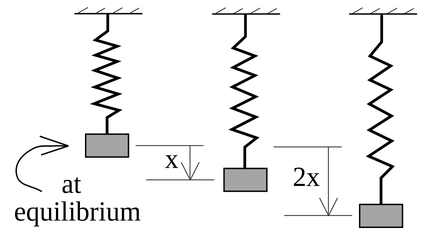
Vibratory Motion -- Conceptual Solutions



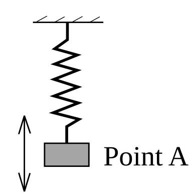
1.) An ideal spring attached to a mass *m = .3 kg* provides a force equal to -kx, where *k = 47.33 nt/m* is the spring's *spring constant* and *x* denotes the spring's displacement from its equilibrium position. Let's assume that when such a spring is displaced a distance *x = 1 meter*, the period of oscillation (this is defined as the amount of time required for the system to oscillate through one complete cycle) is *T = .5 seconds per cycle.*

a.) When the mass is displaced a distance *2x = 2 meters*, what is its new period?

Solution: The temptation is to assume that because the displacement is doubled, the period will be doubled. That doesn't happen. Why? Because the farther you pull the spring, the bigger the restoring force (remember, the restoring force is a *linear function* of *x*). So although the mass has farther to travel in the *2x* situation, the greater average force over any given half cycle keeps the period the same.

b.) Given the numbers in the original statement of the set-up, would it have been possible for the period to have been *any other number* other than *.5 seconds per cycle*? Explain.

Solution: It turns out that the period of an ideal spring is solely determined by the spring constant *k* and the mass attached to the spring. As both of those values are fixed in the problem, the period *must* be the value calculated.

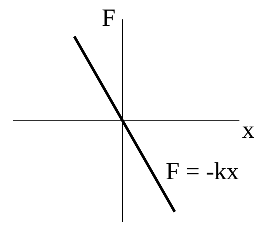


2.) A vertical spring/mass system oscillates up and down. At *t = 0*, the mass is at *equilibrium* moving downward. Through how many cycles will the system have moved by the time the mass has passed by that point five times, not including its first passage at *t = 0*?

Solution: The mass moves down to the bottom of its motion, then comes back up passing equilibrium for the first time after having traveled through *half a cycle*. It proceeds upward to the top of its motion, then down again passing equilibrium for its second time after having completed another half a cycle. In other words, it passes equilibrium twice per cycle. Passing equilibrium (or, for that matter, any point other than one of the extremes) five times means it will have traveled through 2.5 cycles.

3.) When you attach a mass to an ideal spring, the force *F* provided to the mass by the spring will be proportional to the displacement *x* of the mass/spring system from its equilibrium position. Algebraically, that proportionality can be written as an equality equal to *F = -kx*, where *k* is the proportionality constant called *the spring*

constant. One of the things that is interesting about the oscillatory motion of the mass attached to an ideal spring is that the mass's motion will have a single period T. That is, it will always take the same amount of time for the mass to oscillate through one cycle no matter what the initial displacement was.
Having said that:

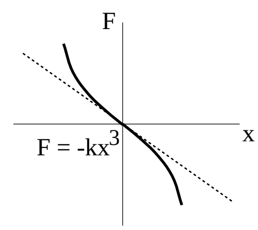


a.) Sketch the *Force versus Displacement* graph for an ideal spring. Remember that the displacement of a spring from its equilibrium position can be either positive or negative.

Solution: See sketch.

b.) Briefly, explain why the period of an ideal spring/mass system doesn't change if the initial displacement of the mass is increased or decreased.

Solution: Increasing the displacement requires the mass to travel farther to get through one cycle, but because the force is proportionally larger, the one-cycle time of travel remains the same no matter what initial displacement you try.



c.) Now for the fun part. Consider a second, *non-ideal* spring whose force expression is $-bx^3$, where b is some spring constant. On the graph you produced in *Part a*, make an approximate sketch of the *Force versus Displacement* graph for this spring force (don't get anal about this--you don't need numbers, just show the trend of the force as x goes positive and negative).

