop have to add to zero (i.e., she knows Kirchoff's Laws), so she starts on the right side of the capacitor and traversing clockwise. She observes a voltage increase across the capacitor of q/C and a voltage drop across the resistor of iR (see circuit). With that, she writes:

$$-iR + q/C = 0.$$

She rewrites this equation as $iR = q/C$, replaces i with dq/dt yielding $(dq/dt)R = q/C$, puts all the q parts on the left side of the equation and all the constants on the right side yielding $dq/q = (1/RC)dt$, integrates both sides to get $\ln(q_{final}) - \ln(q_o) = \ln(q_{final}/q_o) = (1/RC)t$, uses the exponential trick to conclude that $q_{final}/q_o = e^{t/RC}$, comes up with an expression that says $q_{final} = q_o e^{t/RC}$, and loudly exclaims YIKES. Why did she say YIKES, and how can she get out of the difficulty?

Solution: The girl shrieked YIKES because her expression predicts that the charge on the plate should GROW exponentially instead of diminishing exponentially. Oops!

Getting out of the problem is a little dicier than identifying what the problem is . . . unless, of course, someone points out the little bit of nastiness that lurks within (which I'm about to do for you).

The key lies in some seriously sloppy notation and the realization that to solve this problem in general, one has to relate the charge ON THE PLATE q_{plate} to the current flow in the circuit $d(q_{in\ ckt})/dt$.

To begin with, think about signs. The charge on the plate q_{plate} and the capacitor's capacitance C are both inherently positive. That means that when our young scientist wrote the Kirchoff's expression $iR = q_{plate}/C$, she was writing something that demanded that both the right and left side of the expression be positive.

The question is, was she careful about this sign coherence throughout the derivation?

To answer that question, notice that in the original derivation, even though BY DEFINITION current is $dq_{in\ ckt}/dt$, she used dq_{plate}/dt to identify the current in the circuit.

Are these the same? In this case, not quite.

The problem exists in the fact that the rate at which charge leaves the plate, or dq_{plate}/dt, is inherently a NEGATIVE number (q_{plate} is diminishing, so the slope of its q *vs* t graph will be negative and its derivative dq_{plate}/dt will be negative). When she took the perfectly legitimate $iR = q_{plate}/C$ relationship (both sides of which are positive) and substituted in $i = dq/dt$, where the q term was q_{plate} (this had to be the case or all the q terms in the expression wouldn't have been the same critter and she wouldn't have been able to relate all of them to one another), she inadvertently made the iR side of the expression *inherently negative*.

In other words, to keep the signs right, she needed to observe that $-dq_{plate}/dt$ was positive only because the negative sign was included. With that observation, she should have written $i = dq_{in\ ckt}/dt = -\, dq_{plate}/dt$.

If she had done this, she would have ended up with a differential equation that would have been manipulable to $dq/q = -\,(1/RC)dt$, our final expression for q would have been $q_{final} = q_o e^{-t/RC}$. . . and with that exponential decay not a YIKES would have been heard throughout the land.