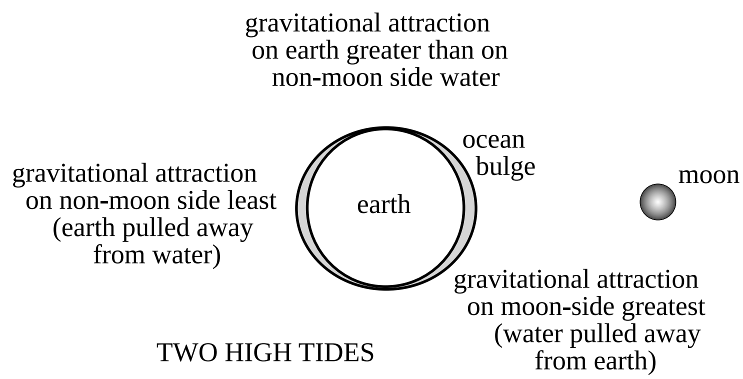
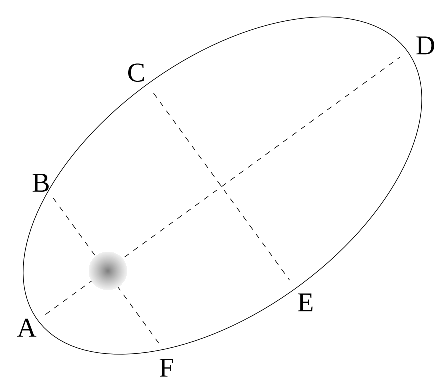
Gravitation -- Conceptual Solutions

1.) It is the gravitational attraction between the moon and the oceans that causes the bulge we associate with high tide. So why do we observe *two* high tides at a given time of the day, one on either side of the earth?

Solution: Gravitational attraction is a function of distance between the objects involved. That means the oceans on the moon-side of the earth will feel a greater gravitational force on them due to the moon than will the oceans on the non-moon side of the earth. What is usually forgotten in all of this analysis is the fact that the earth is *also* being pulled toward the moon . . . even more than are the oceans on the non-moon side of the earth. So what we have with the moon pull are the oceans on the moon side being pulled away from the earth--hence, a high tide--while the earth is being pulled away from the oceans on the non-moon side--hence the second high tide. Bizarre, eh? Bizarre, but true.



2.) The elliptical orbit of a planet moving around its star is shown as a solid line in the sketch. Justifying each response, at which points will the planet have the same:



a.) Momentum?

Solution: *Momentum* is a function of velocity *as a vector*, so the momentum will be the same where the magnitude and direction are the same. Due to symmetry, the magnitude will be the same at points *B*and *F*, *C* and *E*. Unfortunately, the directions will not be the same at any of those points, so there is *no* point at which the momentum will be the same.

b.) Angular momentum?

Solution: Angular momentum will be conserved (i.e., the same everywhere) if the torques acting on the planet are internal to the planet/star system. In this case, the line of the force between the planet and the star goes straight through the star, so the torque on the planet about the star is *zero*. As such, angular momentum will be conserved and the angular momentum will be the same everywhere in the orbit. Notice that that means the *direction* of the angular momentum (it will either be into the page or out of the page, depending upon whether the body is moving clockwise or counterclockwise in its orbit) will not change as the body moves around its orbital path. Notice also that because the distance vector ***r*** between the planet and star varies from point to point, the planet's momentum ***p*** must also vary so that ***rxp*** (i.e., the angular momentum) remains constant.

c.) Mechanical energy?

Solution: Mechanical energy is the sum of the kinetic and potential energy. It will be conserved if the forces acting on the system are conservative, and if we have potential energy functions for those forces. As we generally assume that there is no friction in space, the non-conservative force of friction does no work to remove energy from the system. And as the force on the planet due to the star is gravitational, we have a potential energy function for it. In short, the mechanical energy of the system is conserved.

d.) Orbital velocity?

