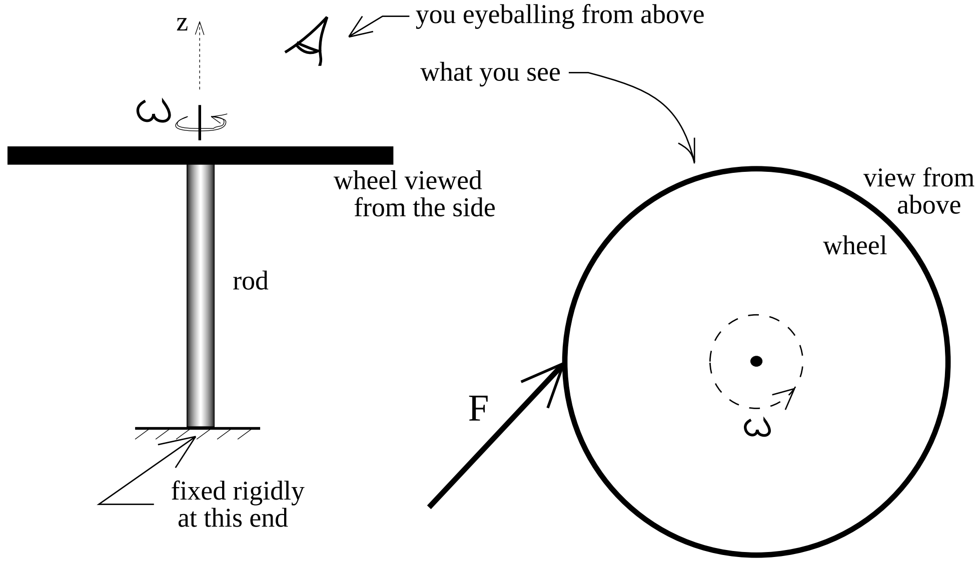
Rotational Motion II -- Conceptual Solutions

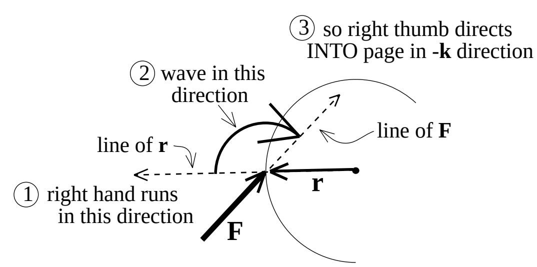
1.) A rotating wheel is supported by a fixed rod oriented as shown. A force *F* is applied to the wheel. At the moment depicted in the sketch:



a.) In what direction is the torque due to ***F***, relative to the wheel's center?

Solution:

Formally, torque is defined as the *cross product* of the applied force ***F*** and the distance ***r*** between the point of interest (in this case, the axis of rotation) and the point at which the force acts. That is, torque $\boldsymbol{\Gamma} = \boldsymbol{r} \times \boldsymbol{F}$. Determining the direction of a *cross product* is usually done with the *right hand rule* (though an easier way will be presented shortly). With the *right hand rule*, the length of the right hand runs along the first vector (***r***), then the hand motions into hitch hiker position as it waves toward the second vector ***(F)***. The extended right thumb defines the direction of the *cross product*. In this case, that will be directed *into* the page in the *-**k** direction*. THE EASY WAY TO DO THIS: Once you come to understand and believe in the *right hand rule*, note that a force that motivates a body to rotate clockwise (as viewed from the *positive* side of the axis of rotation) produces a negative torque, and a force that motivates a body to rotate counterclockwise produces a positive torque. For this problem, the force pushes the wheel clockwise, so its torque will be negative.



b.) In what direction is the wheel's resulting angular acceleration?

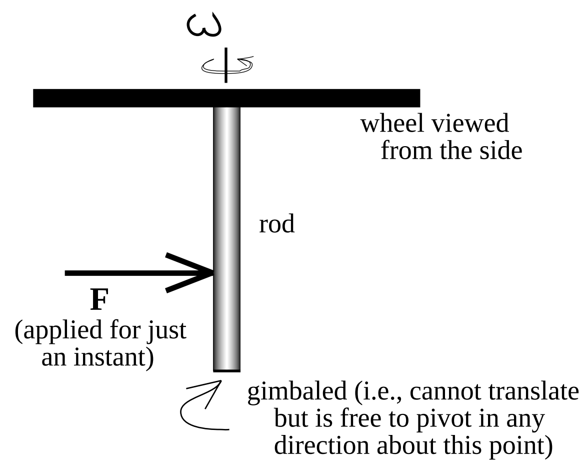
Solution: In the world of rotational motion, net torque and angular acceleration are the counterparts to net force and acceleration. According to Newton, the net force acting on a body is proportional to the body's acceleration (note that that means their directions are the same). Running a parallel for the rotational world, the net torque acting on a body is proportional to the body's *angular* acceleration (that means *their* directions are the same). As there are no other torques acting on the system except that of $\boldsymbol{F}$, we can conclude that the direction of the angular acceleration will be the same as the direction of the torque produced by $\boldsymbol{F}$ (i.e., in the *-**k** direction*).

c.) In what direction is the wheel's angular momentum?

<u>Solution</u>: The angular momentum vector is equal to $\boldsymbol{L} = I\omega$. As such, the direction of the angular momentum vector will always be the same as the direction of the angular velocity vector ω. The easiest way to determine the unit vector attached to ω is to remember that the unit vector will always be perpendicular to the plane in which the rotation occurs (for our situation, that direction will be along the *z-axis*). Further, ω will be positive if the rotation is counterclockwise (as viewed from the positive side of the axis) and negative if the rotation is clockwise. In short, looking from the side of our disk, the angular velocity, hence, angular momentum vector, will be straight up in the *+**k** direction*.