Solution: See sketch.

d.) Attach the non-ideal spring to the same mass you used in *Part a* and *b*. It is possible to displace this spring/mass system so that when released, it oscillates with the same period as was the case with the ideal spring used in *Part a*. Take that displacement, double it, displace the non-linear system that doubled distance, and release. Will the period of the resulting oscillation be greater than, less than, or the same as T?

Solution: This is tricky . . . and good for you to think through. Look at the graph of the ideal spring and think about what happens when you attach a mass to that spring. You displace the mass. The spring provides a certain amount of force. The resulting action produces oscillatory motion whose period is T. *If you double the displacement* with that ideal spring, the force doubles and the *period remains the same*. Now look at the graph of the non-ideal spring and think about what happens when you attach a mass to it. You displace the mass. The spring provides a certain amount of force. The resulting action produces oscillatory motion whose period is T. *If you double the displacement* of that spring, the force doesn't double, it gets bigger still. So what happens to the period? The mass has twice as far to travel, but the force is more than double, so it should take *less time* to make a round trip. In other words, doubling the displacement *decreases* the period. Please note that this is similar to an AP question that was asked several years ago.

4.) Can a spring have a force function of $-kx^4$? Explain.

Solution: The only forces that qualify as periodic (i.e., forces that produce oscillatory motion) are those that ALWAYS motivate the attached mass back toward the equilibrium position of the system. As an example, an *ideal* spring has the force function $F = -kx$. If the displacement x of such a spring is positive, the force direction will be negative (put a positive x into $F = -kx$ and F becomes negative). By the same token, if the displacement is negative, the force direction will be positive (put a negative x into $F = -kx$ and F becomes positive). Force functions that are even powers do not accommodate that constraint. The function $-kx^4$ yields a negative force when x is positive (this is good), but *doesn't* yield a positive force when x is negative (this is bad). In short, this function won't do.

5.) You have access to Gepetto's Workshop, complete with Newton scales, meter sticks, balances--all sorts of science-y things. Someone gives you an ideal spring and asks you to determine its spring constant. How might you do that?

Solution: The spring constant tells you how much *force per length of displacement* is required to displace a spring. That is, it is the ratio of force F and the associated displacement x that comes with the application of that force. Mathematically, this is F/x (note that spring constants are ALWAYS positive). To determine a spring constant experimentally, all you have to do is apply a known force, see how much displacement that application produces, and divide the one into the other. Using the stuff in Gepetto's, there are a couple of ways to do this. One would be to hang a known mass m from the spring and measure the resulting spring displacement x. The force would be the weight of the mass mg, and the spring constant would be mg/x. Another way would be to attach a Newton scale to the hanging spring and measure the force F required to pull the spring down a given distance y. Again, $k = F/y$.

6.) Most people know that frequency measures the number of cycles through which an object oscillates per unit time. What does *angular frequency* measure?

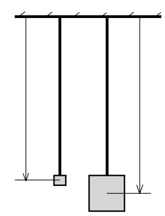
Solution: Just as frequency, measured in *cycles per second*, tells you the number of *cycles* that are swept through by an oscillating object in one second, angular frequency, measured in *radians per second*, tells you the number of *radians* that are swept through by an oscillating object in one second (note that there are 2π radians in one cycle--in fact, the relationship between angular frequency and frequency is $\omega = 2\pi\nu$). An object that oscillates through *2π radians per second* has a frequency of *one cycle per second*. An object that has a frequency of *2 cycles per second* has an angular frequency of *4π radians per second*. This probably seems like a bizarre way to measure oscillatory rates, but it is perfectly sensible when you consider the problem it is meant to remedy. Assume you have a spring/mass system that is oscillating along the *x-axis* around an equilibrium position. The relationship that defines the mass's position is a sine function. Specifically, $x = A \sin \theta$ (note that the argument of the sine function has to have the units of *radians*--an angle). The problem with this relationship is that it isn't explicit in time. So how *can* we get obvious time dependence into the equation? By rewriting θ as a *constant* times *time* (i.e. $\theta = \omega t$--we can do this as long as the constant ω has the units of *radians per second* . . . multiply *radians per second* by *seconds* and you get *radians*), we can write the position function as $x = A \sin \omega t$. Expressed this way, the ω term in the position expression governs how fast the system vibrates. This just makes sense. It takes 2π radians for a sine wave to repeat itself. If ω is large, it only requires a small time t for the system to execute one cycle (i.e., for ωt to equal *2π radians*). This is exactly what you would expect if the oscillatory motion was happening as fast as the large ω suggests (remember, a large ω means the system is oscillating through a lot of radians per second).