Solution: We have already established that the orbital velocity's magnitude is the same at various points, but its direction is not the same at any of those points. In short, there are no points where the orbital velocity is the same.

e.) Angular velocity?

Solution: Angular velocity is the number of radians the planet sweeps through per unit time. It varies from point to point, but it's direction will always be into or out of the page, depending, (see the comment in the angular momentum section). Due to symmetry, it will have the same magnitude at points B and F and points C and E.

f.) Planetary acceleration?

Solution: Again, planetary acceleration is a directional beast, so although the magnitude of the acceleration (due to symmetry) will be the same at points B and F and points C and E, the directions will never match up and there will be no two places where it is the same.

g.) Net force?

Solution: The net force acting on the planet is always between the star and the planet. It's magnitude, due to symmetry, will be the same at points B and F and points C and E, but the directions will be different so the force *as a vector* will never be the same anywhere in the motion.

h.) Torque?

Solution: We have already decided that the torque on the planet due to the existence of the star is zero (the force runs along the line between the two objects--***r*** and ***F*** are parallel to one another, so the cross product is zero). That means that the torque is the same everywhere (i.e., it's zero everywhere)!

3.) Newton's general expression for the gravitational force on any mass m due to the presence of a second mass M is GmM/r^2, where r is the distance between the center of mass of each of the two bodies. What one might deduce from this is that if we put an object at the center of the earth, the distance between the earth's center of mass and the object's center of mass would be zero . . . and the force would be infinite. This obviously makes no sense. Explain your way out of the problem.

Solution: Because the gravitational force is an inverse square with distance, it turns out that the only mass that actually affects the object is the mass that is *inside the sphere* upon which the object sits at a given point in time (the net gravitational effect of all of the mass outside the sphere adds to zero). If the sphere is defined at the earth's surface, that *mass inside* is the entire mass of the earth. If the sphere is defined as having a radius of half the earth's radius, then the mass will be one-eighth the total mass of the earth (the volume of a sphere is a function of r^3). If the sphere is at the center of the earth, there is no mass inside the sphere and the gravitational force produced on the object will be zero. In short, once inside the earth, the force function takes on a new look--a linear one--and the problem is resolved.

4.) For the gravitational force function GmM/r^2, where would you expect its potential energy function to have its zero?

Solution: The zero for a potential energy function associated with a particular conservative force defined as zero where the force function is, itself, zero (in the case of gravity near the surface of the earth, there *is* no zero . . . hence the fact that you can choose your zero level to be anywhere--with the general gravitational expression, that isn't the case). In this case, the force function is zero at infinity, so that's where you would have to define the potential energy function to be zero.

5.) In the case of a circular orbit, what *kind* of force is gravity?

Solution: Gravity is a centripetal force in the case of a circular orbit.

6.) Assuming you were invincible:

a.) What would happen to your weight if you stood on the surface of an imploding star (i.e., a structure whose mass isn't changing but whose outer radius is decreasing)? Explain.

Solution: The mass *inside the sphere upon which you were standing* wouldn't change, but the radius and, hence, distance between your center of mass and the star's center of mass would change. As such, the force (i.e., your weight) would increase as r got small (remember, F_{grav} is proportional to $1/r^2$).

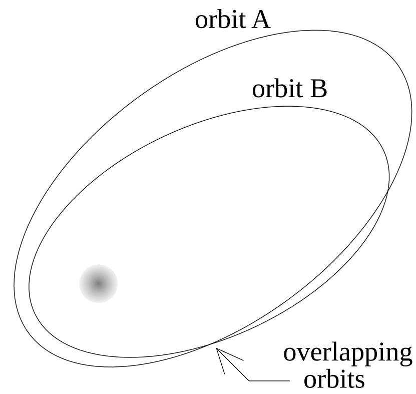
b.) What would happen to your weight if you stayed stationary where the surface had been as the star implodes underneath you?