Note: Although this next question has a seemingly perverse result, its weirdness will be addressed at the end of the chapter. For now, view it as an exercise in the use of the *right hand rule* coupled with a bit of thinking about the rotational version of both *Newton's Laws* and *momentum*.

2.) The rotating wheel shown is supported by a gimbaled rod (that is, the rod cannot physically translate but it can rotate about its end in any direction). Ignore gravity (i.e., think of this as being an experiment done in the weightlessness of space). A force F is applied to the wheel/rod system for just a moment as shown in the sketch.



a.) In what direction is the resulting torque *about the gimbaled end*?

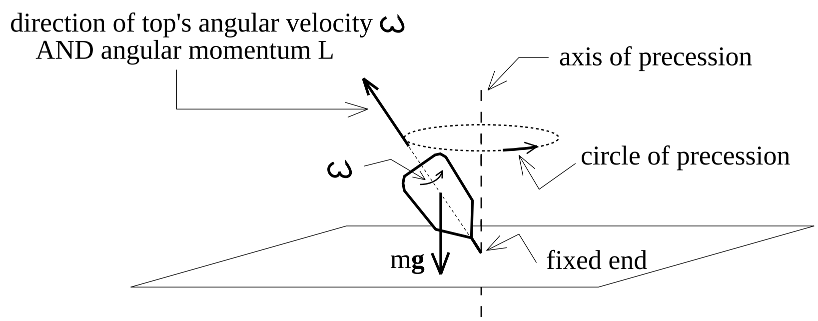
<u>Solution</u>: Torque is $\boldsymbol{r}x\boldsymbol{F}$, where $\boldsymbol{r}$ is a vector that extends *from* the axis of rotation *to* the point where the force acts. In this case, that vector goes from the gimbaled end vertically up to $\boldsymbol{F}$. If you do the *right hand rule* on this, your thumb will point *into* the page. That is the direction of the torque due to $\boldsymbol{F}$ about the end. Note: Just as a force that makes the magnitude of a body's acceleration increase or decrease must be directed along the line of the body's velocity vector, a torque that makes the magnitude of a body's angular velocity increase or decrease must be directed along the line of the body's angular velocity vector. IN THE CASE ALLUDED TO IN THIS QUESTION, the direction of the torque (into the page) and the direction of the wheel's angular velocity ω (up the axis in the *+**k** direction*) ARE NOT THE SAME. That means the torque provided by $\boldsymbol{F}$ will not affect the *magnitude* of the angular velocity of the wheel. What will it affect? You'll see shortly.

b.) In what direction is the system's resulting angular acceleration?

Solution: The direction of the angular acceleration will be the same as the direction of the net torque on the system. According to *Part a*, that's *into the page*.

c.) Assuming the angular velocity starts out in the *+**k** direction* (i.e., the wheel is rotating counterclockwise as viewed from above), what is the system's angular momentum vector going to do due to the application of ***F***?

Solution: This is where things get strange. The direction of the angular momentum vector is the same as the direction of the angular velocity of the wheel. In this case, that is *up the rod* in the *+**k** direction*. Just as the translational version of Newton's Second Law maintains that the *force* on an object is equal to its *change of momentum with time*, the rotational version of Newton's Second Law maintains that the *torque* on a body must equal the *change of a body's angular momentum with time*. That is, $\Gamma = \Delta(I\omega)/\Delta t$. As was noted in *Part a*, pushing on the rod surely isn't going to change the MAGNITUDE of the wheel's already existent angular momentum, so it must change the DIRECTION of the wheel's angular momentum. In other words, applying the force as shown in this case will make the wheel's axis (i.e., the rod) lurch *into* the page. Why? Because that's the direction the new angular momentum vector has to go, relative to the old angular momentum, so that the CHANGE of the angular momentum vector is in the same direction as the external torque. Now *that's* weird ... but, in fact, that's what's happening. (If you have ever played with a TOP, you've probably noticed that it circles--its axis changes direction as shown in the sketch. This precession is due to the phenomenon alluded to above. Gravity acting at the center of mass produces a torque about the fixed end of the TOP. The direction of that torque is perpendicular to the angular momentum vector, so the change of angular momentum motivates the top's axis to circle).

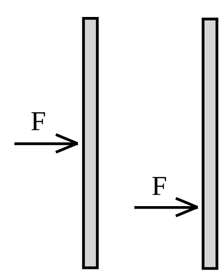


In the orientation shown, m**g** produces a torque about the fixed end that is approximately directed *out of page*. This motivates L to move in that direction and, as a consequence, the top precesses.

3.) Can an object that is not translating have kinetic energy?

Solution: The blade of a table saw does not translate, but it can definitely do damage to you. Rotational kinetic energy is the energy associated with rotational motion.

4.) A meterstick sitting on a frictionless surface has a force *F* applied at its *center of mass*. The same force is then applied to an identical meterstick halfway between its *center of mass* and its end (see sketch).



a.) In the second situation, why might the phrase "the stick's acceleration due to the force F" be somewhat misleading?

Solution: Examining the first stick, F is applied at the stick's *center of mass*, there is no rotation, and the acceleration $a = F/m$ will be applicable to every point on the stick. But because the force on the second stick is not applied at the *center of mass*, that stick will not only accelerate translationally, it will additionally *rotate about its center of mass*. That means each point on the stick will have a different translational acceleration. There is only one part of that stick that accelerates at F/m--the stick's *center of mass*--so the phrase should have been "the acceleration of the stick's *center of mass* due to F."

b.) In the second situation, the phrase *the stick's acceleration due to F* is misleading whereas the phrase *the stick's angular acceleration due to F* is NOT misleading. How so?

