Gauss's Law -- Conceptual Solutions

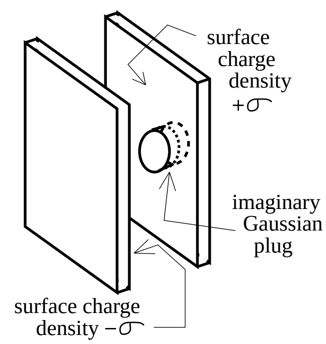
1.) An electric charge exists outside a balloon. The net electric flux through the balloon is zero. Why?

Solution: There will be the same amount of flux *into* the balloon as *out of* the balloon, therefore the net flux will be zero.

2.) A point charge Q is placed off-center inside a sphere. What is the electric flux passing through the sphere?

Solution: The temptation is to try to calculate the flux using $\Phi_E = \int_s \mathbf{E}\cdot d\mathbf{S}$. The clever thing to do is to realize that, according to Gauss's Law, the evaluation of that integral will always equal Q/ε_o, where Q is the charge enclosed within the Gaussian surface.

3.) If you use Gauss's Law to determine the electric field between two parallel, infinitely large (well, big anyway), oppositely charged plates, the electric field function you will derive is σ/ε_o, where σ is the area charge density on the inside surface of the plate (i.e., in the region where the Gaussian plug intersects the plate face). That is, the electric field is a constant--it doesn't change as you travel from one plate to the other. What is unsettling about all this is that the derivation seems to completely ignore the plate that *isn't* involved with the Gaussian surface. After all, Gauss says that the only charge that matters is the charge *inside* the Gaussian surface. In fact, I claim the charge on the second plate *does* have something to do with the problem. Your thrill is to justify my assertion.

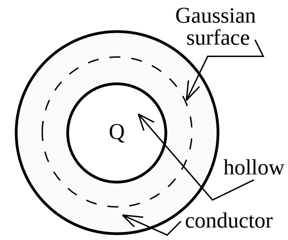


Solution: The key here lies in the fact that if there was just one plate, the free charge on that plate would be distributed over both the front and back surfaces. When a second, oppositely charged plate is brought into the picture, it effectively motivates the charge on the backside of the first plate to the side closest to the second plate, essentially doubling the charge on the inside surface of the first plate. In other words, the charge density used in the parallel plate situation is twice as large as would be the case if the first plate were alone. Hence, the second plate *has* had something to do with the situation even though it appears that Gauss's Law ignores it completely.

4.) In static electric situations, why must the electric field inside a conductor always equal zero?

Solution: Valence electrons in a conductor will move under the influence of an electric field. In static electric situations, that movement will continue until the valence electrons have rearranged themselves so as to nullify the E. field. As such, the net electric field in any conductor will always be zero in static electric situations.

5.) Consider a hollow, electrically neutral, thick skinned, conducting sphere of inside radius a and outside radius b. At the center of the hollow exists a charge $-Q$. If you construct an imaginary Gaussian sphere whose radius falls somewhere between a and b, the apparent presence of a net charge inside the Gaussian surface suggests that there must be an electric field on the Gaussian surface inside the conductor. The problem is that in static electric situations, a conductor is supposed to have *no electric field* inside it. How can you reconcile these two seemingly disparate observations?



Solution: Due to the presence of the $-Q$ charge at the center of the hollow, negative charge in the amount of $-Q$ will be repulsed from the inside wall of the sphere (i.e., at $r = a$) leaving that wall with a net $+Q$ charge on it (the repulsed negative charge will move to the outside surface of the sphere at $r = b$). What is really happening inside the conductor? The electric field generated by the negative charge at the sphere's center and the electric field generated by the induced positive charge on the inside surface of the sphere *add vectorially to zero* (in addition, the bits and pieces of electric field produced inside the conductor due to the charge induced on the outer edge of the conductor will add vectorially to zero . . . remember, we are dealing with a $1/r^2$ type of field). What Gauss's Law correctly maintains, on the other hand, is that with the addition of the induced positive charge on the inside wall of the sphere, the net charge *inside the Gaussian surface* is ZERO. According to Gauss, that means the electric field evaluated at the Gaussian surface must also be zero. Either way, the field is zero inside the conductor.