Solution: For this situation, none of the parameters that govern weight would change. That is, there would still be the same amount of mass *inside the sphere, etc.*, the distance between your center of mass and the star's center of mass would be the same, and your mass would be the same. In short, your weight wouldn't change.

7.) Two moons orbit in *exactly* the same circular orbit around a planet. Is it possible for one of the moons to orbit faster than the other? Explain.

Solution: The answer is *no*. Why? Because an orbital path of a given radius requires a particular velocity to hold. As such, two velocities will not do.

8.) Two moons in different elliptical orbits share part of their respective orbits with one another (that is, assume their orbits partially overlap).



a.) Which of the two orbits will, on average, require the largest average velocity to hold?

Solution: The orbit that is, on average, the closest to the star will require the greatest velocity to hold. That is because at close orbit, there is a greater gravitational force acting on the moon, hence the need for a greater velocity to keep that moon from being sucked, so to speak, into the star. In this case, it looks like *orbit B* will require the greatest average velocity.

b.) The chances are slim that both moons would be found in the overlapping part of their orbits at the same time, but let's assume that that has happened. If the moon in the rear is moving a little faster than the moon in front, will the two ultimately collide? Explain. (Assume there *is* time for the rear moon to catch the front moon before the two orbits diverge.)

Solution: This is a fun question. What happens, and this was actually observed during one of the planetary flybys several years ago, is the following: As the rear, faster moon approaches the front moon, the gravitational attraction between the two speeds the rear moon up and slows the front moon down. With the increased speed, the rear moon's orbit swings wider than it would normally be. With the decrease in speed, the front moon's orbit pulls in closer to the planet. The net effect is that the rear moon will hop scotch the front moon. Once on the other side, gravitational attraction will slow the now front moon, pulling it back into its "normal" orbit, and gravitational attraction will speed up the now rear moon pulling *it* back into its "normal" orbit. In that way the faster moon will be able to pass the slower moon and continue on its way without any collision occurring. Neat, eh?

9.) The acceleration of gravity at the moon's surface is one sixth the gravitational acceleration at the earth's surface. If you moved to twice the moon's radius, what will the gravitational acceleration be?

Solution: The gravitational force function is dependent upon the inverse of the distance between the moon's center and the object, quantity *squared*. By doubling the distance between those two centers of mass, you decrease the gravitational force by a factor of *four*. In terms of gravity as measured on earth, that will be a quarter of a sixth of g, or a *twenty-fourth* of earth's gravitational acceleration.

10.) Assuming the earth was homogenous, how would you expect your weight to change as you moved downward into a vertical mining shaft? Be specific (that is, will it change linearly, exponentially, quadratically, what?). Would things be different if the earth had a thin crust several miles thick with molten magma below it?

Solution: As was alluded to above, the gravitational force exerted upon a body (i.e., the body's weight) is a function of the amount of mass that is *inside the sphere upon which the* *object rests*. That is true assuming you are talking about a homogeneous structure. Doing the problem to derive an expression for that force yields a linear function. If the earth were not homogeneous, you would still expect the weight to be a function that went to zero at the earth's center, but *the mass enclosed within the sphere upon which you rested* would not be a function solely of the cube of the radius (remember, the volume of a sphere is proportional to r^3). That is because the mass density would change with depth, hence the amount of mass enclosed would depend upon where you were within the interior.

11.) Assume the radius of the earth is *r*. At *r*, the gravitational acceleration is *g*. If you could compress all of the earth's material into a ball whose radius was *r/2*, how would the gravitational acceleration at *r* change?

Solution: The amount of mass that would be inside *r* would be the same in both cases, as would the distance between your center of mass and the earth's center of mass. In other words, none of the parameters that govern the gravitational acceleration of a body at *r* would change with the compressing of the earth's surface to *r/2*, so you would still accelerate at 9.8 m/s^2 at *r*. Weird, but true.

12.) Assume two planets have equal, homogeneous mass distributions. If one planet has half the radius of the other, how will the escape velocity differ between the two?