Solution: This is most easily seen by examining the idea of *angular velocity*, then extrapolating to the idea of *angular acceleration*. The angular velocity of an object about its *center of mass* will be the same as the angular velocity about *ANY POINT ON THE OBJECT*. That is, if you sit at the *center of mass* and count the number of radians the object sweeps through as it rotates about you in a given time interval, the number you'll come up with will be the same as the number you'll come up with if you do the same process while sitting at any other point on the object. (That's why angular parameters are so nice.) The same can be said about an object's *angular acceleration*. If you know the angular acceleration relative to one point on the body (say, relative to the *center of mass*), you know the angular acceleration relative to *any* point on the body.

c.) Will the acceleration of each stick's *center of mass* be different in the two situations? If so, how so?

Solution: Just because there is rotation in one case and no rotation in the other doesn't mean the translational version of Newton's Second Law (i.e., $F_{net,x} = ma_x$) is no longer valid. The translational acceleration of either stick's *center of mass* will still equal the total force F acting on the stick in the *x direction* divided by the stick's mass, and that number will be the same in both cases.

d.) Will the stick's angular acceleration about its *center of mass* be different in the two situations? If so, how so?

Solution: There is no angular acceleration for the first stick because there is no torque being applied to that stick about its *center of mass* (a force acting through a point will not produce a torque about that point). As there is a torque on the second stick, the angular accelerations will be "different."

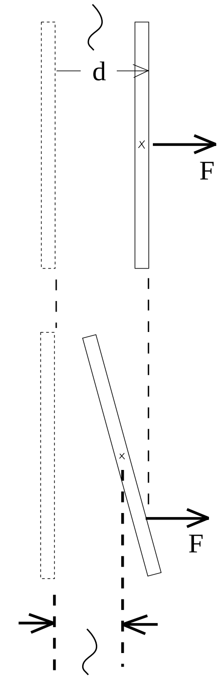
e.) Will the velocity of each stick's *center of mass* be different? If so, how so?

Solution: Because the acceleration of each stick's *center of mass* will be the same, the velocity change will be the same and the two objects will parallel one another as far as *velocity in the x direction* goes. (Note: Whenever you see the word *velocity* alone, the word refers to *translational velocity*. If you want to designate *angular* velocity, you have to do just that by using the word *angular* or *rotational* in front of the velocity term.)

f.) Will the angular velocity about the stick's *center of mass* be different in the two situations? If so, how so?

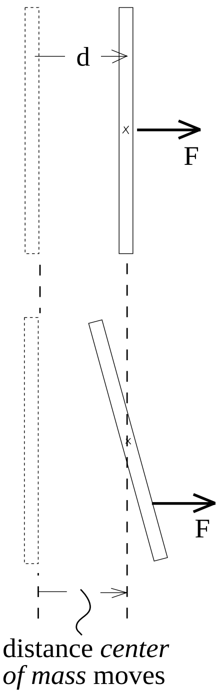
<u>Solution</u>: Because there is a torque in the second case and no torque in the first case, the second case will have an angular acceleration, hence angular velocity, associated with its motion whereas the first won't.

g.) Assume the force in both cases acts over a small displacement *d*. How does the work done in each case compare?



<u>Solution</u>: This is tricky. By the definition of work, if the force acts over the same distance *d* in both cases, the same amount of work is done on both systems. On the other hand, as the first case is solely translational while the second case is both translational *and* rotational, it would seem more energy would be required to support the second case. To add to the confusion, we also have to contend with the fact that the *center of mass* accelerations must be the same in both cases (remember, N.S.L. doesn't cease to exist just because there's rotation in the system--a net force on a body in a particular direction, *no matter where the force is applied to the body in that direction*, is going to accelerate the body's *center of mass* by an amount equal to F/m), so it looks like the same amount of energy is being pumped into both systems when clearly we need more energy in one system than the other. So what's going on? The key lies in a subtlety of geometry. Because the force is applied at the *center of mass* in the first case, the distance the force is applied and the distance the *center of mass* actually moves are the same (see sketch). Also, it will take, maybe, *t seconds* to execute this motion. Because the force is applied at a point *other than the center of mass* in the second case, the distance the force is applied and the distance the *center of mass* actually moves are NOT THE SAME (see second sketch). In fact, the *center of mass* will travel *less distance* in that second case. Additionally, as the *center of mass* accelerations are the same in both cases, the *time* of acceleration in that second case will be *less than* the time of acceleration in the first case (this makes sense if you think about it--the force in the first case is fighting the entire inertia of the mass as it motivates that entire mass forward; the force in the second case is motivating some mass forward while some of the mass rotates backwards--the net effect is that the point at which the force is applied in the second case will shoot forward and reach *d* quicker, so the time of motion will be less). Bottom line: The amount of work done in both cases will be the same, but because the times of acceleration are different, the *center of mass* velocities (hence, the translational kinetic energies) will differ. That energy discrepancy is accounted for in the second case as rotational kinetic energy.

h.) Assume the force in both cases acts over a small *center of mass* displacement *d* (say, 2 centimeters). How does the work done in each case compare?



Solution: This is reminiscent of *Part g*. If the *center of mass* velocities of the two objects are going to parallel one another, the work needed to change each stick's translational kinetic energy must be the same. But if work is also needed to change *rotational velocity*, which happens to be the case in the second situation, then still more energy must be pumped into that system. In other words, the force F must do more work on the second stick than it does on the first stick. This may seem strange, given the fact that the *center of mass*es move the same distance, but it isn't. Think about it. Assume the force on the second stick is applied at *point P*. As that stick rotates, does the stick's *center of mass* or *point P* move farther? *Point P* physically moves farther than the *center of mass* does. That means the force acting at that point acts *over a larger distance* than was the case with the first meterstick. It is the work done by the force acting over that extra distance that powers the rotation.

5.) Why does a homogeneous ball released from rest roll downhill? That is, what is going on that motivates it to roll? (Hint: No, it's not just that there is a force acting! There are all sorts of situations in which forces act and rolling does not occur.)