6.) Assume you have a spherically symmetric charge configuration centered on the *x axis* at $x = a$ that produces an electric field. You draw a Gaussian surface. You find that at every point on the Gaussian surface, the magnitude of the electric field is the same.

a.) What do you know about the surface?

Solution: Due to symmetry, the surface must be spherical and it must be centered at *point* a.

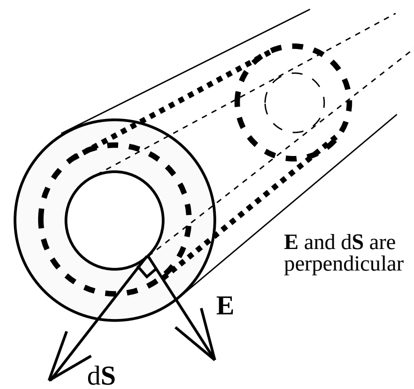
b.) Let's assume that you aren't sure whether the electric field passing through your Gaussian surface is oriented inward or outward. How do you deal with Gauss's Law in this situation? That is:

i.) How do you deal with the *dot product* found on the left side of the Gauss's Law relationship?

Solution: The easiest way to do this is to *always* assume that the electric field is oriented outward. In that way, the dot product will always be positive as both the electric field and $d\mathbf{S}$ will always be in the same direction. Whether this is the right assumption in a given problem will be evident when you *do* the problem, as will be made clear in the next part of this question.

ii.) How do you deal with the *charge enclosed* term found on the right side of Gauss's Law?

Solution: Unless you've run into a very strange Gauss's Law problem, the *signs* of the charges (i.e., positive or negative) involved in a charge configuration will be given. The "direction problem" alluded to in *Part 6b-i* will be taken care of as long as you include each charge's sign in the $q_{enclosed}$ part of Gauss's Law. How so? After doing the dot product inside the integral on the left side of Gauss's Law, the electric field term becomes a magnitude. Magnitudes should be positive. If you attend to the positive and negative signs associated with the charges summed in the $q_{enclosed}$ term on the right side of Gauss's Law, and if you find that you are looking at an electric field magnitude that is negative, you know one thing for sure--you've assumed the *wrong direction* for ***E***. To rectify the situation, all you have to do is acknowledge that fact, and you are done. (Note: In other words, a negative *E* doesn't mean the electric field is in the negative direction, at least not if you are working with Cartesian coordinates. It means the field is *inward* instead of *outward*. The only time a negative sign signifies a negative electric field direction is if you are working with radial symmetry in polar-spherical coordinates--then -***r*** really does mean a vector in the *negative radial direction*. Fortunately, this is not something you need to worry about . . . it just happens to be true.)



7.) Assume you have a cylindrically symmetric charge configuration that produces an electric field at some point in the vicinity of the configuration. You draw an appropriate Gaussian surface. From what you know about Gauss's Law:

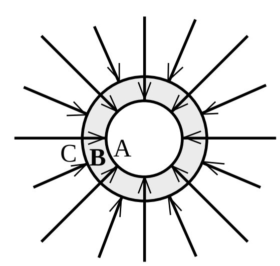
a.) What shape must that surface take?

Solution: The surface must be on the curved part of a cylinder (versus on one end of the cylinder) that is co-axial with the charge configuration.

b.) Is there a place on the Gaussian surface where the electric field is zero? If not, is there a place where the dot product between E and dS is zero? Explain.

Solution: There really is no place where the electric field is, by definition, zero ... unless you are inside a conductor or at infinity. As for the $\boldsymbol{E}\cdot d\boldsymbol{S}$ question, that dot product is zero at the end-faces of the Gaussian cylinder. How so? ***E*** is perpendicular and $d\boldsymbol{S}$ is parallel to the axis at each end-face. The *dot product* of two vectors at right angles to one another is zero.

8.) To the right is a cut-away cross-section of a thick-skinned sphere. Assume the inside radius is a and the outside radius is b. The lines shown are electric field lines.



a.) Using what you know about Gauss's Law, what can you tell me about *area A*?

Solution: As there are no electric field lines in the region, Gauss's Law maintains that there is no free charge in that region.

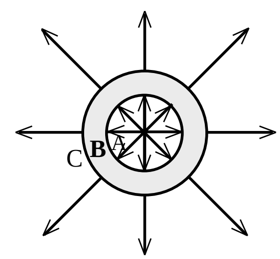
b.) Using what you know about Gauss's Law, what can you tell me about *area B*?