Solution: To determine the escape velocity for an object, we have to use the conservation of energy and the fact that an object would theoretically have to travel to infinity (i.e., where the potential energy was zero) to completely escape the earth's gravitational pull. As the potential energy at a planet's surface is $-GmM/r$, where *m* is the projectile mass, *G* is the universal gravitational constant ($6.67\times10^{-11}\ nt\cdot m^2/kg^2$), *M* is the planet's mass, and *r* is the distance from the planet's center to the starting point for the projectile (in this case, the planet's radius). If the initial kinetic energy is $.5m(v_{escape})^2$, conservation of energy yields $v_{escape} = (2GM/r_e)^{1/2}$. Halving the radius will increase the argument inside the square root by a factor of 2, but it will change the volume, hence mass, by 1/8 (the volume of a sphere is a function of r^3 . . . $(1/2)^3 = 1/8$). In other words, the net effect of halving the radius will be to decrease the escape velocity by 1/2 (i.e., $[(1/8)(2)]^{1/2} = 1/2$).

13.) How would the earth's motion be affected if the sun was magically transformed into a black hole? This is not a supernova situation. The sun isn't blowing up, it's just suddenly, magically dropped its radius to, maybe, one mile across.

Solution: If Newton's universal gravitational force (i.e., $-GmM/r^2$)is a true reflection of reality, nothing would change in the expression and the force would remain the same. What makes a Black Hole so devastating to its surroundings is the fact that, because it has collapsed, objects *can* get in closer to their center than they normally would have. (Actually, it's a lot more complex than that, but that's the Newtonian slant on the question).

14.) Assume there are three equal radius, equal mass moons orbiting a planet. One moon is homogeneous, the second has almost all of its mass at its center, and the third has almost all of its mass in a thin crust at its surface. Ignoring any effect from the planet, which will have the greatest acceleration of gravity at its surface?

Solution: They'll all be the same as they'll all have the same "mass enclosed."

15.) As far as fuel goes, would it take more to go from Earth to Jupiter or from Jupiter to Earth?

Solution: Having to escape the more massive planet will require the most fuel, so going from Jupiter to Earth will require more fuel.

16.) Why doesn't the moon fall into the earth?

Solution: If the moon was force-free, it would move in a straight line. The earth's gravitational attraction has captured the moon, so to speak, motivating it out of straight line motion and into a nearly circular motion. If you could somehow stop that motion, the moon would fall directly *into* the earth. But as it has tangential motion also, the moon essentially falls *around the earth*.

17.) Due to the gravitational force the earth exerts on the moon, the moon essentially *falls around the earth*. But according to Newton's Third Law, that means there must be an equal and opposite gravitational force that the moon exerts on the earth. So why doesn't the earth fall around the moon?

Solution: In fact, the moon DOESN'T fall around the earth. It falls around the center of mass of the moon/earth system. Because the earth is so much more massive than the moon, that *center of mass* point is about two-thirds of the way out from the earth's center. As such, the moon moves around that point in an obvious circle, and the earth moves around that point in a not-so-obvious circle. In short, the earth's reaction to the presence of the moon is to execute a bit of a wobbling.

18.) Consider a planet with half the earth's mass and a third of its radius.

a.) Is the planet's density the same as the earth's?

Solution: Density is *mass/volume*. In other words, (density)(volume) = (mass). If the mass halves, the volume would have to halve for the density to remain the same. That isn't happening, so the densities of the two planets are not the same.

b.) What would the gravitational acceleration be on the planet's surface?

Solution: The acceleration term from Newton's universal gravitational expression $-GM/r^2$ suggest that halving the mass while thirding the radius would yield a gravitational acceleration that was $(1/2)(1/(1/3)^2) = 9/2$.

19.) If you know the period T (i.e., the time for one revolution) of a satellite in circular motion about an object of known mass M, can you determine the satellite's radius r of motion? If so, how so? If you additionally know the radius, is there some clever way you can determine its speed v?