Solution: What motivates objects to roll is *torque*. In the case of the ball, you can either look at the torque relative to the *center of mass* (in that case, the torque will be produced by the friction between the ball and the incline) or the torque relative to the point of contact with the incline (in that case, the torque will be produced by gravity acting at the body's *center of mass*). In all cases, rolling doesn't happen unless there is a net torque acting on the object.

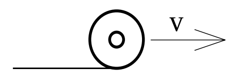
6.) A spinning ice skater with his arms stretched outward has kinetic energy, angular velocity, and angular momentum. If the skater pulls his arms in, which of those quantities will be conserved? For the quantities that aren't conserved, how will they change (i.e., go up, go down, what?)? Explain. (Hint: I would suggest you begin by thinking about the *angular momentum*.)

Solution: As the guy pulls his arms in, there is no torque about his axis of rotation (the muscular force he applies to himself is along a line through his axis of rotation, so it produces no torque about that axis). As a consequence, ANGULAR MOMENTUM must be CONSERVED. In this case, though, the *moment of inertia* decreases as he pulls his arms in (his overall mass is getting closer to his axis of rotation diminishing his rotational inertia). As the constant angular momentum L is a function of moment of inertia I and angular velocity ω, a decrease in *moment of inertia* means an INCREASE in ANGULAR VELOCITY. In other words, as anyone who has ever watched an ice skating exhibition knows, when the arms come in, the rotation speeds up. In short, while L remains the same, I decreases and ω increases proportionally. As for kinetic energy, that is a function of angular velocity *squared* (i.e., $KE_{rot} = .5I\omega^2$). So although I goes down, ω goes up proportionally *and is squared*. The net effect is that his ROTATIONAL KINETIC ENERGY will INCREASE. (Where does the energy come from? His muscles provide it by burning chemical energy inside his body.)

7.) An object rotates with some angular velocity. The angular velocity is halved. By how much does the rotational kinetic energy change?

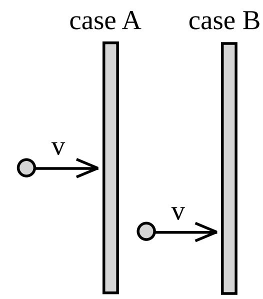
Solution: Rotational kinetic energy is equal to $KE_{rot} = .5I\omega^2$, so halving ω drops the kinetic energy by a factor of 4.

8.) If you give a roll of relatively firm toilet paper an initial push on a flat, horizontal, hardwood floor, it may not slow down and come to a rest as expected but, instead, pick up speed. How so?



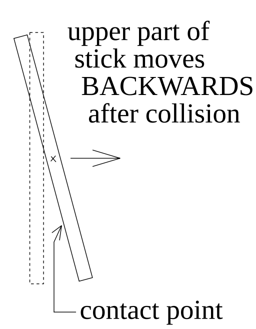
Solution: This is a fun one. As the roll lays down more and more t.p., its *center of mass* lowers. As the *center of mass* drops, gravitational potential energy is converted into kinetic energy and the body continues to roll on.

9.) A meterstick of mass m sits on a frictionless surface. A hockey puck of mass $2m$ strikes the meterstick perpendicularly at the stick's *center of mass* (call this *case A*). A second puck strikes an identical meterstick in the same way on an identical frictionless surface, but does so halfway between the stick's *center of mass* and its end (call this *case B*).



a.) Is the average force of contact going to be different in the two cases? If so, how so?

Solution: The temptation is to think that because the collision velocities are the same, the average contact forces will be the same. It turns out that that isn't the case. First, some observations: To begin with, because the mass of the puck is *double* the mass of the meterstick, it is safe to assume that when the collision occurs, the puck will not rebound but will, instead, continue moving in its original direction with diminished speed. This will be true in both cases. It is also safe to assume that the contact point on the meterstick will leave the puck with a velocity that is greater than that of the puck (i.e., the two will separate . . . in a way, that is the only way it *can* be). It should also be noted that the force of contact will not be constant over time (that is the reason the question alludes to *average force*). At first brush, the force will be slight, growing as the impact deepens (remember, a collision between two solid objects typically occurs over a period of, maybe, several hundredths of a second). The key to untangling this question is to note that if the meterstick is easily motivated to its separation speed, the force will not have the time required to grow to the same extent that it would if the meterstick had been more inert (i.e., more difficult to motivate to separation speed). So what's happening in each case? In *case B*, the puck hits the meterstick away from the stick's *center of mass*. That



means the stick both rotates and translates. As for translation, part of the stick will accelerate out away from the puck while the upper section of the stick will lag behind having rotated around the *center of mass* (see the sketch). As such, the effective inertia of the stick will be less than if the entire stick was required to accelerate uniformly as would be the situation in *case A*. As for the rotation, it will be relatively easy to motivate the meterstick into rotational motion (the moment of inertia for a rod or stick is $(1/12)mL^2$, where $L = 1$ *meter* for a meterstick and, hence, $I = m/12$--this will be small in comparison to the meterstick's translational inertia m). In short, the net inertia (rotational and translational) that must be overcome to make the meterstick separate from the puck will be relatively small in *case B*. In *case A*, on the other hand, the puck hits the meterstick dead center. There is no rotation, but the force of the collision has to overcome the entire inertia of the meterstick in motivating it away from the puck. Why? Because the meterstick will have to uniformly accelerate along its entire length. In other words, the stick will put up more resistance to changing its motion than would be the situation in *case B* and, as a consequence, will absorb more force before separation occurs. In fact, the farther out from the stick's *center of mass*, the less the average contact force will be.