Solution: There are inward electric field lines suggesting that there are free electrons (i.e., negative charge) on the inside surface (i.e., at $r = a$).

c.) Using what you know about Gauss's Law, what can you tell me about *area C*?

Solution: There are even more inward electric field lines in *area C* suggesting that along with the negative charge found on the inside surface (i.e., at $r = a$), there is additional free negative charge on the outside surface at $r = b$.

9.) To the right is a cut-away cross-section of a hollow pipe. Assume the inside radius is a and the outside radius is b. The lines shown are electric field lines.



a.) Using what you know about Gauss's Law, what can you tell me about *area A*?

Solution: Given that there are electric field lines inside the hollow, there must be free charge distributed down the axis, probably on a wire. Additionally, as the lines are outward, the charge must be positive.

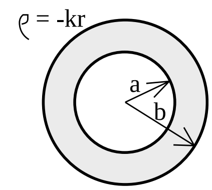
b.) Using what you know about Gauss's Law, what can you tell me about *area B*?

Solution: Huzzah! There is no electric field in *area B*. That means one of two things. The pipe could be an insulator with a surface charge density $-\sigma$ on its inside surface, where $-\sigma$ is equal in magnitude to the charge density down the axis, or *area B* could be a conductor. If the latter is the case, the surface charge density $-\sigma$ suggested above would have to have been *induced* on the inside surface of the pipe. In either case, the net charge *inside a Gaussian surface* positioned in *area B* is zero, and as such, there is no resulting electric field.

c.) Using what you know about Gauss's Law, what can you tell me about *area C*.

Solution: If the pipe had been an insulator with a surface charge density placed on its inside surface, the net charge inside a Gaussian surface positioned outside the pipe would still be zero and the electric field in that region would also be zero. If, on the other hand, the pipe was a conductor and the charge on its inside surface was induced, then an equal and opposite amount of charge would reside on the outside surface, the net charge inside the Gaussian surface positioned outside the pipe would simply equal the charge down the axis (the induced charge quantities would add to zero), and an electric field would exist outside the pipe. As the latter is the case, we can conclude that the pipe is, indeed, a conductor, and that the only place there is free charge (versus induced charge) is down the axis.

10.) A hollow, thick-skinned, spherical shell has an inside radius of a and an outside radius of b. Inside the shell exists a volume charge density $-kr$, where $k = 1$ with the appropriate units. On the shell's surface exists a surface charge density that is positive. The total free charge inside the sphere is *less than* the total free charge on the surface.

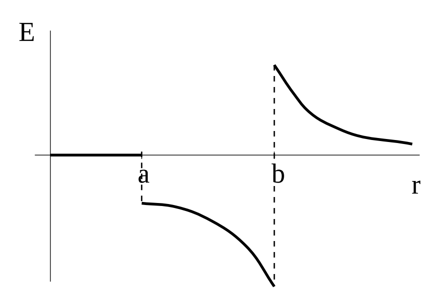


a.) What are *k's* units?

Solution: A volume charge density has the units of *charge per unit volume*, or *Coulombs per meter cubed*. The constant must be such that when it multiplies meters (i.e., r), it yields *Coulombs per meter cubed*. That constant is *Coulombs per meter to the fourth*.

b.) Make a ten-second sketch of the general shape of $E(r)$ for this charge configuration. (Hint: Do this in sections--that is, think about what's going on for $r < a$, then for $a < r < b$, etc.)

Solution: I asked for a quick sketch here to keep you from being too meticulous. My hope was that you tried to *visualize* the relationship between Gauss's Law, geometry, charge densities, and electric fields before doing the math. Treating each individual section as a separate entity, my thoughts follow. For $r < a$: There is no charge in this region, so no electric field. For $a < r < b$: A negative, linear volume charge density is going to do the same thing a constant charge density would do, except faster. That is, the field will grow inside the sphere more quickly as r increases. We know that a *constant* volume charge density function produces an electric field that is directly proportional to r (the fraction of charge inside a Gaussian surface inside a solid sphere is $[(4/3)(\pi r^3)/(4/3)(\pi R^3)]Q = (r^3/R^3)Q$ . . . using this as $q_{enclosed}$ in Gauss's Law with its left side equaling $E(4\pi r^2)$ yields an electric field that is proportional to r). As such, there is a good chance that a *linear* density function will produce an electric field that varies as r^2. In fact, that is the case. What's more, because the charge is negative, the direction of the field will be radially inward. I will call this the *negative* direction for my graph. For $r > b$: The negative electric field due to the negative volume charge density inside the sphere will be overcome by the positive electric field produced by the positive surface charge density on the outer surface of the sphere (the question states that the total positive charge involved in the surface charge density is greater than the total negative charge involved in the volume charge density), so the net electric field outside the sphere will be positive. Furthermore, charge uniformly distributed over a spherical shell (this could be either a differential shell inside the solid or the shell associated with the solid's outer surface) will always produce an electric field *outside*