Solution: In cases like this when you really don't know whether there is a reasonable answer or not, the best thing to do is to write down as many true relationships as you can. In this case, because the motion is centripetal, we can use the gravitational force in N.S.L. and write $GmM/r^2 = ma_c = mv^2/r$. From that we get $v = (Gm/r)^{1/2}$. As for the period, the magnitude of the velocity will be the distance traveled during one revolution (i.e., $2\pi r$) divided by the period T. That is, $v = (2\pi r)/T$. Combining $v = (Gm/r)^{1/2}$ and $v = (2\pi r)/T$

yields $(2\pi r)/T = (Gm/r)^{1/2}$. The only unknown in that relationship is r, which means we can determine r given only T and M (note that the gravitational constant G is just that, a constant). Note that the clever way to determine the velocity is with the relationship $v = (2\pi r)/T$.

20.) Assuming the earth's mass is approximately 80 times the moon's mass, and noting that the acceleration of gravity on the moon is approximately 1/6 that of the acceleration of gravity on earth, what can you say about the relationship between the moon's and earth's radius?

Solution: This is one of those problems in which you just have to root around for expressions that might give you equalities you can use. In this case, it turns out that a N.S.L. approach will work. You dropped a mass M near the moon's surface. Using N.S.L., we can write: $Gm_m M/r_m{}^2 = Ma_m$. You dropped a mass M near the earth's surface. Using N.S.L., we can write: $Gm_e M/r_e{}^2 = Ma_e$. In both cases, the M's cancel. If we divide the expressions, one into the other, the G's cancel out. Manipulating what is left, we get $r_e/r_m = [(a_m m_e)/(a_e m_m)]^{1/2}$. Substituting in $m_e = 80m_m$ and $a_e = 6a_m$ and the a and m terms cancel out leaving us with $r_e/r_m = 3.65$. Kindly note that the actual radius of the earth is approximately 6.37×10^6 meters while the actual radius of the moon is approximately 1.74×10^6 meters. The ratio of the two is approximately 3.65.

21.) Two moons orbit a large planet in approximately circular paths. The first moon is twice as far from the planet's center as is the second moon.

a.) What is the ratio of their speeds?

Solution: Using N.S.L. and the fact that the gravitational force here is centripetal, we have $-Gm_{moon}m_{planet}/r^2 = -m_{moon}a_c = m_{moon}(v^2/r)$. From this, we can deduce that $v = (Gm_{planet}/r)^{1/2}$. In other words, the velocity is proportional to $(1/r)^{1/2}$. Doubling the radius doesn't halve the velocity, it drops the velocity by the square root of a half, or by a factor of *.7*.

b.) What is the ratio of their total mechanical energies?

Solution: The total energy of a satellite (a moon in this case) moving in a circular path will be $-Gm_{moon}m_{planet}/(2r)$. With one planet twice as far out, its total mechanical energy will be half as much.

c.) How would the answers have changed (bigger, smaller, what?) if the distances given had been from the planet's surface?

Solution: The distance you need for the gravitational force expression is the distance from the planet's center to the moon (actually, it's the moon's center, but if we assume it is a small moon, we can just say *the moon*). Doubling that distance doubles both the distance from the planet's *center* to its *surface* AND from the planet's *surface* to the *moon*. Doubling just the distance from the *surface* to the *moon*, as was required in the second scenario, will not put the second moon out as far as was required in the original scenario. With the moon in closer, its velocity must be greater than in the original scenario (the closer in to the planet, the greater the gravitational force on an orbiting object and, as a consequence, the greater the object's speed must be to keep from falling into the planet). The same is true of its kinetic energy.

22.) When a satellite orbits a planet in an elliptical orbit:

a.) Does the planet do work on the satellite as it moves?

Solution: There is a gravitational force on the satellite. That force does act over a distance, so work is done.