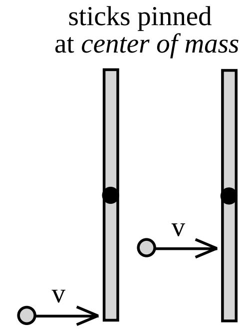
b.) Is the puck's after-collision velocity going to be different in the two cases? If so, how so?

Solution: There are two ways to do this. Both make use of the fact that in *Part a* above, we concluded that the *average contact force* on the puck AND the *time of contact* was smaller in *case B* than in *case A*. With the average contact force on the puck being less in *case B*, the average acceleration will also be less in *case B*. And if that acceleration occurs over a smaller time, then the net change of puck velocity will be less and, hence, the final puck velocity will be greater in *case B* than in *case A*. The other option is to note that with the average contact force and time of contact both being smaller in *case B*, the impulse applied to the puck in *case B* will be smaller and, hence, so will its momentum change. With both systems having the same initial momentum (i.e., 2mv), that means *case B's* final velocity will be closer to its initial velocity than will be the situation in *case A*, and its final velocity will be larger.

c.) Is the puck's after-collision angular velocity (relative to the stick's *center of mass*) going to be different in the two cases? If so, how so?

Solution: You don't tend to think of a translating puck as having angular velocity about some point, but it will as long as the line of its motion doesn't pass through that point. In this case, the angular velocity of the puck about the stick's *center of mass* in *case A* will be zero because the line of that puck's motion *will* pass through the stick's *center of mass*. In the *case B*, though, the puck's angular velocity about the stick's *center of mass* will be $\omega = v/(.25 \text{ meters})$. So what is the *after-collision* angular velocity dependent upon? The torque applied to the puck about the stick's *center of mass*. The torque in the first situation will be zero (again, you are dealing with a force that acts through the stick's *center of mass*) whereas the torque on the second puck will be non-zero. With the two torques being different, the angular accelerations about each *center of mass* will differ and, as a consequence, the final angular velocities about each *center of mass* will be different.

10.) A meterstick of mass m is pinned at its *center of mass* on a frictionless surface. A puck whose mass is $10m$ strikes and sticks to the meterstick at the .33 meter mark (i.e., .17 meters from the pin). Call this *case A*. A second meterstick experiences exactly the same situation except that its puck strikes and sticks at its end. Call this *case B*.



a.) Is energy conserved through either collision?

<u>Solution</u>: Energy is practically never conserved during a collision (it's close, maybe, when you're talking about charged subatomic particles interacting with one another, but in the real world, at least some energy in a collision is given up as heat or sound or in the rearrangement of material we call *deformation*). As such, energy would not be conserved in this collision (note: to add a little extra twist, some physics problems maintain that a collision is *elastic* . . . meaning energy is supposed to be conserved--this is always a contrived situation).

b.) In which case will the final angular speed be larger, and by how much?

<u>Solution</u>: This isn't an intuitively obvious question (well, the question may be but the answer isn't). At first glance, I'd say the greater angular velocity would belong to *case B*. It turns out that that isn't right. To see this, think about the puck for a second. Being a point mass, it's moment of inertia is $I = mr^2$. The fact that I is proportional to r^2 means that the farther out you go, the greater the puck's resistance to changing its angular motion. The torque applied to the puck when it hits a distance r meters from the pin will be rF (this is the magnitude of $\mathbf{r} \mathbf{x} \mathbf{F}$ when $\mathbf{r}$ and $\mathbf{F}$ are perpendicular to one another). The rotational version of Newton's Second Law states that $\Gamma = I\alpha$, or $rF = (mr^2)\alpha$, so evidently the relationship between r and α is $\alpha = (F/m)(1/r)$. With the angular acceleration and r being inversely related, the farther a given force is applied from the pin, the smaller the angular acceleration. So what does all this mean? It means we can expect that the angular acceleration will be *less* the farther out the hit occurs and, as a consequence, the angular velocity will be less the farther out the hit occurs. Does this make sense? It does *if the forces are the same*. Are they? Not if you believe the arguments that were made in the preceding problem--*Problem 9a*. So how do we proceed from here? Enter the *conservation of angular momentum* (the crowd gasps). We will do the analysis in pieces by closely examining *case B*. Observe: 1.) The meterstick begins with no angular momentum (it isn't initially moving at all). 2.) The puck has initial angular momentum, relative to the *center of mass* (remember, an object moving in a straight line has angular momentum unless its line of travel passes through the reference point). 3.) The torque on the puck due to the collision is internal to the system (that is, it's due to the puck's interaction with the meterstick), and the torque on the meterstick due to the collision is also internal to the system. 4.) The pin force provides no torque to the system about the pin, so all of the torques acting in the system are internal. 5.) *Angular momentum* is conserved when all the torques on a system are internal. 6.) The system's initial angular momentum is all wrapped up in the angular momentum of the puck. That quantity is mvd, where d is the distance between the collision point and the meterstick's *center of mass*. 6.) Noting that ω is the system's final angular velocity, the system's final angular momentum is the sum of $I_{puck}\omega$ (equal to $(m_{puck}d^2)\omega$ =

$(10md^2)\omega$) and $I_{meterstick}\omega$ (equal to $[(1/12)m_{meterstick}L_{meterstick}{}^2]\omega$). With $L = 1$ *meter* for the meterstick's length, and noting that $1/12 = .0833$, this last term becomes $.0833m\omega$. Putting it all together by equating the initial and final angular momenta yields a relationship between ω and the impact distance d in terms of the initial puck velocity v. Specifically, the relationship becomes $mvd = 10md^2\omega + .0833m\omega$. Solving yields $\omega = [10d/(10d^2 + .0833)]v$. Substituting $d = .17$ *meters* yields an angular velocity of 4.57v. Substituting $d = .5$ *meters* yields an angular velocity of 1.94v. Great jumping huzzahs! The hit farther out produces the smaller angular velocity, just as predicted with all the hand waving.