the shell that looks as though it were produced by a point charge at the center of the sphere. The size of the point charge will always equal the net charge inside the shell, and the direction of the field will depend upon whether that charge is net positive or net negative. As such, the electric field in the region outside our thick-skinned sphere will be positive and will fall off as $1/r^2$ (remember, $1/r^2$ is how point charges act). The sketch is shown above.

11.) A hollow pipe has an inside radius of a and an outside radius of b. Between a and b, the volume charge density is $-kr$, where $k = 1$ with the appropriate units.

a.) What are *k's* units?

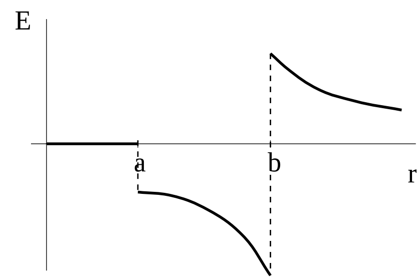
Solution: The fact that the geometry is cylindrical here and not, as was the case in *Problem 10*, spherical, makes no difference. Volume charge density is *always* equal to *Coulombs per meter cubed*. As such, the solution to *10a* holds here, also.

b.) Is the pipe an insulator or a conductor?

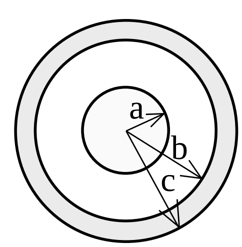
Solution: You should be able to tell by the way the problem is worded. *You can't distribute free charge inside a conductor*. If you try, the charge won't sit still. It will redistribute itself on the outside surface in an attempt to get as far away from the other free charge as possible. Sooo, if you see a material with any kind of *volume* charge density, know that it *has* to be an insulator.

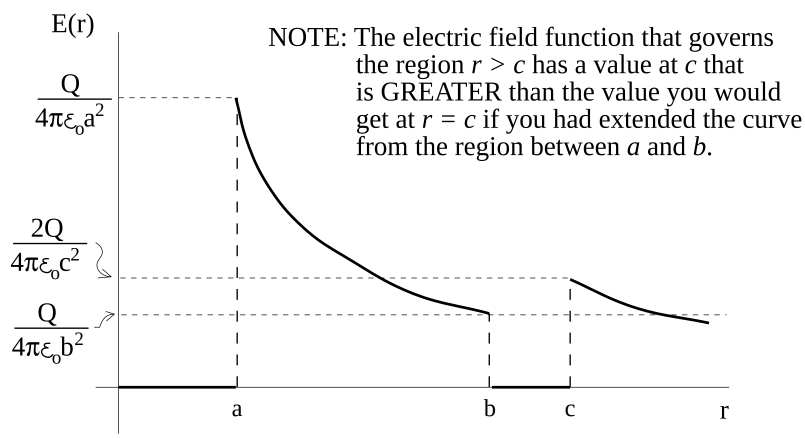
c.) Make a ten-second sketch of the general shape of $E(r)$ for this charge configuration.