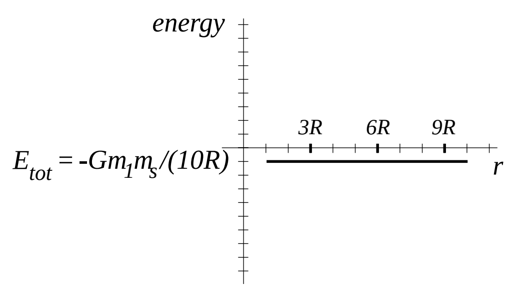
b.) What is the net work done per orbit? Explain.

Solution: Because the work done when the satellite moves toward the planet is equal and opposite the work done when the satellite moves away from the planet, the net work per full orbital rotation is zero. That means that energy is, to a very, very good approximation, conserved (we can ignore friction as it is minuscule out in space).

23.) An object positioned a distance *10R* units out from the center of a star of radius *R* begins its fall from rest toward the star. As the freefall proceeds

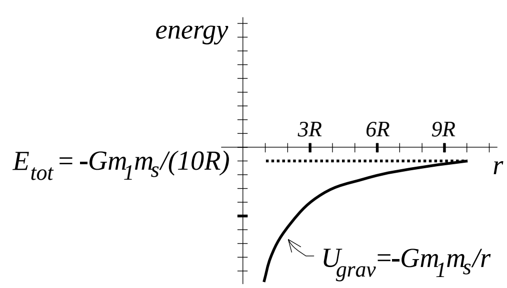
a.) As a function of position, what does a graph of the body's total mechanical energy look like?

Solution: This should be a no-brainer. There is no friction out in space, so the total mechanical energy shouldn't change at all. The constant will numerically equal the amount of gravitational potential energy the body had at its start position (at that position, it had no kinetic energy as it was at rest), or $-Gm_1m_s/(10R)$. The graph is shown to the right.

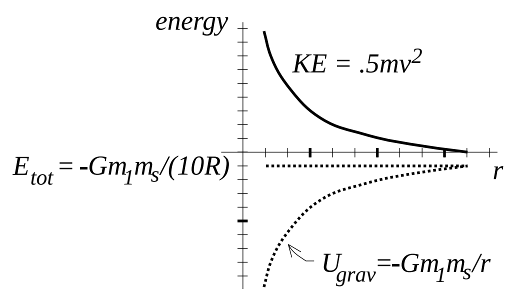


b.) As a function of position, what does a graph of the body's gravitational potential energy look like?

Solution: The gravitational potential energy function is $-Gm_1m_s/r$, where r is the distance to the object from the star's center. Notice that the potential energy function is negative. That denotes what is called *a bound state*. A bound state is a situation in which the object *needs* energy if it is to escape the gravitational attraction of the field-producing body (the star in this case). In general, negative potential energy functions mean you are dealing with a body that is attracted (bound) by the associated force-field. The graph of the gravitational potential energy function is shown.



c.) As a function of position, what does a graph of the body's kinetic energy look like?



Solution: We could try to use N.S.L. to derive the velocity of the body as a function of position *r* so that we could determine a general expression for $.5mv^2$, but that is a very unpleasant, messy proposition. A considerably easier way is to utilize the symmetry that exists between the potential and kinetic energy in this energy-conserved situation (that is, when potential energy goes down, kinetic energy must go up in an equal amount, and vice versa). Using that information and the graph from *Part b* above, the graph of the body's kinetic energy is shown to the right.

24.) Stop a circling satellite dead in its tracks, then release it. It will freefall toward the earth. Why? Because there is a gravitational force on the satellite due to the presence of the earth. It isn't a big force (gravity diminishes the farther you get from the earth's center), but it exists. With that in mind, why do things (sandwiches, hammers, whatever) appear weightless to people who are working in a space station that circles the earth?

Solution: Everything is free falling around the earth--sandwiches, hammers, the people inside the space station, the space station itself--everything. What's more, because everything is accelerating at the same rate, nothing accelerates away from anything else, hence the appearance that the occupants of the space station are floating around inside the station with all of their tools floating around next to them.