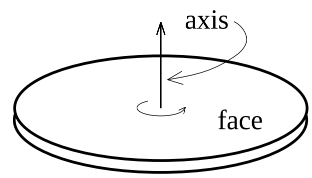
11.) A rotating ice skater has 100 joules of rotational kinetic energy. The skater increases her *moment of inertia* by a factor of 2 (i.e., she extends her hugely muscular arms outward). How will her rotational speed change?

Solution: As there are no external torques acting, angular momentum must be conserved. That means the product of the moment of inertia I and angular velocity ω will never change. Be that the case, if I increases by a factor of 2, ω must decrease by a factor of 2.

12.) It is easier to balance on a moving bicycle than on a stationary one. Why?

Solution: If your weight is off-center while sitting on a stationary bicycle, the bike will rotate about the ground and come crashing down. If you do the same thing on a moving bicycle, it will take a considerably greater off-set to make the bike go down. Why? A rotating wheel has angular momentum directed along its axle. Changing the direction of that angular momentum vector--something that would have to happen if the bike were to fall over--requires a sizable torque. But unless you tilt the bike considerably, no such torque is available. In other words, when you off-set your weight just a bit, the wheel's angular momentum fights the change of axis orientation allowing you time to re-set your weight appropriately.

13.) A disk lying face-up spins without translation on a frictionless surface. At its *center of mass*, its angular velocity about an axis perpendicular to its face is measured and found to equal N. Its angular momentum at that point is measured to be M.

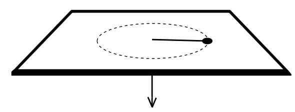


a.) Is there any other *point P* on the disk where the angular velocity about P is equal to N? Explain.

Solution: Assume you are sitting in a chair just above the *center of mass* of the disk (if you'd like, think of the disk as a huge *merry go round*). Assume also that the chair's position is oriented in a fixed direction (to do this on the *merry go round*, you would need a chair that was fixed so that it didn't rotate with the *merry go round*). The disk's angular velocity in this case tells you the number of *radians* you will observe the disk sweep through *per unit time* as it moves around underneath you. In most cases, angular velocity quantities are quoted relative to the axis of rotation, which is usually through the mass's *center of mass*. But if you were hovering above some point other than the *center of mass* (you'd have to be moving with the disk to do this, but assume you could), sitting in your fixed-direction chair, how many radians of the disk would sweep under you per unit time. The answer is *the same number of radians per second as you would have observed while sitting over the center of mass*. This makes perfect sense if you notice the following: no matter where you are located on the disk, it will always take the same amount of time for the entire disk to rotate around underneath you once (i.e., through 2π radians). As such, the angular velocity *relative to any point on the disk* will be the same as the angular velocity about any *other* point on the disk.

b.) Is there any other *point P* on the disk where the angular momentum about *P* is equal to *M*? Explain.

Solution: You might think that because the angular velocity about any point on a rotating object is the same as about any other point, the angular momentum--an angular velocity related quantity--would also be the same. The problem is that angular momentum is not only associated with angular velocity, it is also associated with *moment of inertia*. As the *moment of inertia* is going to increase as one gets farther away from the *center of mass*, the angular momentum is also going to increase. By how much? The *moment of inertia* is generally a function of r^2, so you would expect the angular momentum to increase as the *square of the distance* between the point and the disk's center.



14.) A string threaded through a hole in a frictionless table is attached to a puck. The puck is set in motion so that it circles around the hole. The string is pulled, decreasing the puck's radius of motion. When this happens, the puck's angular velocity increases. Explain this using the idea of:

a.) Angular momentum.

Solution: Because there are no external torques acting on the puck about the hole (in fact, there are no torques acting at all as the tension is along the line of $\boldsymbol{r}$), angular momentum will be conserved. As the radius of the circling puck diminishes, the puck's *moment of inertia* $I = mr^2$ also diminishes. For angular momentum (i.e., $I\omega$) to remain constant, therefore, ω must increase.

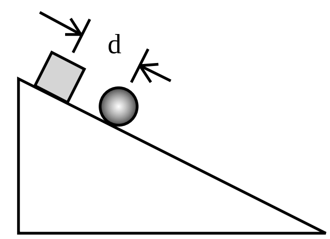
b.) Energy.

Solution: By pulling the string down through the hole, you are doing work ($\boldsymbol{F}\cdot\boldsymbol{d}$) on the puck. The energy must show itself somehow. It does so with an increase in the puck's rotational kinetic energy ($.5I\omega^2$). With the moment of inertia diminishing (r is getting smaller), this energy increase shows itself as an increase in the angular velocity of the puck.

15.) When a star supernovas, it blows its outer cover outward and its core inward. For moderately large stars (several solar masses), the implosion can produce a structure that is so dense that one solar mass's worth of material would fit into a sphere of radius *less than 10 miles*. All stars rotate, so what would you expect the rotational speed of the core of a typical star to do when and if the star supernovaed? Explain using appropriate conservation principles.

Solution: As there are no external torques being applied during a supernova, the star's angular momentum will be conserved. If its mass is compressed into a very small ball, its *moment of inertia* will drop precipitously. To keep the angular momentum constant, its angular velocity must go up as much as the *moment of inertia* goes down. Stars that do this, called *neutron stars* or *pulsars*, typically have rotational speed upwards of 60 revolutions per second (think about it--an object that is 10 miles across rotating 60 times every second . . .). Pretty amazing.

16.) A cube and a ball of equal mass and approximately equal size are d units apart on a very slightly frictional incline plane (frictional enough for the ball to grab traction but not frictional enough to take discernible amounts of energy out of the system). By the time the ball gets to the bottom of the ramp, will the distance d be larger, smaller, or the same as it was at the beginning of the run? Use *conservation principles* to explain.