Solution: Again, the hope is that you have visualized the geometry and charge/electric field relationship in your head. My analysis follows. For $r < a$: There is no charge in this region, so no electric field. For $a < r < b$: You would probably expect that a negative, linear volume charge density would *not* act in the same way as would a similar density function acting in a spherical charge configuration. Why? Because the volume of a sphere increases as r^3 (the volume of a sphere is $(4/3)\pi r^3$) whereas the volume of a cylinder increases as r^2 (the volume of a cylinder is $2\pi r^2 L$). It turns out, though, that that isn't the case. If we follow what was done in *Question 10* and determine how the electric field acts when the volume density is a constant, we find that the fraction of charge inside a cylinder is $[(\pi r^2 L)/(\pi R^2 L)]Q = (r^2/R^2)Q$. "Ah," you say. The spherical counterpart was a function of r^3! True. But the surface area of the Gaussian surface used on the left side of the Gauss's Law expression is $4\pi r^2$ for spherical symmetry and $2\pi rL$ for cylindrical symmetry. Bottom line: Both spherical and cylindrical symmetries produce electric fields for constant volume charge distributions that are functions of r (of course, it probably would have been easier to use Gauss's Law and just derive the expression rather than intuit the answer using the arguments presented above, but that's life). As such, you'd expect their *linear* density functions to produce electric fields that vary as r^2. In fact, that is the case. As in the spherical situation, the negative charge inside the solid produces what I will define as a negative electric field. For $r > b$: The negative electric field due to the negative volume charge density inside the cylinder will be overcome by the positive electric field produced by the positive surface charge density on the outer surface of the cylinder (the question states that the total positive charge involved in the surface charge density is greater than the total negative charge involved in the volume charge density), so the net electric field outside the cylinder will be positive. Furthermore, charge uniformly distributed over a cylindrical shell (this could be either a differential shell inside the solid or the shell associated with the solid's outer surface) will always produce an electric field *outside the shell* that looks as though it were produced by a line of charge along the cylinder's central axis. As such, the electric field in the region outside our thick-skinned cylinder will be positive and will fall off as $1/r$ (remember, $1/r$ is how a line of charge acts). The sketch is shown above and looks just like the graph shown for the spherical situation, except the outside field drops off more slowly (as $1/r$ instead of $1/r^2$).



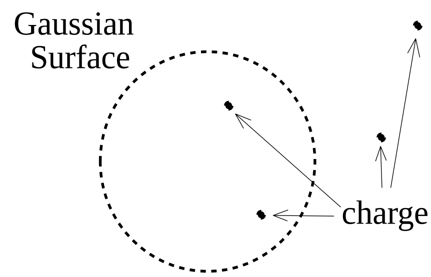
12.) An old AP test problem created the following scenario. *A charge Q is placed on a conducting sphere of radius a. Outside the sphere is a conducting shell of inside radius b and outside radius c (as shown) which also has a charge Q placed on it. Generate the graph of E(r).* The correct graph is shown . What is interesting is that AP graders took off points (or, more accurately, didn't give all the points possible) if the electric field evaluated at $r = c$ wasn't BIGGER than the value the field would have had if the electric field function for the $a < r < b$ region was extended out to $r > c$. My question to you is, *given the problem as stated, what were they thinking?* That is, what assumption would *you* have to make *that wasn't stated in the problem* that would lead you to conclude that finding?