7.) A fixed length of string is cut and loops are made at both ends. The upper end-loop is attached to a ceiling hook while the lower end-loop is used to support a hook-mass m. The mass is pulled to the side and released making a pendulum that swings back and forth. The period is measured as T. The original mass is removed and a second hook-mass from the same mass set, this one of mass $10m$, is placed on the string and made to swing back and forth with the same amplitude. The new period is found to be larger than T.

a.) Does this mean the pendulum swings faster or slower?

Solution: A larger period means it takes a longer time for the body to travel through one oscillation. That means it swings more slowly than before.

b.) Some students look at the data and conclude that the pendulum's period is a function of the bob's mass. In fact, this isn't true! What is probably causing the disparity in the periods?



Solution: The larger mass is probably longer, so the distance from the ceiling support to the center of mass of the pendulum bob has lengthened. As the period of a simple pendulum is dependent upon the length of the pendulum arm, that would explain the difference in periods.

8.) Newton's Second Law is used to sum up the forces acting on an oscillating mass. The resulting expression is then manipulated and found to have the form $(d^2x/dt^2) + bx = 0$. Having access to this expression:

a.) What can you say about the system's angular frequency?

Solution: The angular frequency will simply be the square root of the constant b. This was derived in your book for a spring, but it works for any system that oscillates with simple harmonic motion. In fact, that is one way to determine if you *have* simple harmonic motion--sum the forces, then manipulate the expression to see if you can get it into the form shown above.

b.) What can you say about the system's frequency?

Solution: If you know a system's angular frequency, you know its frequency from $\omega = 2\pi\nu$.

c.) What can you say about the system's period?

Solution: If you know a system's frequency, you know its period from $\nu = (1/T)$.

9.) What is the single characteristic that is common to all vibrating (oscillatory) systems?

Solution: All oscillatory systems must have a restoring force involved somehow. That is, there must be a force in the system that ALWAYS motivates (i.e., accelerates) the system back toward its equilibrium position.

10.) The acceleration of gravity on earth is approximately six times that of the acceleration on the moon. A pendulum on earth has a period of *1 second per cycle*. Will the pendulum's period change if it is used on the moon? If so, how so?

Solution: Period is the inverse of frequency ($T = 1/\nu$). Frequency is directly proportional to the angular frequency ($\omega = 2\pi\nu$). The angular frequency for a pendulum is $(g/L)^{1/2}$, where L is the pendulum arm and g is the effective acceleration of gravity. If the acceleration of gravity drops by 6, the angular frequency changes by $(6)^{1/2}$. That means the frequency changes by $(6)^{1/2}$, which means the period changes by $1/(6)^{1/2}$.

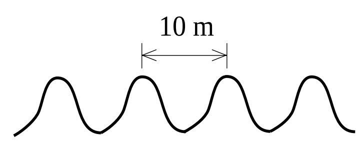
11.) Double the length of a pendulum arm. How will the pendulum's frequency change? How will the pendulum's period change?

Solution: The frequency ($\nu = \omega/2\pi$) of a pendulum is $(g/L)^{1/2}/(2\pi)$. Doubling the length changes the frequency by $(1/2)^{1/2}$, or .707 of the original. The period is the inverse of the frequency, so it will change by $1/(1/2)^{1/2}$, or 1.4 of the original period.

12.) How are frequency and period related?

Solution: As has been said in the last several problems, the frequency is the inverse of the period.

13.) You are sitting on a jetty. You notice ocean waves are coming in approximately 10 meters apart. It takes 30 seconds for *three crests* to pass you by. What is the frequency, period, and angular frequency of the wave train?