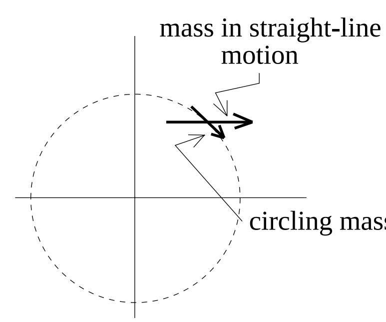


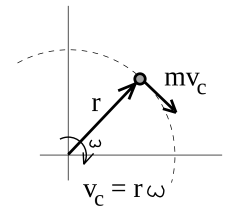
Solution: This is most easily seen by looking at the system from the perspective of energy. If both bodies drop a distance h in the same amount of time (i.e., if d remains the same throughout), the total kinetic energy should be the same for both at the end of the drop (remember, friction isn't removing a discernible amount of energy from the system). Unfortunately, there are two ways kinetic energy can show itself, as translational kinetic energy and as rotational kinetic energy. The *block's* kinetic energy is all translational. The *ball's* kinetic energy is part translational, part rotational. In other words, the block is going to pick up more translational kinetic energy than will the ball and, hence, will pick up more translational velocity than will the ball. In short, d should diminish with time.

17.) Assume global warming is a reality. How will the period of the earth's rotation change as the Arctic ice caps melt?

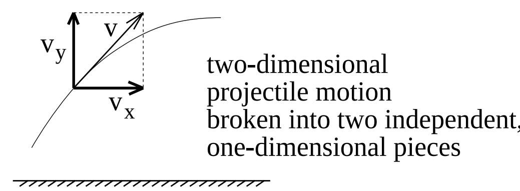
Solution: In the last chapter, we concluded that if the Arctic ice cap melts, the released water would flow outward away from the axis of rotation (i.e., toward the earth's equator) and the earth's moment of inertia I about its axis of rotation would increase. As angular momentum ($\mathbf{L} = I\omega$) would be conserved in this operation (there are no external torques acting on the system), an increase of the earth's moment of inertia would elicit a decrease in the earth's angular velocity.

18.) A point mass m moving along a circular path of radius r passes a second point mass m moving in the x *direction* (see sketch). Is it possible for the two objects to have the same angular momentum and, if so, what conditions must be met for this to happen?

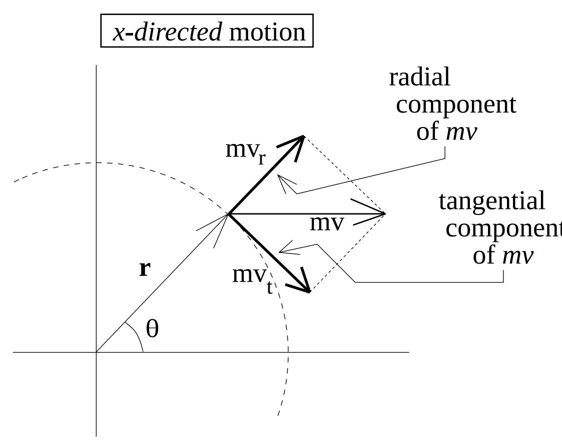




<u>Solution</u>: Let's start with the easy stuff. The mass that is circling about the origin clearly has an angular momentum associated with its motion. There are two ways to determine the magnitude of that angular momentum. The first is to deal solely with rotational parameters. In that case, $L = I\omega$. Remembering that the moment of inertia for a point mass is mr^2, and noting that the angular velocity ω is related to the magnitude of the instantaneous velocity by $\omega = v_c/r$ (I'm obviously defining the magnitude of the circling mass's velocity to be v_c), we can write $L = I\omega = (mr^2)(v_c/r) = mrv_c$. As a quick review, the second way to do this is to determine the magnitude of the cross product of $\boldsymbol{r}\mathrm{x}\boldsymbol{p}$. Noting that the *line of* **r** and the *line of the momentum m***v**$_c$ are perpendicular, that operation yields $L = r(mv_c) \sin 90^o = mrv_c$ . . . and isn't that nice. We get the same result both ways. Now for the fun part. Because people don't intuitively associate angular properties with bodies moving in straight-line motion, one of the more obscure ideas students run into in dealing with the world of angular motion is the idea that a body moving in a straight line can have *angular momentum*. The reason I've included this problem is because there is a sane, conceptually appealing reason for concluding that this must be so. To see it, all you need is the right perspective. That's what I'm about to give you. But first, a small but important digression. Remember back when we talked about projectile motion. In those discussions, it was pointed out that two dimensional motion is really nothing more than *x-type* motion and *y-type* motion happening independently to the same body at the same time (see sketch). So when you attacked a projectile problem, how did you proceed? You wrote out an equation that had to do with the *x motion* (remember, with no friction, $a_x = 0$) completely ignoring what was happening in the *y direction* where the acceleration was $a_y = -9.8\ m/s^2$. You could do this kind of selective thinking because the two directions were independent of one another.