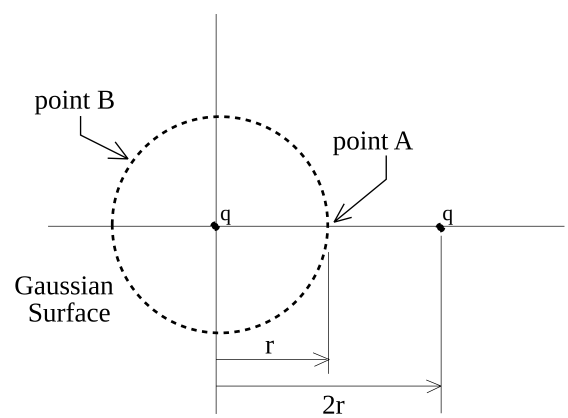


Solution: This is a Gauss's Law problem only in the sense that you would probably *use* Gauss's Law to determine the electric field function for $a < r < b$ and $r > c$ (you should know by now that the electric field inside a conductor is zero). In the former case, the electric field would be $Q/(4\pi\varepsilon_o r^2)$. In the latter case, it would be $2Q/(4\pi\varepsilon_o r^2)$. The difference in the two expressions is based on the fact that there is twice as much charge inside a Gaussian sphere located in the region $r > c$ than in the region $a < r < b$. The question is, *is this charge-doubling enough to insure that E(r = b) is less than E(r = c)?* In fact, it isn't. Sure, there is twice the charge available to create the net field, but if $r = c$ is huge, that doubled charge would be spread so thin over the surface that the charge density would be tiny and the net electric field at $r = c$ would also be tiny. So what's the deal? It turns out that the people who wrote this AP problem evidently assumed that students would look at the sketch, assume it was to scale, see that $r = c$ is just a little bigger than $r = b$, and conclude that the $1/r^2$ part of the electric field relationship would *not* overpower the *2Q* constraint. That's the only thing they *could* have been thinking. Why? Because if *c* is really, really big in comparison to *b*, the electric field at *c* will be a whole lot smaller than the extrapolated value from the electric field function that exists at *b*. If you don't believe me, make $b = 1$ and $c = 100$ and compare the electric fields at those two distances. In short, although they evidently docked people for this "error," it was a dubious point. (Translation: this was, in my opinion, a miserably written question because it required students to make an assumption that might have been marginally legitimate but would definitely have been obscure).

14.) Assume you have four charges scattered indiscriminately in space. You create an imaginary Gaussian surface so that two of the charges are inside the surface and the other two are outside the surface. According to Gauss's Law, the net electric flux through the surface is due solely to the charge enclosed inside the surface.



a.) Don't the charges outside the surface contribute to the electric field on the Gaussian surface?



Solution: This question can, potentially, bring up all sorts of uncertainties about Gauss's Law (that's the reason I've included it). To untangle the mess, let's look at a simple example. Take two equal charges. One is placed at the origin of a coordinate system. The other is placed at $2r$. A circular Gaussian surface of radius r is centered on that first charge. For convenience, two positions on the Gaussian surface are labeled. So what is the net electric field on the Gaussian surface at *point A*? As the distance between *A* and the two equal, like charges is the same, the net electric field at that point will be *zero!* Evidently, the electric field at *point A* IS affected by the charge outside the Gaussian surface . . .

b.) If the answer to *Part a* is *yes*, how can Gauss's Law ignore the charge outside the Gaussian surface?

Solution: According the *Part a*, the *electric field* at a particular point on a Gaussian surface is affected by charge both inside and outside the Gaussian surface. Is this a problem for Gauss's Law? NO! Why? Because Gauss's Law is a tool that is used to determine electric field functions for SYMMETRIC charge configurations. If a configuration is not symmetric, Gauss's Law is still a true law . . . it just isn't a very useful, true law. How so? Think about the two-charge example we used in *Part a*. Because the outside charge exists, there is no electric field at *point A*. There is, on the other hand, a relatively large electric field at *point B* (at least, larger than if the outside charge didn't exist). That means that although there would be no electric flux passing through a differential surface area at *point A*, there would be a relatively large electric flux passing through a differential surface area at *point B* (again, larger than expected if the outside charge didn't exist). All Gauss's Law says is that if you add up all the differential fluxes through all the differential surfaces on the Gaussian surface, that *net flux* will be proportional to the charge enclosed within the Gaussian surface. Charge outside the Gaussian surface will affect the electric field at every point on the Gaussian surface, but those externally produced electric fields will always interact with the internally produced electric fields to produce a net flux that is proportional only to the charge inside the Gaussian surface. It may not be immediately obvious, but that's the way things work. Note: If you forget the electric field part and just focus on the electric flux part, things are obvious. Any electric field lines that pass through a Gaussian surface will produce electric flux through that surface. If the charge that produces a set of electric field lines is outside the surface, those electric field lines will both pass *into* as well as *out of* the surface. As a consequence, the net flux due to charge outside the Gaussian surface will add to zero, and the only charge that affects the net flux will be the charge enclosed inside the Gaussian surface. As for useability, the only time Gauss's Law is useful is when the symmetry is such that the electric field is the SAME at every point on the Gaussian surface (this is true of spherical symmetry--for cylindrical symmetry you can also have surfaces in which $\mathbf{E}\cdot d\mathbf{S}$ is zero). If the magnitude of the electric field *isn't* constant over the surface, you can't pull the E term out of the flux integral and, as a consequence, you won't be able to generate an expression for E. Hence, Gauss's Law will hold in such situations, but it wouldn't be at all useful.

15.) Assume you have a solid sphere of radius a in which charge is distributed uniformly throughout the volume. You create an imaginary Gaussian surface, symmetrically centered, such that its radius is $r < a$ (i.e., it's inside the sphere). According to Gauss's Law, the electric flux through the surface is due solely to the charge enclosed inside the Gaussian surface.