Solution: Three crests corresponds to two full cycles passing by (look at the sketch). Two cycles passing by every 30 seconds yields a frequency of *2 cycles per 30 seconds*, or *1/15 cycle per second*. The inverse of that, *15 seconds per cycle*, is the period. As for the angular frequency, $\omega = 2\pi\nu = 2\pi(1/15) = 6.28/15 = .418$ *radians per second.*

14.) A spring with spring constant $k = .25$ *newtons per meter* vibrates with frequency $\nu = .5$ *hertz*. Across the lab, a string with a small mass $m = .15\ kg$ attached to it makes a simple pendulum.

a.) If the frequency of the pendulum and the frequency of the spring are to be the same, approximately how long must the string be?

Solution: The frequency of a spring is $(k/m)^{1/2}/(2\pi)$ while the frequency of a simple pendulum is $(g/L)^{1/2}/(2\pi)$. Equating the two, dropping away the common 2π terms, and squaring both sides of the relationship leaves us with $(k/m) = (g/L)$. Evidently, $L = mg/k = (.15\ kg)(9.8\ m/s^2)/(.25\ kg\cdot m/s^2/m) = 5.88$ *meters*.

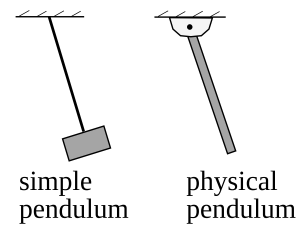
b.) Why are you being asked for an approximate answer? That is, given what you know, why can't you give an exact answer?

Solution: Everything done in the real world that is based on theoretical calculations is an approximation. We've assumed that the spring is ideal, that the pendulum bob is a point mass, that the pendulum arm is massless, and probably most importantly, that the pendulum oscillation is very, very small (see *Part c* for more on this). These may be reasonable assumptions to make, but they will introduce some hopefully small discrepancies in the final product.

c.) For the frequency to be good, is there any limit on the size of the oscillations of the pendulum?

Solution: The theory used to derive the angular frequency expression for the pendulum assumed that the oscillations were small enough so that, to a very good approximation, $\sin\theta = \theta$ (otherwise, the second order differential equation was not in the form characteristic of simple harmonic motion). Depending, of course, on how anal you want to be, that approximation is only good for angles less than, say, .1 radians (about 5^o).

15.) What is the difference between a simple pendulum and a physical pendulum of same mass and length? What approach would you use to derive from scratch an expression for the period of either?



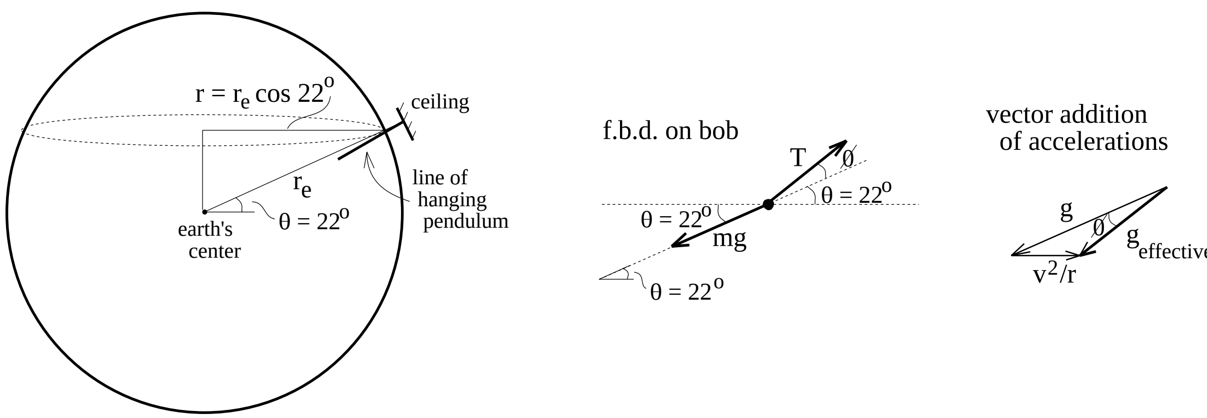
Solution: A simple pendulum is a point mass attached to a massless string whereas a physical pendulum is usually comprised of a massive, extended pendulum arm that *is* the pendulum (that is, there isn't usually a final, concentrated mass at the end of the pendulum arm). The way you get the characteristic equation for any pendulum is to use Newton's Second Law (that is, sum up the torques about the pin and set that equal to $I_{pin}(d^2\theta/dt^2)$). Manipulate that expression into the characteristic form for simple harmonic motion (i.e., (angular acceleration) + (constant)(angular displacement) = 0). Noting that you will probably have to take a small angle approximation to turn the resulting $\sin\theta$ into the displacement term θ, the constant in front of the displacement term in that final relationship will equal the angular frequency *squared*. From that you can get the frequency ($\omega = 2\pi\nu$) and period ($T = 1/\nu$).