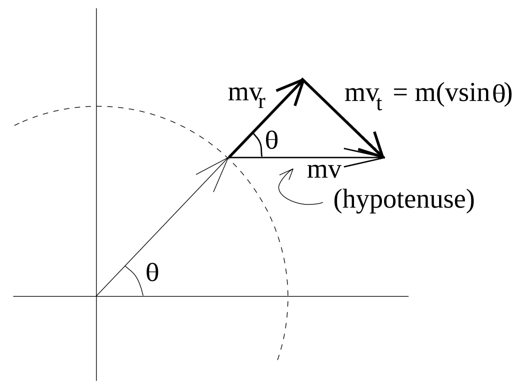


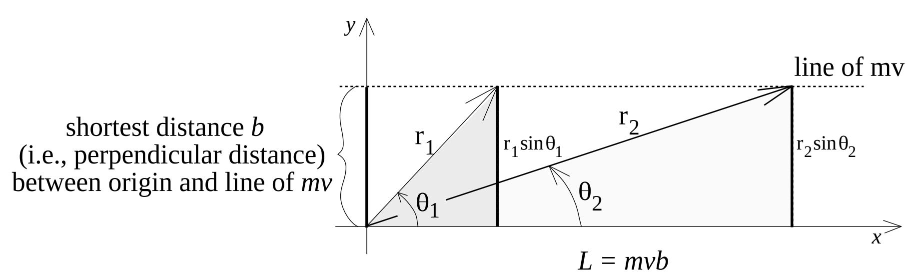
Well, we are about to do a similar thing here. Instead of thinking of the *x-directed motion* as straight-line motion (you might want to look at the sketch now), I want you to think of it as a combination of two independent bits of motion--*radial motion* (i.e., motion along a radial vector that extends from the origin to the mass) and *tangential motion* (i.e., motion that is tangent to a circle upon which the mass resides at a particular point in time). This rather strange combination is shown in the sketch (note that the tangential component of the mass's momentum mv_t and the radial component of the mass's momentum mv_r vectorially add together to equal the mass's total momentum mv). As bizarre as it may seem (to reiterate), what the sketch is suggesting is that you can think about straight-line motion as a combination of radial and tangential components of motion that are happening at the same time (just as projectile motion can



be thought of *x-type* and *y-type* motion independently happening at the same time). Once you've bent your mind around this idea, it's not too big a jump to see that there is really no difference, at least instantaneously, between the tangential component of the *x-directed* momentum (i.e., mv_t) and the momentum of the circling mass mv_c. Their magnitudes may or may not be the same, but their directions are, at least instantaneously, parallel to one another.

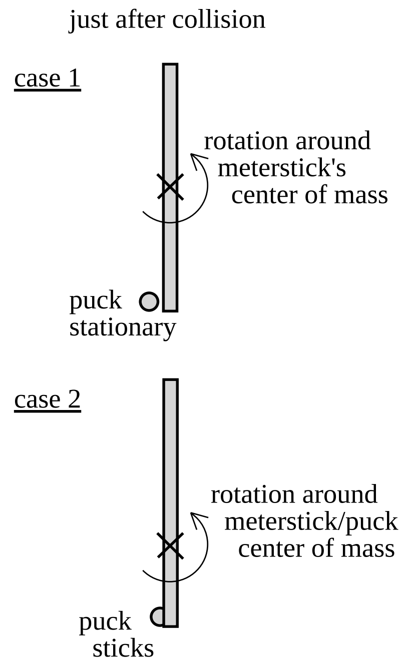
If you are willing to accept this proposition--that *x-directed* straight-line motion can be viewed as having a *circular (i.e., angular) component* to it--then it is not at all difficult to see how that same motion might have angular momentum attributed to it. And in fact, that is exactly the case. The *x-directed* motion has an angular momentum whose magnitude is equal to the magnitude of $\mathbf{r} \times \mathbf{p} = \mathbf{r} \times (m\mathbf{v})$. Noting that the component of $m\mathbf{v}$ perpendicular to $\mathbf{r}$ is $mv_t = mv \sin\theta$ (see sketch), the evaluation of the cross product using the perpendicular component approach becomes $L = rp_t = r(mv_t) = r(mv \sin\theta) = rm(v \sin\theta)$. (Note that this was EASY to compute mathematically--you could have mindlessly, mechanically done it whether you understood the concepts being discussed here or not--the reason you've been slogging through all of this hand waving is because mindless calculations are just that, mindless--the trick is to understand why the moves are legitimate). When we compare that (i.e., $rm(v \sin\theta)$) to the circling mass's angular momentum (i.e., mrv_c), it becomes obvious that when $v_c = v_t = v \sin\theta$, the two angular momenta will be the same. Oh, and there is one more interesting bit of whimsy that should be noted. No matter what $\mathbf{r}$ happens to be at a given instant, $r \sin\theta$ will ALWAYS equal the distance labeled as b in the sketch (note that b is the perpendicular distance--read this *shortest distance*--between the point about which you are taking the angular momentum--in this case, that would be the origin of our coordinate axis--and the line of mv). Put a little differently, no matter what r and θ are, it will always be true that $r \sin\theta = b$ (i.e., that perpendicular distance) and $L_{puck} = r(mv \sin\theta) = mv(r \sin\theta) = mvb$.





Bottom line: The magnitude of the angular momentum of any object moving in straight line motion will equal the momentum mv of the object times the shortest distance b between the *line of the momentum vector* and the point about which the momentum is being taken.

19.) Two experiments are done involving a puck sliding over a frictionless surface and striking a meterstick at its end. In the first case, the puck stays motionless after the hit (that is, the puck hits and, as a consequence of the contact, loses all of its kinetic energy). In the second case, the puck sticks to the meterstick. If you wanted to derive an expression for, say, the angular velocity of the meterstick after the collision, there would be one major difference in the way you would set the two problems up. What might that difference be?



Solution: Finding the *angular velocity* of the meterstick after the collision is essentially an *angular momentum* problem (all the torques during the collision are internal to the system, so angular momentum through the collision will be conserved). In the first case, the meterstick will not end up attached to the puck and, as a consequence, will rotate about its own center of mass after the collision. For that problem, it would be wise to do everything relative to the meterstick's center of mass (that is, calculate the initial angular momentum ***rxp*** for the puck relative to the meterstick's mass center, etc.). In the case of the puck sticking to the meterstick, the meterstick's final rotation will be about the center of mass of the meterstick/puck system. In that case, it would be wise to determine the new center of mass and do the entire problem relative to that point. The sketches highlight the points of importance.