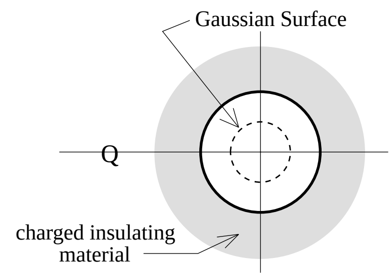
a.) Don't the charges outside the surface contribute to the electric field on the Gaussian surface? If so, how can Gauss ignore them?

Solution: Although it may be a temptation to suggest that the solution to this is exactly the same as the combined solutions to *Problem 14a* and *14b*, that wouldn't be the case. In this situation, the electric field at any point on the Gaussian surface due to charge outside the Gaussian surface will equal ZERO. That is, charge outside the Gaussian sphere will NOT affect the electric field on the surface. Why? The charge symmetry makes this problem just like the "if you are inside the earth, will the mass outside the sphere upon which you reside produce any gravitational force on you (this assumes the sphere is symmetrically placed relative to the center of the earth)?" problem we did in the gravitation section. If you will remember, the conclusion we drew in that section was that because the gravitational field due to a point mass is a $1/r^2$ creature, the gravitational pulling and tugging of all of the point masses outside a symmetrically placed sphere will effectively cancel one another out. Well, it turns out that the electric field produced by a point charge is also a $1/r^2$ creature, so the net force (hence net electric field) due to a continuous distribution of symmetrically placed point charges outside a Gaussian surface will not affect the electric field *on* or *inside* the Gaussian surface in the same way.

b.) As far as Gauss's Law goes, how is this situation different from the four-charge situation alluded to at the beginning of *Problem 14*?

Solution: The obvious difference is that of symmetry. In the previous problem, there was none. Here, there is spherical symmetry. Note that this part of this problem was included to nudge students who saw no difference between *Problem 14* and *Problem 15* into thinking again.

16.) A hollow ball has a charge filled, insulating material coating its outside surface (see sketch). A single point charge Q sits outside the complex somewhere along the *x axis*. Eva maintains that there is an electric field on the *x axis* INSIDE THE HOLLOW of the ball. Gunther say not. He puts a Gaussian sphere inside the hollow, observes that the charge enclosed is zero (this is true), observes that the electric flux through the surface is zero (this is also true), and deduces that the electric field on the surface must be zero. It turns out that this last deduction is wrong (i.e., Eva was right), but Gunther's argument still seem persuasive. What is he missing?



Solution: Forget Gauss's Law for the time being. The easiest way to see what is really happening in this problem is to consider the idea of *superposition of fields*. The charged, insulating material symmetrically distributed around the ball will produce no net electric field inside the hollow (i.e., on the Gaussian surface), but the charge Q WILL produce a net field inside the hollow. Superimposing the two fields produces a non-zero field inside the hollow. So how did Gauss's Law manage to confuse Gunther? The problem lay in understanding the difference between an *electric flux* and an *electric field*. There is, indeed, no electric flux through the Gaussian surface. This is conceptually obvious as all of the electric field lines that pass into the Gaussian sphere pass out of the sphere. What *isn't* obvious is what that says about the electric field as evaluated *on* the Gaussian sphere. Specifically, WHEN THERE IS A LACK OF SYMMETRY, the integral $\boldsymbol{E} \cdot d\boldsymbol{S}$ WILL equal zero (i.e., Gauss's Law will still rule) but the magnitude of E will *not be constant* over the Gaussian surface.

To see how this works, consider the sketch to the right. Note that although $\boldsymbol{E}$ due to $\boldsymbol{Q}$ on the left side of the Gaussian surface is greater the $\boldsymbol{E}$ on the right side of the surface, the surface area penetrated by the nearer-side $\boldsymbol{E}$ is SMALLER than the surface area penetrated by the far-side $\boldsymbol{E}$. The sketch shows this clearly. In other words, a positive flux generated by a smaller electric field passing through a bigger area can be offset by a negative flux generated by a larger field passing through a smaller area. In short, Gauss's Law holds *at all times*, it's just only useful in situations where there is charge symmetry. Such situations are the only times we can used it to deduce something about an *E field's* magnitude.