16.) You live in California (Los Angeles). You're a physics teacher, complete with sadistic streak. You have your students calculate the theoretical period of a pendulum system. They determine that value to be T. You then claim that no matter how good and precise your students' set-up is, its period will never exactly equal the theoretically calculated value, *even if your students do the experiment in a vacuum*. What are you talking about? (Note: This isn't obvious--think about the parameters that determine a pendulum's period, and how they might be off). Once you've figured out the problem, approximate by how much your theoretical period will be off (note that the latitude of LA is approximately 22^o). Is this going to be noticeable?

Solution: The period of a pendulum is determined by the expression $2\pi(L/g)^{1/2}$. Which of those variables might be a bit hinky due to the fact that you are in LA (no, this is not the set-up for a west coast joke, though there are undoubtedly some doozies out there just waiting to happen)? The key is found in the fact that the gravitational acceleration is predicated on the assumption that the earth is not spinning. In fact, the earth *is* spinning. We dealt with this problem in the Newton's Laws chapter. What was said at that time was the following:

Think about holding the string end of a simple pendulum while standing on the edge of a rotating *merry go round*. From your perspective, what does the pendulum bob appear to do? It seems to push out away from the center (actually, it's really trying to follow straight-line motion--your holding it, via the string, applies a force that pulls it into circular motion . . . hence the feeling that it pushes you outward). If the earth were stationary, a pendulum would be gravitationally attracted to the center of the earth and the string of a freely hanging pendulum (i.e., one that was not swinging) would orient itself between the pendulum's contact point (i.e., where the pendulum is attached to, say, the ceiling) and the earth's center. The problem is that the earth is rotating. This means that along with the tension force required to counteract gravity, there must be a tension *component* that pulls the pendulum bob into circular motion. The consequence is that the line of the pendulum will not be toward the earth's center but will be off a bit toward the equator (see sketch below).

The long and the short of all of this is that the acceleration the bob feels along the line of its swing is not g but the vector sum of g and the centripetal acceleration $v^2/(r_{circle})^2$ (see sketches below). That means the g term in the $2\pi(L/g)^{1/2}$ expression isn't really going to be 9.8 m/s^2 but something less. I wouldn't suggest you take the time to do the calculations, but you could use the relationships suggested by the sketches shown to determine exactly how big the *effective* acceleration term is and, in doing so, would find that the discrepancy is indeed minuscule.



17.) Consider the expression $x = A \sin(\omega t + \delta)$.

a.) What does the A term do for you ?

Solution: The maximum value a sine function can have is *one*. If you want your oscillation to have a maximum displacement other than *one*, you have to multiply the sine term by that maximum displacement. In short, A is the amplitude of the motion as measured relative to the equilibrium point.

b.) What does the ω term do for you?

Solution: It gives you a feel for how fast the oscillation is occurring. An angular frequency of *6.28 radians/second* (i.e., *2π rad/sec*) is the same as *1 cycle/second*. As was pointed out earlier, the angular frequency term is necessary so that time dependence can be inserted into an expression that must, inherently, be a function of an angle (hence, ωt replaces θ in the sine function).

c.) What does the δ term do for you?

Solution: This is called the *phase shift*. Without it, the expression would require that at $t = 0$, the displacement x would have to be zero. Additionally, it would require that just after $t = 0$, the motion proceed into the positive region of the oscillation. What the phase shift allows you to do is to move the axis, so to speak, allowing the displacement to be whatever you want at $t = 0$.

d.) What does the expression in general do for you?

Solution: This expression tells you where the oscillating body is, relative to its equilibrium position, at any point in time.